On the birational gonalitys of smooth curves

Abstract. Let $C$ be a smooth curve of genus $g$. For each positive integer $r$ the birational $r$-gonality $s_r(C)$ of $C$ is the minimal integer $t$ such that there is $L \in \text{Pic}^t(C)$ with $h^0(C, L) = r + 1$. Fix an integer $r \geq 3$. In this paper we prove the existence of an integer $g_r$ such that for every integer $g \geq g_r$ there is a smooth curve $C$ of genus $g$ with $s_{r+1}(C)/(r+1) > s_r(C)/r$, i.e. in the sequence of all birational gonalitys of $C$ at least one of the slope inequalities fails.

1. Introduction. Let $C$ be a smooth curve of genus $g$. For each positive integer $r$ the birational $r$-gonality $s_r(C)$ of $C$ is the minimal integer $t$ such that there is $L \in \text{Pic}^t(C)$ with $h^0(C, L) = r + 1$ ([1], §2). In this paper we prove the following result.

**Theorem 1.** Fix an integer $r \geq 3$. Then there exists an integer $g_r$ such that for every integer $g \geq g_r$ there is a smooth curve $C$ of genus $g$ with $s_{r+1}(C)/(r+1) > s_r(C)/r$.

Theorem 1 means that for the curve $C$ at least one slope inequality fails. For any integer $r \geq 1$ the $r$-gonality of $C$ is the minimal degree of a line bundle $L$ on $C$ with $h^0(C, L) \geq r + 1$. Obviously $s_r(C) \geq d_r(C)$ if $r \geq 2$. Equality holds if $d_r(C) < r \cdot d_1(C)$ and $C$ has no non-trivial morphism onto a smooth curve of positive genus. In [6] H. Lange and G. Martens studied...
the slope inequality for the usual gonality sequence of smooth curves (it may fail for some \( C \), but not for a general \( C \)).

We work over an algebraically closed base field with characteristic zero.

2. Working inside a Hirzebruch surface. Fix \( e \in \mathbb{N} \). Let \( F_e \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-e)) \) denote the Hirzebruch surface ([4], Chapter V, §2). We call \( \pi : F_e \rightarrow \mathbb{P}^1 \) a ruling of \( F_e \). We have \( \text{Pic}(F_e) \cong \mathbb{Z}^2 \) and take as a basis of \( \text{Pic}(F_e) \) a fiber \( f \) of \( \pi \) and a section \( h \) of \( \pi \) with \( h^2 = -e \) (\( \pi \) and \( h \) are unique if \( e > 0 \)).

For any finite set \( S \subset F_e \) let \( 2S \) denote the first infinitesimal neighborhood of \( S \) in \( F_e \), i.e. the closed subscheme of \( F_e \) with \((\mathcal{I}_S)^2 \) has its ideal sheaf. We have \((2S)_{\text{red}} = S \) and \( \deg(2S) = 3 \cdot \sharp(S) \). Fix an integer \( a \geq 0 \). The line bundle \( \mathcal{O}_{F_e}(ah + bf) \) is spanned (resp. very ample) if and only if \( b \geq ea \) (resp. \( b > ea \) and \( a > 0 \)) ([4], V.2.18). We have \( h^1(F_e, \mathcal{O}_{F_e}(ah + bf)) = 0 \) if and only if \( b \geq -1 \). If \( b \geq ae \), then

\[
h^0(F_e, \mathcal{O}_{F_e}(ah + bf)) = (a + 1)(2b - ea + 2)/2
\]

([5], Proposition 2.3). Assume \( a > 0 \) and \( b \geq ae \); if \( e = 0 \), then assume \( b > 0 \). Fix any \( Y \in |\mathcal{O}_{F_e}(ah + ea)| \). Since \( \omega_{F_e} \cong \mathcal{O}_{F_e}(-2h + (-e - 2)f) \), the adjunction formula gives

\[
\omega_Y \cong \mathcal{O}_Y((a - 2)h + (ea - e - 2)f).
\]

Hence \( p_a(Y) = 1 + a(2a - e - 2)/2 \). We have

\[
h^0(F_e, \mathcal{O}_{F_e}(ah + ea)) = (ea + 2)(a + 1)/2.
\]

To prove Theorem 1 for the integer \( r \) we will use as \( C \) the normalization of a nodal curve \( Y \in |\mathcal{O}_{F_e}(ah + ea)| \), where \( e := r - 1 \).

