SOMJATE CHAIYA and AIMO HINKKANEN

Location of the critical points of certain polynomials

Abstract. Let $\mathbb{D}$ denote the unit disk $\{z : |z| < 1\}$ in the complex plane $\mathbb{C}$. In this paper, we study a family of polynomials $P$ with only one zero lying outside $\mathbb{D}$. We establish criteria for $P$ to satisfy implying that each of $P$ and $P'$ has exactly one critical point outside $\mathbb{D}$.

1. Introduction. Let $P$ be a polynomial in the complex plane $\mathbb{C}$. We denote the degree of $P$ by $\deg P$. We say that $\alpha$ is a critical point of $P$ if $P'(\alpha) = 0$. Throughout this paper, if not otherwise stated, when we talk about the number of zeros of a polynomial in a domain, we mean the number of zeros counting multiplicities. As the critical points of $P$ are the zeros of $P'$, this applies also to the number of critical points. There are several known results involving the critical points of polynomials. The most classical one is the Gauss–Lucas Theorem, [8, p. 25].

Gauss–Lucas Theorem. Let $P$ be a polynomial of degree $n$ with zeros $z_1, z_2, \ldots, z_n$, not necessarily distinct. The zeros of the derivative $P'$ lie in the convex hull of the set $\{z_1, z_2, \ldots, z_n\}$.

Another classical theorem concerning the location of the critical points is the Walsh’s Two-Circle Theorem, [9].
Walsh’s Two-Circle Theorem. Let $P$ be a polynomial of degree $n \geq 2$. Let $n_1$ and $n_2$ be positive integers with $n_1 + n_2 = n$, let $\alpha_1$ and $\alpha_2$ be two distinct complex numbers, and let $r_1, r_2$ be positive real numbers. Let $C_1 = \{ z : |z - \alpha_1| \leq r_1 \}$, $C_2 = \{ z : |z - \alpha_2| \leq r_2 \}$, and let $C = \{ z : |z - \alpha_0| \leq r \}$, where $\alpha_0 = \frac{\alpha_2 n_1 + \alpha_1 n_2}{n}$ and $r = \frac{n_1 r_2 + n_2 r_1}{n}$.

Assume that $P$ has $n_1$ and $n_2$ zeros in $C_1$ and $C_2$ respectively. Then all critical points of $P$ lie in $C_1 \cup C_2 \cup C$.

In this paper we are interested in the location of the critical points of a certain type of polynomials. If $P$ has a zero lying outside the closed unit disk $D = \{ z \in \mathbb{C} : |z| \leq 1 \}$, by the Gauss–Lucas Theorem, it follows that the zeros of its derivative are in the convex hull of the zeros of $P$, which includes a region outside $D$. But we do not know how many zeros of $P'$ are outside $D$. We may ask the question of under what conditions does $P$ have only one critical point outside the closed unit disk? A consequence of Walsh’s theorem gives a partial answer to the question. That is,

Theorem ([5, see (4.1.1) on p. 117]). If $S \in \{ C_1, C_2, C \}$ is a disjoint component of $C_1 \cup C_2 \cup C$, then $S$ contains exactly

$$ n(S) = \begin{cases} n_j - 1 & \text{if } S = C_j \\ 1 & \text{if } S = C \end{cases} $$

critical points of $P$.

Let $P$ be a polynomial of degree $n \geq 2$ that has only one zero, say $\alpha_n$, that lies outside the closed unit disk $\overline{D} = \{ z \in \mathbb{C} : |z| \leq 1 \}$. By taking $r_2 \to 0^+$ we see by the above theorem that if $|\alpha_n| > \frac{n + 1}{n+1}$, then $P$ has exactly one critical point $\alpha$ in $C = \{ z : \left| z - \left( \frac{n+1}{n} \right) \alpha_n \right| \leq \frac{1}{n} \}$ while $C$ does not intersect $\overline{D}$. Hence $P$ has exactly one critical point outside $\overline{D}$ whenever $|\alpha_n| > \frac{n+1}{n+1}$.

Here we give a general criterion for determining the number of critical points outside $\overline{D}$.

