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Properties of functions concerned with
Carathéodory functions

ABSTRACT. Let $\mathcal{P}_n$ denote the class of analytic functions $p(z)$ of the form
$p(z) = 1 + c_n z^n + c_{n+1} z^{n+1} + \ldots$ in the open unit disc $U$. Applying the result by S. S. Miller and P. T. Mocanu (J. Math. Anal. Appl. 65 (1978),
289–305), some interesting properties for $p(z)$ concerned with Carathéodory
functions are discussed. Further, some corollaries of the results concerned
with the result due to M. Obradović and S. Owa (Math. Nachr. 140 (1989),
97–102) are shown.

1. Introduction. Let $\mathcal{A}_n$ denote the class of functions $f(z)$ of the form

\begin{equation}
    f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k \quad (n = 1, 2, 3, \ldots)
\end{equation}

which are analytic in the open unit disc $U = \{ z \in \mathbb{C} \mid |z| < 1 \}$. If a function
$f(z) \in \mathcal{A}_n$ satisfies

\begin{equation}
    \text{Re} \left( \frac{zf'(z)}{f(z)} \right) > 0 \quad (z \in U),
\end{equation}

then $f(z)$ is said to be starlike with respect to the origin in $U$. We denote
by $\mathcal{S}_n^*$ the subclass of $\mathcal{A}_n$ consisting of functions $f(z)$ which are starlike with
respect to the origin in $U$. From the definition of the class $\mathcal{S}_n^*$, we see that

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if \( f(z) \in A_n \) satisfies
\[
\left| \frac{zf'(z)}{f(z)} - 1 \right| < 1 \quad (z \in \mathbb{U}),
\]
then \( f(z) \in S_n^* \). We denote by \( T_n^* \) the subclass of \( S_n^* \) consisting of \( f(z) \) satisfying (1.3).

Obradović and Owa [5] have shown the following result:

**Theorem A.** If \( f(z) \in A_1 \) satisfies \( f(z)f'(z) \neq 0 \) for \( 0 < |z| < 1 \) and
\[
\left| 1 + \frac{zf''(z)}{f'(z)} \right| < \frac{5}{4} \left| \frac{zf'(z)}{f(z)} \right| \quad (z \in \mathbb{U}),
\]
then \( f(z) \in T_1^* \).

In order to discuss our results, we have to recall here the following lemma due to Miller and Mocanu [3] (also due to Jack [2]):

**Lemma 1.1.** Let
\[
w(z) = a_n z^n + a_{n+1} z^{n+1} + \ldots \quad (a_n \neq 0)
\]
be analytic in \( \mathbb{U} \). If there exists a point \( z_0 \in \mathbb{U} \) on the circle \( |z| = r < 1 \) such that
\[
\max_{|z| \leq |z_0|} |w(z)| = |w(z_0)|,
\]
then we can write
\[
z_0 w'(z_0) = mw(z_0),
\]
where \( m \) is real and \( m \geq n \).

**Example 1.1.** We consider the function \( w(z) \) given by
\[
w(z) = z^n + \frac{e^{i\theta}}{n+1} z^{n+1} \quad (n = 1, 2, 3, \ldots).
\]
Then, it follows that
\[
\max_{|z| \leq |z_0|} |w(z)| = \max_{|z| \leq |z_0|} |z|^n \left| 1 + \frac{e^{i\theta}}{n+1} \right| \leq r^n \left( 1 + \frac{r}{n+1} \right)
\]
for \( z_0 = re^{-i\theta} \in \mathbb{U} \). This shows that \( |w(z)| \) attains its maximum value at a point \( z_0 \in \mathbb{U} \) on the circle \( |z| = r \). For such a point \( z_0 = re^{-i\theta} \), we have that
\[
\frac{z_0 w'(z_0)}{w(z_0)} = \frac{z_0^n (n + e^{i\theta} z_0)}{z_0^n (1 + \frac{e^{i\theta} z_0}{n+1})} = \frac{(n+1)(n+r)}{n+1+r} = m \geq n.
\]
Let \( P_n \) be the class of functions \( p(z) \) of the form

\[
(1.10) \quad p(z) = 1 + \sum_{k=n}^{\infty} c_k z^k \quad (c_n \neq 0)
\]

which are analytic in \( U \). We also denote by \( Q_n \) the subclass of \( P_n \) consisting of \( f(z) \) which satisfy

\[
(1.11) \quad |p(z) - 1| < 1 \quad (z \in \mathbb{U}).
\]

Since \( p(z) \in Q_n \) shows that \( \text{Re} \ p(z) > 0 \ (z \in \mathbb{U}) \), \( p(z) \in Q_n \) is said to be a Carathéodory function in \( U \) (see Carathéodory [1]).