Notation 1. For all integers \( a \geq 1 \) and \( e \geq 1 \) set \( g_{a,e} := 1 + a(2a - e - 2)/2 \).

Notice that if \( a \geq 2 \), then \( g_{a,e} - g_{a-1,e} = ae - e - 1 \).

Lemma 1. Assume \( e \geq 2 \). Fix integers \( a, x \). If \( x = 0 \), assume \( a \geq 1 \). If \( x > 0 \), assume \( a \geq 5 \) and \( 3x \leq (ea - 2e + 1)(a - 1)/2 \). Fix a general \( S \subset F_e \) such that \( \sharp(S) = x \). Then

\[
h^1(F_e, \mathcal{I}_{2S}(ah + ea)) = 0, \ h^0(F_e, \mathcal{I}_{2S}(ah + ea)) = (ea + 2)(a + 1)/2 - 3x,
\]

a general \( Y \in |\mathcal{I}_{2S}(ah + ea)| \) is integral, nodal and with \( \text{Sing}(Y) = S \).

Proof. First assume \( x = 0 \). Since \( \mathcal{O}_{F_e}(ah + ae) \) is spanned, Bertini’s theorem gives that a general \( Y \in |\mathcal{O}_{F_e}(ah + ae)| \) is smooth. Since

\[
h^0(F_e, \mathcal{O}_{F_e}(h + ef)) + h^0(F_e, \mathcal{O}_{F_e}((c-1)h + (c-1)rf)) < h^0(F_e, \mathcal{O}_{F_e}(ch + cf))
\]

for every integer \( c \in \{1, \ldots, a-1\} \) and \( |\mathcal{O}_{F_e}(uh + vf)| \) has \( h \) in the base locus if \( u > 0 \) and \( v < eu \), \( Y \) is also irreducible.

Now assume \( x > 0 \). Fix a general \( S \subset F_e \) such that \( \sharp(S) = x \). Since

\[
3x \leq h^0(F_e, \mathcal{O}_{F_e}((a - 2)h + e(a - 2)f)),
\]
\( e \geq 2 \) and \( a - 2 \geq 3 \), a theorem of A. Lafce gives
\[
h^1(Fe, \mathcal{I}_{2S}((a - 2)h + e(a - 2)f)) = 0
\]
([5], Proposition 5.2 and case \( m = 2 \) of Theorem 7.2). Hence
\[
h^1(Fe, \mathcal{I}_{2S}((a - i)h + e(a - i)f)) = 0
\]
for \( i = 0, 1 \). Hence
\[
h^0(Fe, \mathcal{I}_{2S}(ah + eaf)) = (ea + 2)(a + 1)/2 - 3x.
\]

