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On limiting values of Cauchy type integral in a harmonic algebra with two-dimensional radical

Abstract. We consider a certain analog of Cauchy type integral taking values in a three-dimensional harmonic algebra with two-dimensional radical. We establish sufficient conditions for an existence of limiting values of this integral on the curve of integration.

1. Introduction. Let $\Gamma$ be a closed Jordan rectifiable curve in the complex plane $\mathbb{C}$. By $D^+$ and $D^-$ we denote, respectively, the interior and the exterior domains bounded by the curve $\Gamma$.

N. Davydov [1] established sufficient conditions for an existence of limiting values of the Cauchy type integral

\[
\frac{1}{2\pi i} \int_{\Gamma} \frac{g(t)}{t - \xi} dt, \quad \xi \in \mathbb{C} \setminus \Gamma,
\]

on $\Gamma$ from the domains $D^+$ and $D^-$. This result stimulated a development of the theory of Cauchy type integral on curves which are not piecewise-smooth.

In particular, using the mentioned result of the paper [1], the following result was proved: if the curve $\Gamma$ satisfies the condition (see [2])

\[
\theta(\varepsilon) := \sup_{\xi \in \Gamma} \frac{\theta_\varepsilon(\varepsilon)}{\varepsilon} = O(\varepsilon), \quad \varepsilon \to 0
\]

2010 Mathematics Subject Classification. 30G35, 30E25.

Key words and phrases. Three-dimensional harmonic algebra, Cauchy type integral, limiting values, closed Jordan rectifiable curve.
(here \( \theta_\varepsilon(\xi) := \operatorname{mes}\{t \in \Gamma : |t - \xi| \leq \varepsilon\} \), where \( \operatorname{mes} \) denotes the linear Lebesgue measure on \( \Gamma \), and the modulus of continuity

\[
\omega_g(\varepsilon) := \sup_{t_1, t_2 \in \Gamma, |t_1 - t_2| \leq \varepsilon} |g(t_1) - g(t_2)|
\]

of a function \( g : \Gamma \to \mathbb{C} \) satisfies the Dini condition

\[
\int_0^1 \frac{\omega_g(\eta)}{\eta} d\eta < \infty,
\]

then the integral (1) has limiting values in every point of \( \Gamma \) from the domains \( D^+ \) and \( D^- \) (see [3]). The condition (2) means that the measure of a part of the curve \( \Gamma \) in every disk centered at a point of the curve is commensurable with the radius of the disk.

In this paper we consider a certain analogue of Cauchy type integral taking values in a three-dimensional harmonic algebra with two-dimensional radical and study the question about an existence of its limiting values on the curve of integration.

2. A three-dimensional harmonic algebra with a two-dimensional radical. Let \( \mathbb{A}_3 \) be a three-dimensional commutative associative Banach algebra with unit 1 over the field of complex numbers \( \mathbb{C} \). Let \( \{1, \rho_1, \rho_2\} \) be a basis of algebra \( \mathbb{A}_3 \) with the multiplication table:

\[
\rho_1 \rho_2 = \rho_2 \rho_1 = 0, \quad \rho_2 \rho_1 = \rho_1 \rho_2.
\]

\( \mathbb{A}_3 \) is a harmonic algebra, i.e. there exists a harmonic basis \( \{e_1, e_2, e_3\} \subset \mathbb{A}_3 \) satisfying the following conditions (see [5], [6], [7], [8], [9]):

\[
e_1^2 + e_2^2 + e_3^2 = 0, \quad e_j^2 \neq 0 \quad \text{for } j = 1, 2, 3.
\]

P. Ketchum [5] discovered that every function \( \Phi(\zeta) \) analytic with respect to the variable \( \zeta := xe_1 + ye_2 + ze_3 \) with real \( x, y, z \) satisfies the equalities

\[
\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) \Phi(\zeta) = \Phi''(\zeta) (e_1^2 + e_2^2 + e_3^2) = 0
\]

owing to the equality (4). I. Mel’nichenko [7] noticed that doubly differentiable in the sense of Gateaux functions form the largest class of functions \( \Phi \) satisfying the equalities (5).

All harmonic bases in \( \mathbb{A}_3 \) are constructed by I. Mel’nichenko in [9].