**Theorem 1.1.** Let $Q(z) = c \prod_{k=1}^{n} (z - \alpha_k)$ be a polynomial of degree $n \geq 2$, where $c \neq 0$. Suppose that $\alpha_k \notin \overline{D}$ for $1 \leq k \leq m$, and the remaining points $\alpha_k$ are in $\overline{D}$. If we have

$$ \sum_{k=m+1}^{n} \frac{1}{1 + |\alpha_k|} > \sum_{k=1}^{m} \frac{1}{|\alpha_k| - 1}, $$

then $Q$ has exactly $m$ critical points outside $\overline{D}$, counting multiplicities. If, in addition, all the points $\alpha_k$ lying on the unit circle are simple zeros of $Q$, then $Q'$ has no zeros on the unit circle.
Note that if $Q$ has only one zero $\alpha_n$ lying outside $\mathbb{D}$ with $|\alpha_n| > \frac{n+1}{n-1}$, which is the same condition as discussed previously, then by Theorem 1.1, $Q$ has exactly one critical point outside $\mathbb{D}$. From Theorem 1.1, we can deduce that the result still holds even though $|\alpha_n| \leq \frac{n+1}{n-1}$ if $Q$ satisfies an additional condition.

**Corollary 1.2.** Let $Q(z) = c \prod_{k=1}^{n}(z - \alpha_k)$ be a polynomial of degree $n \geq 2$, where $c \neq 0$. Suppose that $\alpha_1 = \alpha$, $\alpha_2 = \alpha^{-1}$, where $\alpha$ is real and $|\alpha| > 1$, and all the remaining points $\alpha_k$, if any, are in $\mathbb{D}$. Then $Q$ has exactly one critical point outside $\mathbb{D}$, counting multiplicities. If, in addition, all the points $\alpha_k$ that are on the unit circle are simple zeros of $Q$, then $Q$ has exactly $n - 2$ critical points in $\mathbb{D}$, counting multiplicities.

A polynomial $P$ is said to be anti-reciprocal if $P(z) = -z^{\deg P} P(z^{-1})$. If $P$ is anti-reciprocal, then so is $cP$ for any non-zero complex number $c$. Note that if $P$ is anti-reciprocal, then 1 is a zero of $P$, we have $P(0) \neq 0$, and for $\alpha \neq 0$, we have $P(\alpha) = 0$ if, and only if, $P(\alpha^{-1}) = 0$. Furthermore, $\alpha$ and $\alpha^{-1}$ have the same multiplicity as zeros of $P$, as we see (for $\alpha \neq \pm 1$) by writing $P(z) = (z - \alpha)^m(z - 1/\alpha)^n g(z)$, where $g(\alpha)g(1/\alpha) \neq 0$ and using $P(z) = -z^{\deg P} P(z^{-1})$. Therefore, if the leading coefficient of $P$ is real and each zero of $P$ is real or has modulus 1, then the coefficients of $P$ are real. If $P$ is an anti-reciprocal polynomial with exactly one zero, counting multiplicities, lying outside $\mathbb{D}$, and which furthermore is real, then $P$ satisfies the assumptions of Corollary 1.2, and so $P$ has only one critical point outside $\mathbb{D}$. Indeed, if $P$ is anti-reciprocal with exactly one zero, say $\alpha$, which is furthermore simple, outside $\mathbb{D}$, then $P$ has exactly one zero (namely, $1/\alpha$) in $\mathbb{D}$, and all the other zeros of $P$ must lie on $\partial \mathbb{D}$. In Theorem 1.3, we prove that if $P$ satisfies certain additional conditions, then not only does $P$ have only one zero outside $\mathbb{D}$ but the same is also true for $P''$.

**Theorem 1.3.** Let $Q$ be an anti-reciprocal polynomial with real coefficients of degree $n \geq 3$. Suppose that the zeros of $Q$ are simple and that $\alpha > 1$ is the only zero of $Q$ lying outside $\mathbb{D}$. Then each of the polynomials $Q'$ and $Q''$ has exactly one zero outside $\mathbb{D}$, counting multiplicities.

We can construct a family of anti-reciprocal polynomials satisfying Theorem 1.3. Let $P$ be a polynomial with real coefficients, and set $P^*(z) := z^{\deg P} P(z^{-1})$. Suppose that $P$ has a real zero greater than 1, that the remaining zeros of $P$ are in $\mathbb{D}$ (so $P(1) \neq 0$), and that $P^* \neq P$. Boyd [1, p. 320] showed that the polynomial

$$Q(z) = z^n P(z) - P^*(z)$$

satisfies the assumptions of Theorem 1.3 provided that $n > \deg P - 2P'(1)/P(1)$ and that all zeros of $P$ are simple. The polynomial in (1) was originally introduced by R. Salem [6, Theorem IV, p. 166], [7, p. 30]. Therefore, this gives the following corollary.