Fix \( P \in F_e \setminus S \) and a general \( A \in |\mathcal{O}_{F_e}(h + e)f)| \) containing \( P \). The curve \( A \) is smooth if \( P \notin h \), while \( A = h \cup F \) with \( F \in |\mathcal{O}_{F_e}(f)| \) if \( P \in h \). In all cases we see that \( \mathcal{O}_A(ah + eaf) \) is spanned at \( P \) (in the case \( P \in h \) use the following facts: \( \mathcal{O}_h(ah + eaf) \cong \mathcal{O}_h, \mathcal{O}_{\mathbb{P}^1}(a) \) is spanned). Since
\[
h^1(Fe, \mathcal{O}_{F_e}((a - 1)h + e(a - 1)f)) = 0, \quad P \in A \) and \( \mathcal{O}_A(ah + eaf) \) is spanned at \( P \), the exact sequence
\[
0 \to \mathcal{I}_{2S}((a - 1)h + e(a - 1)f) \to \mathcal{I}_{2S}((a - 1)h + e(a - 1)f) \\
\to \mathcal{O}_A(ah + eaf) \to 0
\]
gives that \( \mathcal{I}_{2S}(ah + eaf) \) is spanned at \( P \). Since this is true for all \( P \notin S \), Bertini’s theorem gives \( \text{Sing}(Y) = S \). In particular \( Y \) has no multiple component. Fix \( P \in S \). Since \( S \) is general, we have \( P \notin h \). Since \( |\mathcal{O}_{F_e}(h + eaf)| \) induces a morphism with injective differential at \( P \), \( |\mathcal{O}_{F_e}(2h + 2af)| \) spans the jets at \( P \) of \( \mathcal{O}_{F_e} \) up to order 2. Hence we may find \( Y' \in |\mathcal{O}_{F_e}(2h + 2ef)| \) with an ordinary node at \( P \). Since
\[
h^1(Fe, \mathcal{I}_{2S}((a - 2)h + e(a - 2)f)) = 0,
\]
we have
\[
h^1(Fe, \mathcal{I}_{\{P\} \cup 2(S \setminus \{P\})}((a - 2)h + e(a - 2)f)) = 0.
\]
Hence
\[
h^0(Fe, \mathcal{I}_{\{P\} \cup 2(S \setminus \{P\})}((a - 2)h + e(a - 2)f))
\]
\[
= h^0(Fe, \mathcal{I}_{2(S \setminus \{P\})}((a - 2)h + e(a - 2)f)) - 1.
\]
Hence there is \( Y'' \in |\mathcal{I}_{2(S \setminus \{P\})}((a - 2)h + e(a - 2)f)| \) such that \( P \notin Y'' \). Hence \( Y'' \cup Y' \) has an ordinary node at \( P \). Since \( Y'' \cup Y' \in |\mathcal{I}_{2S}(ah + eaf)| \), \( S \) is finite and \( Y \) is general, \( Y \) is nodal. Recall that \( \text{Sing}(Y) = S \) and that \( S \) is general. Since \( S \) is general, no pair of points of \( S \) is on the same fiber of the ruling of \( F_e \). Hence no fiber of \( F_e \) may be an irreducible component of \( Y \). Since \( \mathcal{O}_{F_e}(ch + ef) \cdot \mathcal{O}_{F_e}((a - c)h + e(a - c)f) = ec(a - c) \), we immediately see that \( Y \) is irreducible. \( \square \)

**Lemma 2.** Assume \( e \geq 2 \). Fix integers \( a, x \). If \( x = 0 \), assume \( a \geq 1 \). If \( x > 0 \), assume \( a \geq 5 \) and \( 3x \leq (ea - 2e + 1)(a - 1)/2 \). Fix a general \( S \subset F_e \) such that \( \sharp(S) = x \) and a general \( Y \in |\mathcal{I}_{2S}(ah + eaf)| \). Let \( u : C \to Y \) denote the normalization map. The line bundle \( u^*(\mathcal{O}_Y(f)) \) is spanned
and $h^0(C, u^*(\mathcal{O}_Y(f))) = 2$. Let $\rho : C \to \mathbb{P}^1$ be the morphism induced by $|u^*(\mathcal{O}_Y(f))|$. Then $\rho$ is not composed with an involution, i.e. there are no $(C', \rho', \rho'')$ with $C'$ a smooth curve, $\rho' : C \to C'$, $\rho'' : C' \to \mathbb{P}^1$, $\rho = \rho'' \circ \rho'$, $\deg(\rho') \geq 2$ and $\deg(\rho'') \geq 2$.