2. Conditions for the classes \( Q_n \) and \( T_n^* \). Applying Lemma 1.1, we discuss some conditions for \( p(z) \in P_n \) to be in the class \( Q_n \).

**Theorem 2.1.** If \( p(z) \in P_n \) satisfies

\[
(2.1) \quad \text{Re} \left( p(z) + \alpha \frac{zp'(z)}{p(z)} \right) < \sqrt{\alpha n} |p(z)| \quad (z \in \mathbb{U})
\]

for some real \( \alpha > 0 \), then \( p(z) \in Q_n \).

**Proof.** Note that \( p(z) \neq 0 \ (z \in \mathbb{U}) \) with the condition (2.1). Let us define the function \( w(z) \) by

\[
(2.2) \quad p(z) = 1 + w(z) \quad (z \in \mathbb{U})
\]

for \( p(z) \in P_n \). Then \( w(z) \) is analytic in \( U \) and

\[
(2.3) \quad w(z) = c_n z^n + c_{n+1} z^{n+1} + \ldots.
\]

It follows that

\[
(2.4) \quad p(z) + \alpha \frac{zp'(z)}{p(z)} = 1 + w(z) + \frac{\alpha w'(z)}{1 + w(z)}
\]

and that

\[
(2.5) \quad \left| \frac{1}{|p(z)|} \text{Re} \left( p(z) + \alpha \frac{zp'(z)}{p(z)} \right) \right|
= \frac{1}{|1 + w(z)|} \text{Re} \left( 1 + w(z) + \frac{\alpha w'(z)}{1 + w(z)} \right) < \sqrt{\alpha n}
\]

for \( z \in \mathbb{U} \).

We suppose that there exists a point \( z_0 \in \mathbb{U} \) such that

\[
(2.6) \quad \max_{|z| \leq |z_0|} |w(z)| = |w(z_0)| = 1.
\]
Then, Lemma 1.1 gives us that \(w(z_0) = e^{i\theta}\) and \(z_0 w'(z_0) = me^{i\theta}\) \((m \geq n)\).
For such a point \(z_0\), we have that
\[
\frac{1}{|p(z_0)|} \text{Re} \left( p(z_0) + \alpha \frac{zp'(z_0)}{p(z_0)} \right) = \frac{1}{|1 + e^{i\theta}|} \text{Re} \left( 1 + e^{i\theta} + \alpha me^{i\theta} \frac{1}{1 + e^{i\theta}} \right)
\]
\[= \frac{1}{\sqrt{2(1 + \cos \theta)}} \left( 1 + \frac{\alpha m}{2} \right) \]
\[= \frac{1}{\sqrt{2}} \left( \sqrt{1 + \cos \theta} + \frac{\alpha m}{2\sqrt{1 + \cos \theta}} \right) \geq \sqrt{\alpha m} \geq \sqrt{\alpha n}.
\]
This contradicts the condition (2.1). Therefore, there is no such point \(z_0 \in U\). This means that \(p(z) \in Q_n\).

\[\square\]

**Corollary 2.1.** If \(f(z) \in A_n\) satisfies \(f(z)f'(z) \neq 0\) for \(0 < |z| < 1\) and
\[
\text{Re} \left\{ (1 - \alpha) \frac{zf'(z)}{f(z)} + \alpha \left( 1 + \frac{zf''(z)}{f'(z)} \right) \right\} < \sqrt{\alpha n} \left| \frac{zf'(z)}{f(z)} \right| \quad (z \in \mathbb{U})
\]
for some real \(\alpha > 0\), then \(f(z) \in T^*_n\).

**Proof.** Letting \(p(z) = \frac{zf'(z)}{f(z)}\) in Theorem 2.1, we have that
\[
p(z) + \alpha \frac{zp'(z)}{p(z)} = (1 - \alpha) \frac{zf'(z)}{f(z)} + \alpha \left( 1 + \frac{zf''(z)}{f'(z)} \right).
\]
The proof of the corollary follows from the above. \(\square\)

Next we derive

**Theorem 2.2.** If \(p(z) \in P_n\) satisfies \(\text{Re} \, p(z) \neq 0\) \((z \in \mathbb{U})\) and
\[
\text{Re} \left( p(z) + \alpha \frac{zp'(z)}{p(z)} \right) < \left( 1 + \frac{\alpha n}{4} \right) \text{Re} \, p(z) \quad (z \in \mathbb{U})
\]
for some real \(\alpha > 0\), then \(p(z) \in Q_n\).