Corollary 1.4. Let $P$ be a polynomial with real coefficients such that $P^* \neq P$. For $n > \deg P - 2P'(1)/P(1)$, let $Q$ be defined as in (1). Suppose that $P$ has a real zero greater than 1, that the remaining zeros of $P$ are in $\mathbb{D}$, and that all zeros of $P$ are simple. Then each of $Q$, $Q'$, and $Q''$ has exactly one zero outside $\mathbb{D}$, counting multiplicities.

2. Proof of Theorem 1.1.

Lemma 2.1. Let $Q(z) = c \prod_{k=1}^{n} (z - \alpha_k)$ be a polynomial of degree $n \geq 2$, where $c \neq 0$. Suppose that $\alpha_k \notin \mathbb{D}$ for $1 \leq k \leq m$, and that the remaining points $\alpha_k$ are in $\mathbb{D}$. If we have

$$\sum_{k=1}^{m} \frac{1}{1 - |\alpha_k|} + \sum_{k=m+1}^{n} \frac{1}{1 + |\alpha_k|} > 0,$$

then there is a positive $\delta$ such that for any $r \in (1, 1 + \delta)$, we have

$$\Re \left\{ \frac{zQ'(z)}{Q(z)} \right\} > 0 \text{ on } |z| = r.$$

Furthermore, we have $\Re \left\{ \frac{zQ'(z)}{Q(z)} \right\} > 0$ whenever $|z| = 1$ and $Q(z) \neq 0$.

Proof. By an elementary calculation, we can show that if $|z| > 1$ and $\alpha_k \neq 0$, then $\Re \left\{ \frac{z}{z - \alpha_k} \right\} > \frac{1}{1 + |\alpha_k|}$ for $m + 1 \leq k \leq n$, the two sides being equal if $\alpha_k = 0$. Also, if $|z| = 1$ then $\Re \left\{ \frac{z}{z - \alpha_k} \right\} \geq 1 - |\alpha_k|$ for $1 \leq k \leq m$.

Let

$$\varepsilon = \sum_{k=1}^{m} \frac{1}{1 - |\alpha_k|} + \sum_{k=m+1}^{n} \frac{1}{1 + |\alpha_k|} > 0.$$

Since $\Re \left\{ \frac{z}{z - \alpha_k} \right\}$ is a continuous function except at $z = \alpha_k$ and since $|\alpha_k| > 1$ for $1 \leq k \leq m$, there exists a positive constant $\delta$ with $1 + \delta < \min\{|\alpha_k| : 1 \leq k \leq m\}$ such that

$$\sum_{k=1}^{m} \Re \left\{ \frac{z}{z - \alpha_k} \right\} > \sum_{k=1}^{m} \frac{1}{1 - |\alpha_k|} - \frac{\varepsilon}{2}$$

on $|z| = r$, for all $r \in (1, 1 + \delta)$. Therefore, if $r \in (1, 1 + \delta)$ and $|z| = r$, we have

$$\Re \left\{ \frac{zQ'(z)}{Q(z)} \right\} = \sum_{k=1}^{n} \Re \left\{ \frac{z}{z - \alpha_k} \right\} > \sum_{k=1}^{m} \frac{1}{1 - |\alpha_k|} - \frac{\varepsilon}{2} + \sum_{k=m+1}^{n} \frac{1}{1 + |\alpha_k|} = \frac{\varepsilon}{2}.$$

This proves Lemma 2.1. \qed

Now we are ready to present a proof of Theorem 1.1.
Proof of Theorem 1.1. We are to show that $zQ'(z)$ and $Q(z)$ have the same number of zeros lying in $\mathbb{D}$. By Lemma 2.1, there is $\delta > 0$ such that, for all $r \in (1, 1 + \delta)$, we have $\text{Re} \left\{ \frac{zQ'(z)}{Q(z)} \right\} > 0$ on $|z| = r$. So, for each fixed $r \in (1, 1 + \delta)$, we have

$$\left| 1 - \frac{zQ'(z)}{Q(z)} \right| < 1 + \left| \frac{zQ'(z)}{Q(z)} \right|,$$

hence $|zQ'(z) - Q(z)| < |Q(z)| + |zQ'(z)|$, on $|z| = r$. Then, by Rouche’s theorem [4, Theorem 3.6, p. 341], $zQ'(z)$ and $Q(z)$ must have the same number of zeros lying in \{ $z : |z| \leq r$ \} for all $r \in (1, 1 + \delta)$. This proves the first part of the theorem.