**Proof.** Obviously $u^*(\mathcal{O}_Y(f))$ is spanned. Since $ae + 1 - e - 2 \geq e(a - 2) - 1$, Serre’s duality gives

$$h^1(F_e, \mathcal{O}_{F_e}(-ah-(ae+1)f)) = h^1(F_e, \mathcal{O}_{F_e}((a-2)h+(ae+1-e-2)f)) = 0.$$  

Hence $h^0(Y, \mathcal{O}_Y(f)) = 2$. Since $h^i(F_e, \mathcal{O}_{F_e}) = 0$, $i = 1, 2$, $\omega_{F_e} \cong \mathcal{O}_{F_e}(-2h + (-e - 2)f)$, $Y$ is nodal and $S = \text{Sing}(Y)$, we have

$$H^0(Y, \omega_Y) \cong H^0(F_e, \mathcal{O}_{F_e}((a-2)h + (ae-e-2)f))$$

and $H^0(C, \omega_C)$ is induced (after deleting the base points) from

$$H^0(F_e, \mathcal{I}_S((a-2)h + (ae-e-2)f)).$$

Hence $h^0(C, u^*(\mathcal{O}_Y(f))) = 2 = h^0(Y, \mathcal{O}_Y(f))$ if and only if

$$h^1(C, u^*(\mathcal{O}_Y(f))) = x + h^1(Y, \mathcal{O}_Y(f)),$$

i.e. if and only if $h^1(F_e, \mathcal{I}_S((a-2)h + (ae-e-3)f)) = 0$. The last equality is true, because $S$ is general and $x \leq (a-1)(ea-2e-2)/2 = h^0(F_e, \mathcal{I}_S((a-2)h + (ae-e-3)f)).$

For any $P \in F_e$ let $F_P$ be the fiber of the ruling of $F_e$ containing $P$. We fix $P \in F_e \setminus h$ such that $F_P \cap S = \emptyset$. Let $Z \subset F_P$ be the degree two effective divisor with $Z$ as its support. Take any $S_1 \subset F_P \setminus \{P, h \cap F_P\}$ such that $\sharp(S_1) = a - 2$ and set $Z' := Z \cup S_1$. Taking the inclusion $F_P \hookrightarrow F_e$, we may also see $Z'$ as a degree a zero-dimensional subscheme of $F_e$.

**Claim.** $h^1(F_e, \mathcal{I}_{Z'}(ah + ae)) = 0$.

**Proof of the Claim.** Set $T := h \cup F_P \subset |\mathcal{O}_{F_e}(h + f)|$. Since $S \cap h = \emptyset$ and $S \cap F_P = \emptyset$, we have $S \cap T = \emptyset$. Hence $(2S \cup Z') \cap T = Z'$. We proved during the proof of Lemma 1 that $h^1(F_e, \mathcal{I}_{Z'}((a-1)h + (a-1)ef))) = 0$. Hence $h^1(F_e, \mathcal{I}_{2S}((a-1)h + (ae-e+e-1)f)) = 0$. Notice that

$$\mathcal{I}_{2S}((a-1)h + (ae-e+e-1)f) \cong \mathcal{I}_{Z'}(ah + ae)((-T)).$$

Since $h^1(F_e, \mathcal{I}_{Z'}(ah + ae)) = 0$ (Lemma 1), the Claim is true if

$$h^1(T, \mathcal{I}_{Z', T}(ah + ae)) = 0.$$

The nodal curve $T$ has two irreducible components, $h$ and $F_P$, and both components are isomorphic to $\mathbb{P}^1$. Since $Z' \cap h = \emptyset$, we have $Z' \cap h \cap F_P = \emptyset$ and hence the $\mathcal{O}_T$-sheaf $\mathcal{I}_{Z', T}(ah + ae)$ is a line bundle. Since $Z' \cap h = \emptyset$ and $\mathcal{O}_h(ah + ae) \cong \mathcal{O}_h$, we have $\mathcal{I}_{Z', T}(ah + ae)|h \cong \mathcal{O}_h$. Since $\deg(Z') = a$, we have $\mathcal{I}_{Z', T}(ah + ae) \cap F_P \cong \mathcal{O}_{F_P}$. Hence a Mayer–Vietoris exact sequence gives $h^1(T, \mathcal{I}_{Z', T}(ah + ae)) = 0$, concluding the proof of the Claim.

