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Strongly gamma-starlike functions of order alpha

ABSTRACT. In this work we consider the class of analytic functions $G(\alpha, \gamma)$, which is a subset of gamma-starlike functions introduced by Lewandowski, Miller and Złotkiewicz in Gamma starlike functions, Ann. Univ. Mariae Curie-Skłodowska, Sect. A 28 (1974), 53–58. We discuss the order of strongly starlikeness and the order of strongly convexity in this subclass.

1. Introduction. Let $\mathcal{H}$ denote the class of analytic functions in the unit disc $\mathbb{D} = \{ z : |z| < 1 \}$ on the complex plane $\mathbb{C}$. For $a \in \mathbb{C}$ and $n \in \mathbb{N}$ we denote by

$$ \mathcal{H}[a, n] = \{ f \in \mathcal{H} : f(z) = a + a_n z^n + \ldots \} $$

and

$$ \mathcal{A}_n = \{ f \in \mathcal{H} : f(z) = z + a_{n+1} z^{n+1} + \ldots \}, $$

so $\mathcal{A} = \mathcal{A}_1$. Let $\mathcal{S}$ be the subclass of $\mathcal{A}$ whose members are univalent in $\mathbb{D}$. The class $\mathcal{S}_\alpha^* \mathcal{S}$ of starlike functions of order $\alpha < 1$ may be defined as

$$ \mathcal{S}_\alpha^* \mathcal{S} = \{ f \in \mathcal{A} : \Re \frac{zf'(z)}{f(z)} > \alpha, \ z \in \mathbb{D} \}. $$

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The class $S^*_\alpha$ and the class $K_\alpha$ of convex functions of order $\alpha < 1$

$$K_\alpha := \left\{ f \in A : \Re \left( 1 + \frac{zf''(z)}{f'(z)} \right) > \alpha, \ z \in D \right\}$$

$$= \left\{ f \in A : zf' \in S^*_\alpha \right\}$$

were introduced by Robertson in [6], see also [2]. If $\alpha \in [0; 1)$, then a function in either of these sets is univalent, if $\alpha < 0$ it may fail to be univalent. In particular we denote $S^*_0 = S^*$, $K_0 = K$, the classes of starlike and convex functions, respectively. Furthermore, note that if $f \in K_\alpha$, then $f \in S^*_{\delta(\alpha)}$, see [9], where

$$\delta(\alpha) = \begin{cases} \frac{1-2\alpha}{2-2\alpha} & \text{for } \alpha \neq \frac{1}{2}, \\ \frac{1}{2\log 2} & \text{for } \alpha = \frac{1}{2}. \end{cases}$$

Let $SS^*(\beta)$ denote the class of strongly starlike functions of order $\beta$, $0 < \beta < 2$,

$$SS^*(\beta) := \left\{ f \in A : \left| \frac{zf'(z)}{f(z)} \right| < \frac{\beta \pi}{2}, \ z \in D \right\},$$

which was introduced in [8] and [1]. Furthermore,

$$SK(\beta) = \left\{ f \in A : zf' \in SS^*(\beta) \right\}$$

denotes the class of strongly convex functions of order $\beta$. Analogously to (1.1), in the work [5] it was proved that if $\beta \in (0, 1)$ and $f \in SK(\alpha(\beta))$, then $f \in SS^*(\beta)$, where

$$\alpha(\beta) = \beta + \frac{2}{\pi} \tan^{-1} \left( \frac{\beta n(\beta) \sin(\pi(1 - \beta)/2)}{m(\beta) + \beta n(\beta) \cos(\pi(1 - \beta)/2)} \right),$$

and where

$$m(\beta) = (1 + \beta)^{(1+\beta)/2}, \quad n(\beta) = (1 - \beta)^{(\beta-1)/2}.$$
2. Preliminaries. To prove the main results, we need the following Nunokawa’s Lemma.