Consider a harmonic basis

\[
e_1 = 1, \quad e_2 = i + \frac{1}{2} i \rho_2, \quad e_3 = -\rho_1 - \frac{\sqrt{3}}{2} i \rho_2
\]

in \( \mathbb{A}_3 \) and the linear envelope \( E_3 := \{\zeta = xe_2 + ze_3 : x, y, z \in \mathbb{R}\} \) over the field of real numbers \( \mathbb{R} \), that is generated by the vectors \( 1, e_2, e_3 \). Associate with a domain \( \Omega \subset \mathbb{R}^3 \) the domain \( \Omega_\zeta := \{\zeta = xe_2 + ze_3 : (x, y, z) \in \Omega\} \) in \( E_3 \).
The algebra $\mathcal{A}_3$ have the unique maximal ideal $\{\lambda_1 \rho_1 + \lambda_2 \rho_2 : \lambda_1, \lambda_2 \in \mathbb{C}\}$ which is also the radical of $\mathcal{A}_3$. Thus, it is obvious that the straight line $\{ze_3 : z \in \mathbb{R}\}$ is contained in the radical of algebra $\mathcal{A}_3$.

$\mathcal{A}_3$ is a Banach algebra with the Euclidean norm

$$\|a\| := \sqrt{|\xi_1|^2 + |\xi_2|^2 + |\xi_3|^2},$$

where $a = \xi_1 + \xi_2 e_2 + \xi_3 e_3$ and $\xi_1, \xi_2, \xi_3 \in \mathbb{C}$.

We say that a continuous function $\Phi : \Omega \to \mathcal{A}_3$ is *monogenic* in a domain $\Omega \subset E_3$ if $\Phi$ is differentiable in the sense of Gateaux in every point of $\Omega$, i.e., if for every $\zeta \in \Omega$ there exists $\Phi'(\zeta) \in \mathcal{A}_3$ such that

$$\lim_{\varepsilon \to 0^+0^-} \frac{(\Phi(\zeta + \varepsilon h) - \Phi(\zeta))}{\varepsilon} = h\Phi'(\zeta) \quad \forall h \in E_3.$$

For monogenic functions $\Phi : \Omega \to \mathcal{A}_3$ we established basic properties analogous to properties of analytic functions of the complex variable: the Cauchy integral theorem, the Cauchy integral formula, the Morera theorem, the Taylor expansion (see [11]).

3. **On existence of limiting values of a hypercomplex analogue of the Cauchy type integral.** In what follows, $t_1, t_2, x, y, z \in \mathbb{R}$ and the variables $x, y, z$ with subscripts are real. For example, $x_0$ and $x_1$ are real, etc.

Let $\Gamma \subset \mathcal{C}$ be the curve congruent to the curve $\Gamma \subset \mathbb{C}$. Consider the domain $\Pi \subset E_3$ if $\Phi$ is differentiable in the sense of Gateaux in every point of $\Omega$, i.e., if for every $\zeta \in \Omega$ there exists $\Phi'(\zeta) \in \mathcal{A}_3$ such that

$$\lim_{\varepsilon \to 0^+0^-} \frac{(\Phi(\zeta + \varepsilon h) - \Phi(\zeta))}{\varepsilon} = h\Phi'(\zeta) \quad \forall h \in E_3.$$

For monogenic functions $\Phi : \Omega \to \mathcal{A}_3$ we established basic properties analogous to properties of analytic functions of the complex variable: the Cauchy integral theorem, the Cauchy integral formula, the Morera theorem, the Taylor expansion (see [11]).

Consider the integral

$$\Phi(\zeta) = \frac{1}{2\pi i} \int_{\Gamma} \varphi(\tau)(\tau - \zeta)^{-1}d\tau$$

with a continuous density $\varphi : \Gamma \to \mathbb{R}$. The function (6) is monogenic in the domains $\Pi^+$ and $\Pi^-$, but the integral (6) is not defined for $\zeta \in \Sigma$.

For the function $\varphi : \Gamma \to \mathbb{R}$ consider the modulus of continuity

$$\omega_{\varphi}(\varepsilon) := \sup_{\tau_1, \tau_2 \in \Gamma, \parallel \tau_1 - \tau_2 \parallel \leq \varepsilon} |\varphi(\tau_1) - \varphi(\tau_2)|,$$

and a singular integral

$$\int_{\Gamma} (\varphi(\tau) - \varphi(\zeta_0)) (\tau - \zeta_0)^{-1} d\tau := \lim_{\varepsilon \to 0} \int_{\Gamma \cap \Gamma_\varepsilon(\zeta_0)} (\varphi(\tau) - \varphi(\zeta_0)) (\tau - \zeta_0)^{-1} d\tau,$$

where $\zeta_0 \in \Gamma$ and $\Gamma_\varepsilon(\zeta_0) := \{\tau \in \Gamma : \parallel \tau - \zeta_0 \parallel \leq \varepsilon\}$.