The Claim is equivalent to

$$h^0(F_e, \mathcal{I}_{Z' \cup Z'}(ah + ae)) = h^0(F_e, \mathcal{I}_{Z'}(ah + ae)) - a.$$
Set $\Gamma := \bigcup_{Q \in S} F_Q$. We take all $Y \in |\mathcal{I}_{2S}(ah + eaf)|$ containing some $Z'$. The set of all $P \in F_e$ has dimension 2. For fixed $P$ the set of all $S_1 \subset F_P \setminus \{P\}$ with $|\mathcal{I}(S_1)| = a - 2$ has dimension $a - 2$. Each $Y$ may contain only finitely many schemes $Z'$, because each non-constant morphism $C \to \mathbb{P}^1$ has only finitely many ramification points. Varying first $P \in F_e \setminus (h \cup \Gamma)$ and then all $S_1 \subset F_P \setminus (\{P\} \cup \{P\})$ with $|\mathcal{I}(S_1)| = a - 2$, we get that a general $Y \in |\mathcal{I}_{2S}(ah + eaf)|$ contains some $Z'$ for some $P \in F_e \setminus (h \cup \Gamma)$. Let $u : C \to \mathbb{P}^1$ be the normalization of any such $Y$, say containing $Z' = Z \cup S_1$ with $Z \subset F_P$. We saw that $h^0(C, u^*(\mathcal{O}_Y(f))) = 2$. Let $\rho : C \to \mathbb{P}^1$ be the morphism associated to $|u^*(\mathcal{O}_Y(f))|$. Notice that $\rho$ is induced by the ruling $\rho_1 : F_e \to \mathbb{P}^1$. Set $Q := \rho_1(P)$. By the construction $\rho^{-1}(Q) \cong Z \cup S_1$, i.e., the fiber of $\rho$ over $Q$ contains a point with multiplicity two and $a - 2$ points with multiplicity one. Hence there are no $(C', \rho', \rho'')$ with $C'$ a smooth curve, $\rho' : C \to C'$, $\rho'' : C' \to \mathbb{P}^1$, $\rho = \rho'' \circ \rho'$, deg$(\rho'') \geq 2$ and deg$(\rho'') \geq 2$. □

**Lemma 3.** Fix $S, Y, C, u$ as in Lemma 1 and take any spanned line bundle $L$ of degree $> 0$. Fix a general $A \in |L|$ and set $B := u(A)$. Then $S \cap B = \emptyset$ and $h^1(F_e, \mathcal{I}_{S \cup B}((a - 2)h + (ae - e - 2)f)) > 0$.

**Proof.** Since $\deg(L) > 0$, $A \neq \emptyset$. Since $L$ is spanned, $h^0(C, L(-Q)) = h^0(C, L) - 1$ for each $Q \in C$ and in particular for each $Q \in A$. Riemann–Roch gives $h^1(C, \mathcal{O}_C(A \setminus \{Q\})) = h^1(C, \mathcal{O}_C(A))$ for every $Q \in A$. Since $H^0(C, \omega_C) \cong H^0(F_e, \mathcal{I}_S((a - 2)h + (ae - e - 2)f))$, we get

$$h^0(F_e, \mathcal{I}_{S \cup \{P\}}((a - 2)h + (ae - e - 2)f)) = h^0(F_e, \mathcal{I}_{S \cup \emptyset}((a - 2)h + (ae - e - 2)f))$$

for every $P \in B$. Hence $h^1(F_e, \mathcal{I}_{S \cup B}((a - 2)h + (ae - e - 2)f)) > 0$. □