Next suppose that all the zeros $\alpha_k$ that are on the unit circle, if any, are simple. If $Q'$ has a zero $\gamma$ on the unit circle, then $\text{Re} \left\{ \frac{zQ'(\gamma)}{Q(\gamma)} \right\} = 0$, which contradicts the fact that $\text{Re} \left\{ \frac{zQ'(z)}{Q(z)} \right\} > 0$ on $|z| = 1$ outside the zeros of $Q$. Hence $Q'$ has no zeros on $\partial \mathbb{D}$. The proof of Theorem 1.1 is now complete. \qed

For a proof of Corollary 1.2, we note that it follows from the fact that $\text{Re} \left\{ \frac{z}{z-\alpha} + \frac{z}{z-\overline{\alpha}} \right\} = 1$ for all $z$ with $|z| = 1$ and the argument in the proof of Lemma 2.1.

3. Preliminaries for Theorem 1.3. To prove Theorem 1.3, we need the following lemmas.

Lemma 3.1. If $x > 1$ and $y \in [-1, 1)$, then

$$1 + x^4 - 2x(1 + x^2)y + 2x^2(2y^2 - 1) \leq \frac{y}{2(1 - y)} < 2.$$

Proof. This can be proved by using only elementary calculus (see [3, Lemma 5.10, p. 54]). \qed

Lemma 3.2. If $Q$ is an anti-reciprocal polynomial of degree $n \geq 2$ with real coefficients, then

$$\begin{align*}
\left(2\right) & \quad \text{Re} \left\{ \frac{zQ'(z)}{nQ(z)} \right\} = \frac{1}{2} \quad \text{and} \quad \text{Im} \left\{ \frac{z^2Q''(z)}{(n-1)Q(z)} \right\} = \text{Im} \left\{ \frac{zQ'(z)}{Q(z)} \right\} \\
\text{whenever} & \quad |z| = 1 \quad \text{and} \quad Q(z) \neq 0.
\end{align*}$$

Proof. We give a proof that yields the entire statement of this lemma, but we note that the first equality in (2) has been proved in [8, (7.5), p. 229] for reciprocal polynomials $Q$. 

Now, since $Q$ is anti-reciprocal, we have $Q(z) = -z^n Q \left( \frac{1}{z} \right)$. Taking the derivative and multiplying both sides by $z$, we get

$$zQ'(z) = -nz^n Q \left( \frac{1}{z} \right) + z^{n-1} Q' \left( \frac{1}{z} \right) = nQ(z) + z^{n-1} Q' \left( \frac{1}{z} \right).$$

So, we have

$$z^{n-1} Q' \left( \frac{1}{z} \right) = zQ'(z) - nQ(z). \quad (3)$$

After taking the derivative of both sides of this equation, and then multiplying both sides by $z$ and applying the identity (3), we obtain

$$-z^{n-2} Q'' \left( \frac{1}{z} \right) = z^2 Q''(z) + 2(1-n)zQ'(z) + n(n-1)Q(z). \quad (4)$$

Let $z \in \partial \mathbb{D}$ with $Q(z) \neq 0$. Next dividing both sides of (4) by $n(n-1)Q(z)$, we get

$$\frac{-z^{n-2} Q'' \left( \frac{1}{z} \right)}{n(n-1)Q(z)} = \frac{z^2 Q''(z)}{n(n-1)Q(z)} - \frac{2zQ'(z)}{nQ(z)} + 1. \quad (5)$$

By replacing $Q(z)$ on the left side of (5) by $-z^n Q \left( \frac{1}{z} \right)$, the left-hand side becomes

$$\frac{z^{-2} Q''(z)}{n(n-1)z^n Q \left( \frac{1}{z} \right)} = \frac{z^{-2} Q'' \left( \frac{1}{z} \right)}{n(n-1) Q \left( \frac{1}{z} \right)} = \left( \frac{z^2 Q''(z)}{n(n-1)Q(z)} \right).$$