Below, in Theorem 1 in the case where the curve $\Gamma$ satisfies the condition (2) and the modulus of continuity of the function $\varphi$ satisfies a condition of the type (3), we establish the existence of certain limiting values of the integral (6) in points $\zeta_0 \in \Gamma$ when $\zeta$ tends to $\zeta_0$ from $\Pi^+$ or $\Pi^-$ along
To estimate \( I \)\(^{\text{7}} \) \( \parallel \Gamma \parallel \) curve \( \Sigma \) for a curve that is not tangential to the surface \( \Sigma \), outside of the plane of curve \( \Gamma \). For the Euclidean norm in \( \mathbb{A}_3 \) the following inequalities are fulfilled:

\[
\|ab\| \leq 2\sqrt{14}\|a\|\|b\| \quad \forall a, b \in \mathbb{A}_3,
\]

\[
\left\| \int_{\Gamma_\zeta} \psi(\tau)d\tau \right\| \leq 9M \int_{\Gamma_\zeta} \|\psi(\tau)\|d\tau
\]

with the constant \( M := \max\{1, \|e_2\|, \|e_2e_3\|, \|e_3^2\|\} \) for any measurable set \( \Gamma_\zeta \subset \Gamma \) and all continuous functions \( \psi : \Gamma_\zeta \to \mathbb{A}_3 \).

**Lemma 1.** Let \( \Gamma \) be a closed Jordan rectifiable curve satisfying the condition (2) and the modulus of continuity of a function \( \varphi : \Gamma_\zeta \to \mathbb{R} \) satisfies the condition of the type (3). If a point \( \zeta \) tends to \( \zeta_0 \in \Gamma_\zeta \) along a curve \( \gamma_\zeta \) for which there exists a constant \( m < 1 \) such that the inequality

\[
\lim_{\zeta \to \zeta_0, \zeta \in \gamma_\zeta} \int_{\Gamma_\zeta} \left( \varphi(\tau) - \varphi(\zeta_0) \right)(\tau - \zeta)^{-1}d\tau = \int_{\Gamma_\zeta} \left( \varphi(\tau) - \varphi(\zeta_0) \right)(\tau - \zeta_0)^{-1}d\tau.
\]

**Proof.** Let \( \varepsilon := \|\zeta - \zeta_0\| \). Consider the difference

\[
\int_{\Gamma_\zeta} \left( \varphi(\tau) - \varphi(\zeta_0) \right)(\tau - \zeta)^{-1}d\tau - \int_{\Gamma_\zeta} \left( \varphi(\tau) - \varphi(\zeta_0) \right)(\tau - \zeta_0)^{-1}d\tau
\]

\[
= \int_{\Gamma_\zeta^x(\zeta_0)} \left( \varphi(\tau) - \varphi(\zeta_0) \right)(\tau - \zeta)^{-1}d\tau - \int_{\Gamma_\zeta^x(\zeta_0)} \left( \varphi(\tau) - \varphi(\zeta_0) \right)(\tau - \zeta_0)^{-1}d\tau
\]

\[
+ (\zeta - \zeta_0) \int_{\Gamma_\zeta^x(\zeta_0)} \left( \varphi(\tau) - \varphi(\zeta_0) \right)(\tau - \zeta)^{-1}(\tau - \zeta_0)^{-1}d\tau =: I_1 - I_2 + I_3.
\]

To estimate \( I_1 \) we choose a point \( \zeta_1 = x_1 + y_1e_2 \) on \( \Gamma_\zeta \) such that \( \|\zeta - \zeta_1\| = \min_{\tau \in \Gamma_\zeta^x} \|\tau - \zeta\| \). Using the inequalities (7) and (8), we obtain

\[
\|I_1\| = \left\| \int_{\Gamma_\zeta^x(\zeta_0)} \left( \varphi(\tau) - \varphi(\zeta_1) \right)(\tau - \zeta)^{-1}d\tau + \left( \varphi(\zeta_1) - \varphi(\zeta_0) \right)(\tau - \zeta)^{-1}d\tau \right\|
\]

\[
\leq 18\sqrt{14}M \int_{\Gamma_\zeta^x(\zeta_0)} \|\varphi(\tau) - \varphi(\zeta_1)\|d\tau \|\tau - \zeta\|^{-1}d\tau
\]

\[
\leq 18\sqrt{14}M \int_{\Gamma_\zeta^x(\zeta_0)} \|\varphi(\tau) - \varphi(\zeta_1)\|d\tau \|\tau - \zeta\|^{-1}d\tau
\]
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\[ + |\varphi(\zeta_1) - \varphi(\zeta_0)| \left\| \int_{\Gamma^2(\zeta_0)} (\tau - \zeta)^{-1} d\tau \right\| =: I'_1 + I''_1. \]