**Lemma 4.** Take $e, a, x, S, Y, C$ as in Lemma 2. Then $d_1(C) = a$.

**Proof.** The line bundle $u^*(\mathcal{O}_Y(f))$ gives $d_1(C) \leq a$. Assume $z := d_1(C) < a$ and take $L \in \text{Pic}^2(C)$ evincing $d_1(C)$, i.e., evincing the gonality of $C$. Fix a general $A \in |L|$ and set $B := u(A)$. Lemma 3 gives

$$h^1(F_0, \mathcal{I}_{S \cup B}((a - 2)h + (ae - 2e - f)) > 0).$$

Since $L$ is spanned and $A$ is general, we have $S \cap B = B \cap h = \emptyset$. Lemma 2 gives $h^0(C, u^*(\mathcal{O}_Y(f))) = 2$. Let $v : C \to \mathbb{P}^1$ be the morphism induced by $|L|$ and $v' : C \to \mathbb{P}^1$ the morphism induced by $|u^*(\mathcal{O}_Y(f))|$. Since $v'$ is not composed with an involution (Lemma 3), the induced map $(v, v') : C \to \mathbb{P}^1 \times \mathbb{P}^1$ is birational onto its image. Hence for general $B$ we have $|\mathcal{I}(D \cap B)| \leq 1$ for every $D \in |\mathcal{O}_{F_e}(f)|$. Since $h^0(F_e, \mathcal{O}_{F_e}(h + ef)) > z$, there is $A_1 \in |\mathcal{O}_{F_e}(h + ef)|$ containing $B$. Since $B \cap h = \emptyset$ and $|\mathcal{I}(D \cap B)| \leq 1$ for every $D \in |\mathcal{O}_{F_e}(f)|$, $A_1$ is irreducible. Hence $E \cong \mathbb{P}^1$. Since $S$ is general
and $h^0(F_e, \mathcal{O}_{F_e}(h + ef)) = e + 2$, we have $\#(S \cap A_1) \leq e + 1$. Hence

$$\#(A_1 \cap (S \cup B)) \leq z + e + 1 \leq a + e.$$ 

Since $\deg(\mathcal{O}_{A_1}((a - 2)h + (ae - e - 2)f)) = ae - e - 2 \geq a + e - 1$, we have $h^1(A_1, \mathcal{I}_{A_1 \cap (S \cup B), A_1}((a - 2)h + (ae - e - 2)f)) = 0$.

Hence the case $i = 1$ of (1) gives

$$h^1(F_e, \mathcal{I}_{S \setminus S \cap A_1}((a - 3)h + ((a - 1)e - e - 2)f)) > 0.$$ 

Since $S \setminus S \setminus S \cap A_1$ is general and

$x \leq e(a - 2)(ea - 3e + 2)/2 \leq h^0(F_e, \mathcal{O}_{F_e}((a - 3)h + ((a - 1)e - e - 2)f)),$

we have

$$h^1(F_e, \mathcal{I}_{S \setminus S \cap A_1}((a - 3)h + ((a - 1)e - e - 2)f)) = 0,$$

a contradiction. \qed

**Lemma 5.** Fix integers $e \geq 2$ and $a \geq 2$. Fix any integral $Y \in |\mathcal{O}_{F_e}(ah + eaf)|$ and call $u : C \to Y$ the normalization map. Then $s_{e+1+2j}(C) \leq ae + je$ for every integer $j \geq 0$.

**Proof.** We have $h^0(F_e, \mathcal{O}_{F_e}(h + (e + j)ef)) = e + 2 + 2j$, for every integer $j \geq 0$. Since $a \geq 2$, we have $h^0(F_e, \mathcal{I}_Y(h + yf)) = 0$ for any $y$. We have $\mathcal{O}_{F_e}(h + (e + j)f) \cdot \mathcal{O}_{F_e}(ah + eaf) = a(e + j)$. Since for any $j \geq 0$ the linear system $|\mathcal{O}_{F_e}((h + (e + j)f)|$ embeds $F_e \setminus h$, the spanned line bundle $u^*(\mathcal{O}_Y((h + (e + j)f))$ gives $s_{e+1+2j}(C) \leq ae + je$. \qed