Here we have used the fact that since $|z| = 1$ and $Q$ has real coefficients, we have $Q(1/z) = Q(z) = Q(\bar{z})$, and similarly for $Q''$ instead of $Q$. Then from (5) we derive

$$\left( \frac{z^2 Q''(z)}{n(n-1)Q(z)} \right) - \frac{z^2 Q''(z)}{n(n-1)Q(z)} = 1 - \frac{2zQ'(z)}{nQ(z)}.$$

which gives $2i \text{Im} \left\{ \frac{z^2 Q''(z)}{n(n-1)Q(z)} \right\} = 2zQ'(z) - 1$. This implies that $\text{Re} \left( \frac{zQ'(z)}{nQ(z)} \right) = \frac{1}{2}$ and $\text{Im} \left( \frac{z^2 Q''(z)}{n(n-1)Q(z)} \right) = \text{Im} \left( \frac{zQ'(z)}{nQ(z)} \right)$, as desired. \Box

**Lemma 3.3.** Let $Q(z) = \prod_{k=1}^{n} (z - \alpha_k)$ be an anti-reciprocal polynomial of degree $n \geq 3$. Suppose that $\alpha_1 = \tau > 1$, $\alpha_2 = \tau^{-1}$, $\alpha_3 = 1$, and $|\alpha_k| = 1$ for $k > 3$. For $|z| = 1$ with $Q(z) \neq 0$, if $\frac{z^2 Q''(z)}{Q(z)}$ is a real number, then it is positive. In particular, then $Q''(z) \neq 0$.

**Proof.** Since $Q$ is monic and each zero of $Q$ is real or has modulus 1, $Q$ has real coefficients. Let $z$ be a point on the unit circle with $Q(z) \neq 0$. We
have
\[ \frac{z^2 Q''(z)}{Q(z)} = z^2 \left( \left( \frac{Q'}{Q} \right)'(z) + \left( \left( \frac{Q'}{Q} \right)(z) \right)^2 \right) = \left( \frac{zQ'(z)}{Q(z)} \right)^2 - \sum_{k=1}^{n} \frac{z^2}{(z - \alpha_k)^2}. \]

Suppose that \( \frac{z^2 Q''(z)}{Q(z)} \) is a real number. Thus, by Lemma 3.2, \( \frac{zQ'(z)}{nQ(z)} \) is real as well, and so is also \( \sum_{k=1}^{n} \frac{z^2}{(z - \alpha_k)^2} \). Since \( \text{Re} \left\{ \frac{zQ'(z)}{nQ(z)} \right\} = \frac{1}{2} \) on \( |z| = 1 \) when \( Q(z) \neq 0 \), we have

\[ (6) \quad \frac{z^2 Q''(z)}{Q(z)} = \frac{n^2}{4} - \sum_{k=1}^{n} \frac{z^2}{(z - \alpha_k)^2}. \]

Next we want to find an upper bound for the real part of \( \sum_{k=1}^{n} \frac{z^2}{(z - \alpha_k)^2} \) on the unit circle. Let \( z = e^{i\theta} \), where \( \theta \in (0, 2\pi) \) (note that \( z \neq 1 \) since \( Q(1) = 0 \)). If \( \alpha \) is real, we have
\[ \text{Re} \left\{ \frac{z^2}{(z - \alpha)^2} \right\} = 1 - \frac{2\alpha \cos \theta + \alpha^2(2 \cos^2 \theta - 1)}{(1 + \alpha^2 - 2\alpha \cos \theta)^2}. \]

For \( k \geq 3 \), by letting \( \alpha_k = e^{i\theta_k}, \theta_k \in [0, 2\pi) \), we have \( \text{Re} \left\{ \frac{z^2}{(z - \alpha_k)^2} \right\} = \frac{-\cos \theta_k}{2 - 2\cos \beta_k} \), where \( \beta_k = \theta - \theta_k \). Therefore,
\[ \text{Re} \left\{ \sum_{k=1}^{n} \frac{z^2}{(z - \alpha_k)^2} \right\} = \frac{1 + \tau^4 - 2\tau(1 + \tau^2) \cos \theta + 2\tau^2(2 \cos^2 \theta - 1)}{(1 + \tau^2 - 2\tau \cos \theta)^2} - \sum_{k=3}^{n} \cos \beta_k. \]