It follows from Lemma 1.1 [9] that

\[ (\tau - \zeta)^{-1} = \frac{1}{t - \xi} - \frac{z}{(t - \xi)^2} \rho_1 + \left( \frac{i}{2} \frac{y - t_2 - \sqrt{3}z}{(t - \xi)^2} + \frac{z^2}{(t - \xi)^4} \right) \rho_2 \]

for all \( \zeta = x + ye^2 + ze^3 \in \Pi^2_\zeta \) and \( \tau = t_1 + t_2e^2 \in \Gamma_\zeta \), where \( \xi := x + iy \) and \( t := t_1 + it_2 \). The following inequality follows from the relations (9) and (10):

\[ \| (\tau - \zeta)^{-1} \| \leq c(m) \frac{1}{|t - \xi|}, \]

where the constant \( c(m) \) depends only on \( m \).

Using the inequality \( |t - \xi| \geq |t - \xi_1|/2 \) with \( \xi_1 := x_1 + iy_1 \) and the inequality (11), we obtain:

\[ \| I'_1 \| \leq 18\sqrt{14}Mc(m) \int_{\Gamma^2(\zeta_0)} \frac{|\varphi(\tau) - \varphi(\zeta_1)|}{|t - \xi|} d\tau \]

\[ \leq 36\sqrt{14}Mc(m) \int_{\Gamma^2(\zeta_0)} \frac{|\varphi(\tau) - \varphi(\zeta_1)|}{|t - \xi_1|} d\tau \]

\[ \leq 36\sqrt{14}Mc(m) \int_{[0,4\varepsilon]} \frac{\varphi(\eta)}{\eta} d\theta_{\xi_1}(\eta), \]

where the last integral is understood as a Lebesgue–Stieltjes integral.

To estimate the last integral we use Proposition 1 [10] (see also the proof of Theorem 1 [4]) and the condition (2). So, we have

\[ \int_{[0,4\varepsilon]} \frac{\varphi(\eta)}{\eta} d\theta_{\xi_1}(\eta) \leq \int_{0}^{8\varepsilon} \frac{\theta_{\xi_1}(\eta)\varphi(\eta)}{\eta^2} d\eta \leq c \int_{0}^{8\varepsilon} \frac{\varphi(\eta)}{\eta} d\eta \rightarrow 0, \quad \varepsilon \rightarrow 0, \]

where the constant \( c \) does not depend on \( \varepsilon \).

To estimate \( I''_1 \) we introduce the domain \( D^2(\zeta_0) := \{ \tau = t_1 + t_2e^2 : t_1 + it_2 \in D^+, \| \tau - \zeta_0 \| \leq 2\varepsilon \} \) and its boundary \( \partial D^2(\zeta_0) \). Using the inequalities (8) and (11), we obtain:
\[ ||I'_1|| \leq \omega_\varphi(||\zeta_1 - \zeta_0||) \left\| \int_{\Gamma^{2\epsilon}_\zeta(\zeta_0)} (\tau - \zeta)^{-1}d\tau \right\| \]

\[ = \omega_\varphi(||\zeta_1 - \zeta_0||) \left\| \int_{\partial D^{2\epsilon}_\zeta(\zeta_0)} (\tau - \zeta)^{-1}d\tau - \int_{\partial D^{2\epsilon}_\zeta(\zeta_0) \setminus \Gamma^{2\epsilon}_\zeta(\zeta_0)} (\tau - \zeta)^{-1}d\tau \right\| \]

\[ \leq \omega_\varphi(||\zeta_1 - \zeta_0||) \left( 2\pi + 9Mc(m) \int_{\partial D^{2\epsilon}_\zeta(\zeta_0) \setminus \Gamma^{2\epsilon}_\zeta(\zeta_0)} ||d\tau||_{|t - \zeta|} \right) \]

\[ \leq \omega_\varphi(2\varepsilon) \left( 2\pi + 9Mc(m) \frac{1}{\varepsilon} \right) \rightarrow 0, \quad \varepsilon \rightarrow 0. \]

Estimating \( I_2 \), by analogy with the estimation of \( I'_1 \), we obtain:

\[ ||I_2|| \leq c \int_0^{4\varepsilon} \frac{\omega_\varphi(\eta)}{\eta} d\eta \rightarrow 0, \quad \varepsilon \rightarrow 0, \]

where the constant \( c \) does not depend on \( \varepsilon \).