**Lemma 2.1** ([4], [5]). Let $p$ be an analytic function in $|z| < 1$ with $p(0) = 1$, $p(z) \neq 0$. If there exists a point $z_0$, $|z_0| < 1$, such that

$$|\text{Arg} \{p(z)\}| < \frac{\pi \alpha}{2} \text{ for } |z| < |z_0|$$

and

$$|\text{Arg} \{p(z_0)\}| = \frac{\pi \alpha}{2}$$

for some $\alpha > 0$, then we have

$$\frac{z_0 p'(z_0)}{p(z_0)} = ik\alpha,$$

where

$$k \geq \frac{1}{2} \left( a + \frac{1}{a} \right) \text{ when } \text{Arg} \{p(z_0)\} = \frac{\pi \alpha}{2}$$

and

$$k \leq -\frac{1}{2} \left( a + \frac{1}{a} \right) \text{ when } \text{Arg} \{p(z_0)\} = -\frac{\pi \alpha}{2},$$

where

$$\{p(z_0)\}^{1/\alpha} = \pm ia, \text{ and } a > 0.$$

Moreover,

$$\text{Arg} \left\{ 1 + \frac{z_0 p'(z_0)}{p^2(z_0)} \right\} \geq \tan^{-1} \left( \frac{\alpha n(\alpha) \sin(\pi(1 - \alpha)/2)}{m(\alpha) + \alpha n(\alpha) \cos(\pi(1 - \alpha)/2)} \right),$$

where

$$m(\alpha) = (1 + \alpha)^{(1 + \alpha)/2} \quad n(\alpha) = (1 - \alpha)^{(\alpha - 1)/2}.$$

3. Main result.

**Theorem 3.1.** Let $f(z) = z + a_2 z^2 + a_3 z^3 + \ldots$ be an analytic function in $\mathbb{D}$. Suppose also that $0 < \alpha \leq 1$ and $\gamma$ is a positive real number such that $f$ satisfies

$$\text{Arg} \left\{ \left( \frac{zf''(z)}{f'(z)} \right)^{1-\gamma} \left( 1 + \frac{zf''(z)}{f'(z)} \right)^\gamma \right\} < \frac{\alpha \pi}{2} \text{ for } |z| < 1.$$

If the equation, with respect to $x$,

$$x + \frac{2\gamma}{\pi} \tan^{-1} \left( \frac{xn(x) \sin(\pi(1 - x)/2)}{m(x) + xn(x) \cos(\pi(1 - x)/2)} \right) = \alpha,$$

where

$$m(x) = (1 + x)^{(1 + x)/2}, \quad n(x) = (1 - x)^{(x - 1)/2},$$

has a solution $\beta \in (0, 1]$, then $f$ is strongly starlike of order $\beta$. 