**Proof.** Define the function \(w(z)\) by (2.2) for \(p(z) \in P_n\). Then, \(w(z)\) is analytic in \(\mathbb{U}\),
\[
w(z) = c_1 z + c_2 z^2 + \ldots,
\]
and
\[
\text{Re} \left( \frac{p(z) + \alpha \frac{zp'(z)}{p(z)}}{\text{Re} (p(z))} \right) = \text{Re} \left( \frac{1 + w(z) + \alpha zw'(z)}{1 + w(z)} \right) < 1 + \frac{\alpha n}{4}
\]
\((z \in \mathbb{U})\). If we suppose that there exists a point \(z_0 \in U\) on the circle \(|z| = r < 1\) such that
\[
\max_{|z| \leq |z_0|} |w(z)| = |w(z_0)| = 1,
\]
we can write that \( w(z_0) = e^{i\theta} \) and \( z_0 w'(z_0) = me^{i\theta} \). This shows that

\[
(2.11) \quad \frac{\operatorname{Re} \left( p(z_0) + \alpha \frac{z_0 p'(z_0)}{p(z_0)} \right)}{\operatorname{Re} p(z_0)} = \frac{1 + \cos \theta + \frac{\alpha m}{2}}{1 + \cos \theta} \geq 1 + \frac{\alpha m}{4} \geq 1 + \frac{\alpha n}{4}.
\]

Since (2.11) contradicts our condition (2.9), \( |w(z)| < 1 \) for all \( z \in U \). This means that \( p(z) \in Q_n \). \( \square \)

If we take \( p(z) = \frac{zf'(z)}{f(z)} \) in Theorem 2.2, we have

**Corollary 2.2.** If \( f(z) \in A_n \) satisfies \( \operatorname{Re} \left( \frac{zf'(z)}{f(z)} \right) \neq 0 \) (\( z \in U \)) and

\[
(2.12) \quad \operatorname{Re} \left\{ (1 - \alpha) \frac{zf'(z)}{f(z)} + \alpha \left( 1 + \frac{zf''(z)}{f'(z)} \right) \right\} < \left( 1 + \frac{\alpha n}{4} \right) \operatorname{Re} \left( \frac{zf'(z)}{f(z)} \right)
\]

(\( z \in U \)) for some real \( \alpha > 0 \), then \( f(z) \in T^*_n \).

**Corollary 2.3.** If \( f(z) \in A_n \) satisfies

\[
(2.13) \quad \operatorname{Re} \left( \frac{zf''(z)}{f'(z)} \right) < \operatorname{Re} \left( \frac{zf'(z)}{f(z)} \right) + \frac{n - 2}{n} \quad (z \in U),
\]

then \( f(z) \in T^*_n \).

**Proof.** If we write

\[
\frac{zf'(z)}{f(z)} = 1 + w(z) \quad (f(z) \in A_n),
\]

we see that \( w(z) \) is analytic in \( U \) and

\[
w(z) = c_n z^n + c_{n+1} z^{n+1} + \ldots.
\]

For such a function \( w(z) \), we see that

\[
(2.14) \quad \operatorname{Re} \left( \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} \right) = \operatorname{Re} \left( \frac{zw'(z)}{1 + w(z)} - 1 \right) < \frac{n - 2}{2} \quad (z \in U).
\]

Supposing that there exists a point \( z_0 \in \mathbb{U} \) on the circle \( |z| = r < 1 \) such that

\[
\max_{|z| \leq |z_0|} |w(z)| = |w(z_0)| = 1,
\]

we can write that \( w(z_0) = e^{i\theta} \) and \( z_0 w'(z_0) = me^{i\theta} \). Therefore, we have

\[
(2.15) \quad \operatorname{Re} \left( \frac{z_0 f''(z_0)}{f'(z_0)} - \frac{z_0 f'(z_0)}{f(z_0)} \right) = \operatorname{Re} \left( \frac{ke^{i\theta}}{1 + e^{i\theta}} - 1 \right) = \frac{k}{2} - 1 \geq \frac{n - 2}{2},
\]