Using the inequality \(|t - \xi| \geq |t - \zeta_0|/2\), where the point \( \xi_0 := x_0 + iy_0 \) corresponds to the point \( \zeta_0 = x_0 + y_0 e_2 \), and using the relations (7), (8), (11) and (2), by analogy with the estimation of \( I'_1 \), we obtain:

\[ ||I_3|| \leq 9M(2\sqrt{4})^2 \varepsilon \int_{\Gamma^{\epsilon}_\zeta \setminus \Gamma^{2\epsilon}_\zeta(\zeta_0)} ||\varphi(\tau) - \varphi(\zeta_0)|| ||(\tau - \zeta)^{-1}|| ||(\tau - \zeta_0)^{-1}|| ||d\tau|| \]

\[ \leq c \varepsilon \int_{\Gamma^{\epsilon}_\zeta \setminus \Gamma^{2\epsilon}_\zeta(\zeta_0)} \frac{|\varphi(\tau) - \varphi(\zeta_0)|}{|t - \xi||t - \zeta_0|} ||d\tau|| \leq c \varepsilon \int_{\Gamma^{\epsilon}_\zeta \setminus \Gamma^{2\epsilon}_\zeta(\zeta_0)} \frac{|\varphi(\tau) - \varphi(\zeta_0)|}{|t - \zeta_0|^2} ||d\tau|| \]

\[ \leq c \varepsilon \int_{[2\varepsilon, \delta]} \frac{\omega_\varphi(\eta)}{\eta} d\theta_{\xi_0}(\eta) \leq c \varepsilon \int_{[2\varepsilon, \delta]} \frac{\theta_{\xi_0}(\eta) \omega_\varphi(\eta)}{\eta^2} d\eta \]

\[ \leq c \varepsilon \int_{2\varepsilon}^{2d} \frac{\omega_\varphi(\eta)}{\eta^2} d\eta \rightarrow 0, \quad \varepsilon \rightarrow 0, \]

where \( d := \max_{\zeta_1, \zeta_2 \in \Gamma} |\zeta_1 - \zeta_2| \) is the diameter of \( \Gamma \) and \( c \) denotes different constants which do not depend on \( \varepsilon \). The lemma is proved. \( \square \)

Let \( \tilde{\Phi}^{\pm}(\zeta_0) \) be the boundary value of function (6) when \( \zeta \) tends to \( \zeta_0 \in \Gamma_\zeta \) along a curve \( \gamma_\zeta \) for which there exists a constant \( m < 1 \) such that the inequality (9) is fulfilled for all \( \zeta = x + ye_2 + ze_3 \in \gamma_\zeta \).
Theorem 1. Let $\Gamma$ be a closed Jordan rectifiable curve satisfying the condition (2) and the modulus of continuity of a function $\varphi : \Gamma \to \mathbb{R}$ satisfies the condition of the type (3). Then the integral (6) has boundary values $\hat{\Phi}^{\pm}(\zeta_0)$ for all $\zeta_0 \in \Gamma$ that are expressed by the formulas:

$$\hat{\Phi}^{+}(\zeta_0) = \frac{1}{2\pi i} \int_{\Gamma} \frac{(\varphi(\tau) - \varphi(\zeta_0))(\tau - \zeta_0)^{-1}d\tau + \varphi(\zeta_0)}{\Gamma}$$

$$\hat{\Phi}^{-}(\zeta_0) = \frac{1}{2\pi i} \int_{\Gamma} \frac{(\varphi(\tau) - \varphi(\zeta_0))(\tau - \zeta_0)^{-1}d\tau}{\Gamma}.$$ 

The theorem follows from the Lemma 1 and the equalities

$$\frac{1}{2\pi i} \int_{\Gamma} \varphi(\tau)(\tau - \zeta)^{-1}d\tau = \frac{1}{2\pi i} \int_{\Gamma} \frac{(\varphi(\tau)-\varphi(\zeta_0))(\tau - \zeta)^{-1}d\tau + \varphi(\zeta_0)}{\Gamma} \quad \forall \zeta \in \Pi^{+}_\zeta,$$

$$\frac{1}{2\pi i} \int_{\Gamma} \varphi(\tau)(\tau - \zeta)^{-1}d\tau = \frac{1}{2\pi i} \int_{\Gamma} \frac{(\varphi(\tau) - \varphi(\zeta_0))(\tau - \zeta)^{-1}d\tau}{\Gamma} \quad \forall \zeta \in \Pi^{-}_\zeta.$$ 

In comparison with Theorem 1, note that additional assumptions about the function $\varphi$ are required for an existence of limiting values of the function (6) from $\Pi^{+}_\zeta$ or $\Pi^{-}_\zeta$ on the boundary $\Sigma_\zeta$. We are going to state these results in next papers.

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Received September 30, 2011