**Lemma 6.** Fix an integer $e \geq 2$. There is an integer $A_e \geq 5$ with the following property. Fix integers $a, x$ such that $a \geq A_e$ and $0 \leq x \leq ae - e - 2$. Moreover, every base point free linear system on $C$ with degree $\leq ae$ and birationally very ample is induced (after deleting the base points) from a linear subspace of $H^0(F_e, \mathcal{O}_{F_e}(h + ef))$.

**Proof.** Fix an integer $z \leq ae$ such that there is a spanned $L \in \text{Pic}^z(C)$ such that the morphism $v : C \to \mathbb{P}^k$, $k := h^0(C, L) - 1$, induced by $|L|$ is birational onto its image. Fix a general $A \in |L|$ and set $B := u(A)$. Since $L$ is spanned and $A$ is general, we have $S \cap B = \emptyset$ and $B \cap h = \emptyset$. Lemma 3

$$h^1(F_0, \mathcal{I}_{S \cup B}((a - 2)h + (ae - 2e)f)) > 0.$$ 

(a) Since the monodromy group $G$ of the general hyperplane section of $v(C)$ is the full symmetric group $S_z$, $B$ is in uniform position in $F_e$ and in particular for all integers $c, t$ such that $0 \leq c \leq a$ and $t \geq ec$ and any $B' \subset B$, either $h^0(F_e, \mathcal{I}_{B'}(ch + tf)) = \max \{0, (c + 1)(t + 1) - \#(B')\}$ or $h^0(F_e, \mathcal{I}_{B}(ch + tf)) > 0$. In particular, $\#(D \cap B) \leq 1$ for every $D \in |\mathcal{O}_{F_e}(f)|$.

(b) In this step we assume $h^0(F_e, \mathcal{I}_B(h + ef)) > 0$. Let $t$ be the minimal non-negative integer such that $h^0(F_e, \mathcal{I}_B(h + tf)) > 0$. By assumption we have $t \leq e$. Varying $A$ in $|L|$, we get that $|L|$ is obtained (after deleting
the base locus) from a linear subspace of $|\mathcal{O}_{F_c}(h+tf)|$. Since $|\mathcal{O}_{F_c}(h+tf)|$ sends $F_c \setminus h$ onto $\mathbb{P}^1$ if $t < e$, while $v$ is birational onto its image, we get $t = e$. Since $h^0(F_c, \mathcal{I}_B(h + (e - 1)f)) = 0$, step (a) gives $\sharp(D \cap B) \leq e - 1$ for every $\Gamma \in |\mathcal{I}_B(h + (e - 1)f)|$. Since $\sharp(D \cap B) \leq 1$ for every $D \in |\mathcal{O}_{F_c}(1)|$ and $z > e$, $T$ is irreducible. Hence $T \cong \mathbb{P}^1$. Since $\sharp(B) \leq Y \cdot T = ae$, we have $z \leq ae$ and if inequality holds, then $|L|$ is induced without deleting any base point from $|\mathcal{O}_{F_c}(h+ef)|$. Hence $k \leq e + 1$ and $v$ is induced (after deleting the base points) from a linear subspace of $H^0(F_c, \mathcal{O}_{F_c}(h+ef))$. We get that if $L$ evincs $s_{c+1}(C)$ and the assumption of this step holds, then $s_{c+1}(C) = ae$ and $L \cong u^*(\mathcal{O}_Y(h+ef))$.