which contradicts the condition (2.13). This implies that \( f(z) \in T^*_n \). \( \square \)
Example 2.1. Let us consider the function $p(z)$ given by
\begin{equation}
(2.16) \quad p(z) = 1 + a_n z^n \quad (z \in \mathbb{U})
\end{equation}
for some $n \in \mathbb{N} = \{1, 2, 3, \ldots \}$, where $a_n$ satisfies
\[ a_n^3 + 2a_n - 1 \leq 0 \quad (0 < a_n < 1). \]
Then $p(z) \in \mathcal{P}_n$ and $p(z) \neq 0$ ($z \in \mathbb{U}$). It is clear that $p(z)$ satisfies the condition (2.9) in Theorem 2.2 for $z = 0$.

Let us put $z = e^{i\theta}$ for $p(z)$. Then we see that
\begin{equation}
(2.17) \quad \Re \left( p(z) + \alpha \frac{zp'(z)}{p(z)} \right) = 1 + a_n \cos n\theta + \frac{\alpha a_n (a_n + \cos n\theta)}{a_n^2 + 1 + 2a_n \cos n\theta}
\end{equation}
and
\begin{equation}
(2.18) \quad \left( 1 + \frac{\alpha n}{4} \right) \Re p(z) = \left( 1 + \frac{\alpha n}{4} \right) (1 + a_n \cos n\theta).
\end{equation}
This gives us that
\begin{equation}
(2.19) \quad \left( 1 + \frac{\alpha n}{4} \right) \Re p(z) - \Re \left( p(z) + \alpha \frac{zp'(z)}{p(z)} \right)
\end{equation}
\[ = \frac{\alpha n (1 + 2a_n \cos n\theta + a_n^3 \cos n\theta + 2a_n^2 \cos^2 n\theta)}{4(a_n^2 + 1 + 2a_n \cos n\theta)} \geq 0. \]

Therefore, the function $p(z)$ satisfies the condition (2.9) for all $z \in \mathbb{U}$. Indeed, we see that
\[ |p(z) - 1| = |a_n z^n| < a_n < 1 \quad (z \in \mathbb{U}). \]
Furthermore, if we define the function $f(z) \in \mathcal{A}_n$ by
\begin{equation}
(2.20) \quad \frac{zf'(z)}{f(z)} = 1 + a_n z^n
\end{equation}
with some real $a_n$ ($0 < a_n < 1$) satisfying
\[ a_n^3 + 2a_n - 1 \leq 0, \]
then we have that
\begin{equation}
(2.21) \quad f(z) = z e^{\frac{a_n}{2} z^n}
\end{equation}
which satisfies the condition (2.12) in Corollary 2.2.

If we consider the function
\[ g(x) = x^3 + 2x - 1 \quad (0 < x < 1), \]
we see that $g(0) = -1 < 0$ and $g \left( \frac{1}{2} \right) = \frac{1}{8} > 0$. Therefore, there exists some real $x$ ($0 < x < 1$) such that $g(x) \leq 0$. Indeed, we see that
\[ 0.4533 < x < 0.4534. \]
3. Properties for the classes $\mathcal{P}_n$ and $\mathcal{A}_n$. We discuss some properties for functions in the classes $\mathcal{P}_n$ and $\mathcal{A}_n$.

**Theorem 3.1.** If $p(z) \in \mathcal{P}_n$ satisfies

\[
\int_{|z|=r} \left| \operatorname{Re} \left( \frac{zp'(z)}{p(z)} \right) \right| d\theta < \pi
\]

for $z = re^{i\theta}$ $(0 < r < 1)$, then $\operatorname{Re} p(z) > 0$ $(z \in \mathbb{U})$.

**Proof.** It follows from (3.1) that

\[
\int_{|z|=r} \left| \operatorname{Re} \left( \frac{zp'(z)}{p(z)} \right) \right| d\theta = \int_0^{2\pi} \left| \frac{d\arg p(z)}{d\theta} \right| d\theta = \int_{|z|=r} |d\arg p(z)| < \pi.
\]

This implies that $\operatorname{Re} p(z) > 0$ for $|z| = r < 1$. Applying the maximum principle for harmonic functions, we obtain that $\operatorname{Re} p(z) > 0$ $(z \in \mathbb{U})$. □

From Theorem 3.1, we have

**Corollary 3.1.** If $f(z) \in \mathcal{A}_n$ satisfies

\[
\int_{|z|=r} \left| \operatorname{Re} \left( 1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} \right) \right| d\theta < \pi
\]

for $z = re^{i\theta}$ $(0 < r < 1)$, then $f(z) \in \mathcal{S}_n^*$.