Taking \( x = \tau \) and \( y = \cos \theta \) in Lemma 3.1, we see that
\[ \frac{1 + \tau^4 - 2\tau(1 + \tau^2) \cos \theta + 2\tau^2(2 \cos^2 \theta - 1)}{(1 + \tau^2 - 2\tau \cos \theta)^2} = \frac{\cos \theta}{2 - 2\cos \theta} < 2. \]

It is easy to see that \( \frac{-\cos \omega}{2 \cos \omega} \leq \frac{1}{4} \) for all \( \omega \in (0, 2\pi) \). So, we obtain
\[ \text{Re} \left\{ \sum_{k=1}^{n} \frac{z^2}{(z - \alpha_k)^2} \right\} < 2 + \frac{1}{4}(n - 3) = \frac{n + 5}{4}. \]

Hence, from (6), we derive
\[ \frac{z^2 Q''(z)}{Q(z)} = \frac{n^2}{4} - \sum_{k=1}^{n} \frac{z^2}{(z - \alpha_k)^2} > \frac{n^2}{4} - \frac{n + 5}{4} > 0 \]
if \( n \geq 3 \), as desired. This proves Lemma 3.3. □
4. Proof of Theorem 1.3. Let the assumptions of Theorem 1.3 be satisfied. By Corollary 1.2 we know that $Q'$ has only one zero outside $\overline{D}$ and has no zeros on $\partial D$. Let $G(z) = -z^{n-2}Q''(\frac{1}{z})$ and $T(z) = z^{n-1}Q'(\frac{1}{z})$. In order to prove that $Q''$ has exactly one zero outside $\overline{D}$, it is equivalent to show that $G$ has only one zero in $D$. Since $Q'$ has only one zero outside $\overline{D}$ and has no zeros on $\partial D$, $T$ has exactly one zero in $\overline{D}$ and has no zeros on $\partial D$. If we have

\[ |G(z) + 2(n-1)T(z)| < |G(z)| + 2(n-1)|T(z)| \tag{7} \]

on $\partial D$, then, by a form of Rouché’s Theorem [4, Theorem 3.6, p. 341], both $G$ and $T$ have the same number of zeros inside $D$. This will prove the theorem. From (3) and (4), we have

\[ G(z) + 2(n-1)T(z) = z^2Q''(z) - n(n-1)Q(z). \]

Let $z \in \partial D$. It is easy to see that if $Q(z) = 0$, then (7) holds. Now, for $Q(z) \neq 0$, write $\frac{z^2Q''(z)}{(n-1)Q(z)} = a + ib$, where $a, b \in \mathbb{R}$. So $G(z) + 2(n-1)T(z) = (a + ib)(n-1)Q(z)$. Since, by Lemma 3.2, $\text{Im}\left\{ \frac{z^2Q''(z)}{(n-1)Q(z)} \right\} = \text{Im}\left\{ \frac{zQ'(z)}{Q(z)} \right\}$ and $\text{Re}\left\{ \frac{zQ'(z)}{nQ(z)} \right\} = \frac{1}{2}$, we have $zQ'(z) = (\frac{a}{2} + ib)Q(z)$. We also have $|G(z)| = |z^2Q''(z)| = (n-1)|a + ib||Q(z)|$ and, by (3),

\[ 2|T(z)| = 2|zQ'(z) - nQ(z)| = |a - n + 2ib||Q(z)|. \]

Thus, the inequality (7) is equivalent to

\[ |a - n + 2ib| < |a + ib| + |a - n + 2ib| \]

which is clearly true if $b \neq 0$. If $b = 0$, then by Lemma 3.3, we have $a > 0$ and so the inequality above is true. Therefore, the inequality (7) holds on $\partial D$, as desired. The proof of Theorem 1.3 is now complete.

References

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Somjate Chaiya
Department of Mathematics
Faculty of Science
Silpakorn University
Nakorn Pathom 73000
Thailand

Aimo Hinkkanen
Department of Mathematics
University of Illinois at Urbana-Champaign
1409 W. Green St.
Urbana, IL 61801
U.S.A.
e-mail: aimo@math.uiuc.edu

Centre of Excellence in Mathematics
CHE
Si Ayutthaya Rd.
Bangkok 10400
Thailand
e-mail: somjate.c@su.ac.th

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