\textbf{Proof.} Let us put
\begin{equation}
(3.3) \quad p(z) = \frac{zf'(z)}{f(z)}, \quad p(0) = 1, \quad (z \in \mathbb{D}).
\end{equation}
Then we have
\begin{equation}
f(z)f'(z) \left(1 + \frac{zf''(z)}{f'(z)}\right) \neq 0 \quad \text{for} \quad 0 < |z| < 1
\end{equation}
because of the assumption (3.1). Moreover,
\begin{equation}
(3.4) \quad \left(\frac{zf'(z)}{f(z)}\right)^{1-\gamma} \left(1 + \frac{zf''(z)}{f'(z)}\right) = p(z) \left(1 + \frac{zp'(z)}{p^2(z)}\right)^\gamma.
\end{equation}
If there exists a point \(z_0, |z_0| < 1,\) such that
\begin{equation}
|\text{Arg}\{p(z)\}| < \frac{\pi \beta}{2} \quad \text{for} \quad |z| < |z_0|
\end{equation}
and
\begin{equation}
|\text{Arg}\{p(z_0)\}| = \frac{\pi \beta}{2},
\end{equation}
then by Nunokawa’s Lemma 2.1, we have
\begin{equation}
z_0p'(z_0) = p(z_0) = i\beta k,
\end{equation}
where
\begin{equation}
k \geq 1 \quad \text{when} \quad \text{Arg}\{p(z_0)\} = \frac{\pi \beta}{2}
\end{equation}
and
\begin{equation}
k \leq -1 \quad \text{when} \quad \text{Arg}\{p(z_0)\} = -\frac{\pi \beta}{2}.
\end{equation}
For the case \(\text{Arg}\{p(z_0)\} = \pi \beta/2,\) we have from (3.4) and (2.1)
\begin{align*}
\text{Arg}\left\{ \left(\frac{z_0f'(z_0)}{f(z_0)}\right)^{1-\gamma} \left(1 + \frac{z_0f''(z_0)}{f'(z_0)}\right)^\gamma \right\} = \\
\text{Arg}\{p(z_0)\} + \gamma \text{Arg}\left\{ 1 + \frac{z_0p'(z_0)}{p^2(z_0)} \right\} \geq \\
\frac{\pi \beta}{2} + \gamma \tan^{-1}\left(\frac{\beta n(\beta) \sin(\pi(1 - \beta)/2)}{m(\beta) + \beta n(\beta) \cos(\pi(1 - \beta)/2)}\right) = \frac{\alpha \pi}{2}
\end{align*}
because \(\beta\) is the solution of (3.2). For the case \(\text{Arg}\{p(z_0)\} = -\pi \beta/2,\) applying the same method as the above, we have
\begin{align*}
\text{Arg}\left\{ \left(\frac{z_0f'(z_0)}{f(z_0)}\right)^{1-\gamma} \left(1 + \frac{z_0f''(z_0)}{f'(z_0)}\right)^\gamma \right\} < -\frac{\alpha \pi}{2}.
\end{align*}
In both of the above cases we have
\[
\left| \text{Arg} \left\{ \left( \frac{z_0 f'(z_0)}{f(z_0)} \right)^{1-\gamma} \left( 1 + \frac{z_0 f''(z_0)}{f'(z_0)} \right) \right\} \right| \geq \frac{\alpha \pi}{2}
\]
for \( z_0 \in \mathbb{D} \), which contradicts hypothesis (3.1) of the theorem and therefore,
\[ |\text{Arg} \{ p(z) \}| < \frac{\pi \beta}{2} \quad \text{for} \quad |z| < 1, \]
which completes the proof. \( \square \)

Theorem 3.1 says that a function in the class \( G(\alpha, \gamma) \), see (1.4), of \( \gamma \)-strongly starlike functions of order \( \alpha \) is strongly starlike function, see (1.2), of order at least \( \beta \), where \( \beta \) is the solution of (3.2). Note that if \( f \in G(\alpha, 0) \), then \( f \) is strongly starlike of order \( \alpha \). For a related result we refer to [7]. If \( \alpha = 1 \), then Theorem 3.1 becomes the following result on the class \( G(\gamma, 1) \) introduced by Lewandowski, Miller and Złotkiewicz [3].

Corollary 3.1. Assume that \( f \in G(\gamma, 1) \) or that \( f \) satisfies (1.5) and (1.6).
If the equation
\[
x + \frac{2\gamma}{\pi} \tan^{-1} \left( \frac{xn(x)\sin(\pi(1-x)/2)}{m(x) + xn(x)\cos(\pi(1-x)/2)} \right) = 1,
\]
has a solution \( \beta \in (0, 1] \), then \( f \) is strongly starlike of order \( \beta \).

In the corollary below there are examples of the choice \( \alpha, \gamma \) and \( \beta \) which satisfies Corollary 3.1 or Theorem 3.1.

Corollary 3.2. If \( f \in G(\gamma(1;1/2), 1) \), then \( f \in SS^*(1/2) \), where
\[
\gamma(1;1/2) = \frac{\pi}{4 \tan^{-1} \left( \frac{1}{3^{1/4}/3+1} \right)} \approx 3.378.
\]
If \( f \in G(\gamma(3/4;1/2), 3/4) \), then \( f \in SS^*(1/2) \), where
\[
\gamma(3/4;1/2) = \frac{\pi}{8 \tan^{-1} \left( \frac{1}{3^{5/4}/3+1} \right)} \approx 1.689.
\]
If \( f \in G(\gamma(3/5;1/2), 3/5) \), then \( f \in SS^*(1/2) \), where
\[
\gamma(3/5;1/2) = \frac{\pi}{20 \tan^{-1} \left( \frac{1}{3^{1/4}/3+1} \right)} \approx 0.675.
\]
If \( f \in G(\gamma(3/4;2/3), 3/4) \), then \( f \in SS^*(2/3) \), where
\[
\gamma(3/4;2/3) = \frac{\pi}{8 \tan^{-1} \left( \frac{\sqrt{3}}{3^{5/3}/3+\sqrt{81}} \right)} \approx 1.481.
\]
If $\alpha = \beta$, then from (3.2) we get $\gamma = 0$, and $G(0, \alpha) \subset SS^*(\alpha)$, but this case is trivial. In the next theorem we consider the order of strongly starlikeness for functions, in some sense, in the class $G(\alpha, \gamma)$ of negative order $\gamma$.