Further, applying the same method as the proof by Umezawa [5] and Nunokawa [3], we derive the following result:

**Theorem 3.2.** If $f(z) \in \mathcal{A}_1$ satisfies

\[
-\frac{\beta}{4\beta - 1} < \operatorname{Re} \left( \frac{zf''(z)}{f'(z)} \right) < \beta \quad (z \in \mathbb{U})
\]

for some real $\beta \geq \frac{1}{4}$, then $\operatorname{Re} f'(z) > 0$ $(z \in \mathbb{U})$.

**Proof.** We note that if $f'(z_0) = 0$ for some $z_0 \in \mathbb{U}$, then $f(z)$ does not satisfy the condition (3.4). This shows that $f'(z) \neq 0$ for all $z \in \mathbb{U}$. Applying the same method by Umezawa [5] and Nunokawa [3], we have that

\[
\int_{|z|=r} \frac{zf''(z)}{f'(z)} d\theta = \int_{|z|=r} \frac{zf''(z)}{f'(z)} \frac{dz}{iz} = -i \int_{|z|=r} \frac{zf''(z)}{f'(z)} dz = 0.
\]

We denote by $C_1$ the part of the circle $|z| = r$ on which

\[
\operatorname{Re} \left( \frac{zf''(z)}{f'(z)} \right) \geq 0
\]

and

\[
\int_{C_1} d\arg z = x.
\]
On the other hand, let us denote by $C_2$ the part of the circle $|z| = r$ on which

\begin{equation}
\text{Re} \left( \frac{zf''(z)}{f'(z)} \right) < 0
\end{equation}

and

\begin{equation}
\int_{C_2} d\arg z = 2\pi - x.
\end{equation}

Putting

\begin{equation}
y_1 = \int_{C_1} \text{Re} \left( \frac{zf''(z)}{f'(z)} \right) d\theta = \int_{C_1} \left( \frac{d\arg f'(z)}{d\theta} \right) d\theta
\end{equation}

and

\begin{equation}
y_2 = \int_{C_2} \text{Re} \left( \frac{zf''(z)}{f'(z)} \right) d\theta = \int_{C_2} \left( \frac{d\arg f'(z)}{d\theta} \right) d\theta,
\end{equation}

we have that $y_1 - y_2 = 0$.

In view of the condition (3.4), we obtain that $y_1 < \beta x$ and $y_2 < \frac{\beta}{4\beta - 1}(2\pi - x)$.

If $y_1 \geq \frac{\pi}{2}$, then $y_2 = y_1 \geq \frac{\pi}{2}$ and $\frac{\pi}{2} < \beta x$. On the other hand, we have that

\begin{equation}
y_2 < \frac{\beta}{4\beta - 1}(2\pi - x) < \frac{2\pi\beta - \frac{\pi}{2}}{4\beta - 1} = \frac{\pi}{2}.
\end{equation}

This contradicts the inequality $y_2 \geq \frac{\pi}{2}$. Therefore, $y_1 = y_2 < \frac{\pi}{2}$. Consequently, we obtain that

\begin{equation}
y_1 + y_2 = \int_{|z|=r} \left| \text{Re} \left( \frac{zf''(z)}{f'(z)} \right) \right| d\theta = \int_{|z|=r} |d\arg f'(z)| < \pi,
\end{equation}

which implies that $\text{Re} f'(z) > 0$ ($z \in \mathbb{U}$).

Finally, letting $\beta \to \infty$, $\beta = \frac{1}{4}$ and $\beta = \frac{1}{2}$ in Theorem 3.2, we have the following corollary.

**Corollary 3.2.** If $f(z) \in A_1$ satisfies one of the following conditions

\begin{equation}
\text{Re} \left( \frac{zf''(z)}{f'(z)} \right) > -\frac{1}{4} \quad (z \in \mathbb{U}),
\end{equation}

\begin{equation}
\text{Re} \left( \frac{zf''(z)}{f'(z)} \right) < \frac{1}{4} \quad (z \in \mathbb{U}),
\end{equation}

\begin{equation}
\left| \text{Re} \left( \frac{zf''(z)}{f'(z)} \right) \right| < 1 \quad (z \in \mathbb{U}),
\end{equation}

then $\text{Re} f'(z) > 0$ ($z \in \mathbb{U}$).
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