**Theorem 3.2.** Let $f(z) = z + a_2z^2 + a_3z^3 + \ldots$ be an analytic function in $D$. Suppose also that $\gamma$ is a negative real number such that $f$ satisfies

$$\left| \frac{zf''(z)}{f'(z)} \right| < \alpha \pi^2$$

for $|z| < 1$, where $0 < \alpha \leq 1$ and suppose that $\beta$ is the root of the equation

$$\beta + \gamma(1 - \beta) = \alpha$$

in the interval $(0, 1]$. Then we have

$$\left| \frac{zf''(z)}{f'(z)} \right| < \frac{\beta \pi}{2}$$

for $|z| < 1$.

**Proof.** In the first part of the proof we apply the same method as in the proof of Theorem 3.1. Let us put

$$p(z) = \frac{zf'(z)}{f(z)}, \quad p(0) = 1, \quad (z \in D).$$

If there exists a point $z_0 \in D$ such that

$$\left| \frac{zf''(z)}{f'(z)} \right| < \frac{\pi \beta}{2}$$

for $|z| < |z_0|$

and

$$\left| \frac{zf''(z_0)}{f'(z_0)} \right| = \frac{\pi \beta}{2},$$

then by Nunokawa’s Lemma 2.1, we have

$$\frac{z_0p'(z_0)}{p(z_0)} = i\beta k,$$

where

$$k \geq \frac{1}{2} \left( a + \frac{1}{a} \right) \quad \text{when} \quad \arg \{p(z_0)\} = \frac{\pi \alpha}{2}$$

and

$$k \leq -\frac{1}{2} \left( a + \frac{1}{a} \right) \quad \text{when} \quad \arg \{p(z_0)\} = -\frac{\pi \alpha}{2},$$

where

$$\{p(z_0)\}^{1/\beta} = \pm ia, \quad \text{and} \quad a > 0.$$
For the case \( p(z_0)^{1/\beta} = ia, a > 0 \), we have from (3.8) and (3.9)

\[
\text{Arg} \left\{ \left( \frac{z_0 f'(z_0)}{f(z_0)} \right)^{1-\gamma} \left( 1 + \frac{z_0 f''(z_0)}{f'(z_0)} \right) \right\} \\
= \text{Arg} \left\{ p(z_0) \left( 1 + \frac{z_0 p'(z_0)}{p^2(z_0)} \right)^\gamma \right\} \\
= \text{Arg} \{ p(z_0) \} + \gamma \text{Arg} \left\{ 1 + \frac{z_0 p'(z_0)}{p^2(z_0)} \right\} \\
= \frac{\pi \beta}{2} + \gamma \text{Arg} \left\{ 1 + \frac{i \beta k}{(ia)^\beta} \right\} \\
= \frac{\pi \beta}{2} + \gamma \text{Arg} \left\{ 1 + \frac{\beta k}{a^\beta e^{i(1-\beta)/2}} \right\} \\
\geq \frac{\pi \beta}{2} + \gamma \frac{(1-\beta)}{2} \\
= \frac{\alpha \pi}{2},
\]

because \( \beta \) is the solution of (3.6). For the case \( \text{Arg} \{ p(z_0) \} = -\pi \beta/2 \), applying the same method as the above, we have

\[
\text{Arg} \left\{ \left( \frac{z_0 f'(z_0)}{f(z_0)} \right)^{1-\gamma} \left( 1 + \frac{z_0 f''(z_0)}{f'(z_0)} \right) \right\} \leq -\frac{\alpha \pi}{2}.
\]

The above cases show that

\[
\left| \text{Arg} \left\{ \left( \frac{z_0 f'(z_0)}{f(z_0)} \right)^{1-\gamma} \left( 1 + \frac{z_0 f''(z_0)}{f'(z_0)} \right) \right\} \right| \geq \frac{\alpha \pi}{2}, \quad z_0 \in \mathbb{D},
\]

which contradicts hypothesis (3.5) of the theorem and therefore,

\[ |\text{Arg} \{ p(z) \}| < \frac{\pi \beta}{2} \quad \text{for} \quad |z| < 1 \]

which completes the proof. \( \square \)

**Theorem 3.3.** Let \( f(z) = z + a_2 z^2 + a_3 z^3 + \ldots \) be an analytic function in \( \mathbb{D} \). Suppose also that \( 0 < \alpha \leq 1 \) and \( 0 < \gamma \leq 1 \) are such that \( f \) satisfies (3.1). If the equation (3.2) has a solution \( \alpha_0 \in (0,1] \), then \( f \) is strongly convex of order \( \{(1-\gamma)\alpha_0 + \alpha\}/\gamma \).

**Proof.**

\[
\left| \text{Arg} \left\{ \left( 1 + \frac{zf''(z)}{f'(z)} \right)^\gamma \right\} \right| - \left| \text{Arg} \left\{ \frac{zf'(z)}{f(z)} \right\} \right|^{1-\gamma} \leq \left| \text{Arg} \left\{ \left( \frac{zf'(z)}{f(z)} \right)^{1-\gamma} \left( 1 + \frac{zf''(z)}{f'(z)} \right)^\gamma \right\} \right| < \frac{\alpha \pi}{2} \quad \text{for} \quad |z| < 1.
\]
Then by Theorem 3.1 we have
\[
\left| \text{Arg}\left\{ \frac{1}{B} \frac{zf''(z)}{f'(z)} \right\} \right| \\
\leq \left| \text{Arg}\left\{ \left(\frac{zf'(z)}{f(z)}\right)^{1-\gamma} \right\} + \frac{\alpha \pi}{2} \right| \\
< \frac{\pi(1-\gamma)\alpha_0}{2} + \frac{\alpha \pi}{2},
\]
and so
\[
\left| \text{Arg}\left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} \right| < \pi \left\{ \frac{(1-\gamma)\alpha_0 + \alpha}{2\gamma} \right\} for \ |z| < 1. \quad \Box
\]

**Theorem 3.4.** Assume that the equation (3.2) has a solution \( \alpha_0, 0 < \alpha_0 < \alpha \leq 1 \). If \( 0 < \delta < \gamma \), then \( G(\alpha, \gamma) \subset G(\alpha, \delta) \).

**Proof.** Let us suppose that \( f \) is a member of \( G(\alpha, \gamma) \) and let us put
\[
A = \left\{ B^{1-\delta} C^\delta \right\}^{\gamma/\delta},
\]
where
\[
B = \frac{zf'(z)}{f(z)} \quad \text{and} \quad C = 1 + \frac{zf''(z)}{f'(z)}.
\]
Then we have
\[
A = B^{1-\gamma} C^\gamma B^{\gamma/\delta - 1}
\]
and by Theorem 3.1 we obtain
\[
|\text{Arg} \{A\}| = \frac{\gamma}{\delta} \left| \text{Arg} \left\{ B^{1-\delta} C^\delta \right\} \right| \\
= \left| \text{Arg} \left\{ B^{1-\gamma} C^\gamma \right\} + \text{Arg} \left\{ B^{\gamma/\delta - 1} \right\} \right| \\
< \frac{\alpha \pi}{2} + \left( \frac{\gamma}{\delta} - 1 \right) \frac{\alpha_0 \pi}{2} \\
< \frac{\alpha \pi}{2} + \left( \frac{\gamma}{\delta} - 1 \right) \frac{\alpha \pi}{2} \\
\leq \frac{\gamma \alpha \pi}{\delta} \frac{2}{2}.
\]
This shows that
\[
\left| \text{Arg} \left\{ B^{1-\delta} C^\delta \right\} \right| < \frac{\alpha \pi}{2} \quad z_0 \in \mathbb{D}.
\]
Therefore, \( f \in G(\alpha, \delta) \). \quad \Box

**References**


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