(c) From now on we assume $h^0(F_c, \mathcal{I}_B(h+ef)) = 0$. To conclude the proof of the lemma it is sufficient to find a contradiction for $a \gg 0$ and any $x \leq ae - e - 2$. Set $c := \lfloor z/(e + 1) \rfloor$. Set $S_0 := S$ and $B_0 := B$. Fix $A_1 \in |\mathcal{O}_{F_c}(h+ef)|$ such that $a_1 := \sharp(A_1 \cap B_0)$ is maximal. Set $S_1 := S_0 \setminus S_0 \cap A_1$ and $B_1 := B_0 \setminus B_0 \cap A_1$. For each integer $i \geq 2$ define recursively the curve $A_i \in |\mathcal{O}_{F_c}(h+ef)|$, the integer $a_i$, and the sets $S_i, B_i$ in the following way. Fix $A_i \in |\mathcal{O}_{F_c}(h+ef)|$ such that $a_i := \sharp(A_i \cap B_{i-1})$ is maximal. Set $S_i := S_{i-1} \setminus S_{i-1} \cap A_i$ and $B_i := B_{i-1} \setminus B_{i-1} \cap A_i$. Since $h^0(F_c, \mathcal{O}_{F_c}(h+ef)) = e + 2$ and $h^0(F_c, \mathcal{I}_B(h+ef)) = 0$, step (a) gives $a_i \leq e + 1$ for all $i$. Since $h^0(F_c, \mathcal{O}_{F_c}(h+ef)) = e + 2$ and $a_i$ is maximal, either $a_i = e + 1$ or $B_i = \emptyset$. Hence $a_i = e + 1$ for $i \leq c$, $a_{c+1} = z - c(a + 1) \leq e + 1$ and $a_i = 0$ for all $i \geq c + 2$. Assume $a \geq 4e$. Hence $(e+1)^2(a-3) \geq e(e+2)a$. Since $z \geq ea$, we get $c \leq a - 4$. For each integer $i = 1, \ldots, c + 1$ we have an exact sequence

$$0 \to \mathcal{I}_{S_i \cup B_i}((a - 2 - i)f + (e(a - i) - e - 2)f) \to \mathcal{I}_{S_{i-1} \cup B_{i-1}}((a - 1 - i)h + (e(a - i + 1) - e - 2)f) \to \mathcal{I}_{A_i \cap (S_{i-1} \cup B_{i-1} \setminus A_i)}((a - 1 - i)h + (e(a - i + 1) - e - 2)f) \to 0.$$ 

Fix $i \in \{1, \ldots, c\}$. By step (a) we have $\sharp(D \cap B) \leq 1$ for every $D \in |\mathcal{O}_{F_c}(f)|$. Hence $A_i$ is irreducible. Hence $A_i \cong \mathbb{P}^1$. Since $\sharp(D \cap B) \leq 1$ for every $B \in |\mathcal{O}_{F_c}(f)|$ and $B \cap h = \emptyset$, even if $a_{c+1} \leq a$ we may take an irreducible $A_{c+1} \in |\mathcal{O}_{F_c}(f)|$ containing $B_{c+1}$. Assume for a moment $c+1 \leq a - 5$. Since $e \geq 2$, we have $e(a - c + 1) - e - 2 \geq 2e + 1$. Set $x_i := \sharp(S_{i-1} \cap A_i)$. Since $S$ is general, we have $x_i \leq e + 1$. Hence $x_i + a_i \leq 2e + 2$. Since $A_i \cong \mathbb{P}^1$ and

$$\deg(\mathcal{O}_{A_i}((a - 1 - i)h + (e(a - i + 1) - e - 2)f)) = e(a - i + 1) - e - 2 \geq e(a - c + 1) - e - 2 \geq 2e + 1,$$

we have

$$h^1(A_i, \mathcal{I}_{A_i \cap (S_{i-1} \cup B_{i-1} \setminus A_i)}((a - 1 - i)h + (e(a - i + 1) - e - 2)f)) = 0.$$ 

Hence applying (2) first for $i = 1$, then for $i = 2$, and so on up to $i = c + 1$, we get

$$h^1(F_c, \mathcal{I}_{S_{c+1}}((a - 3 - c)f + (e(a - c - 1) - e - 2)f)) > 0.$$
Since $2e \geq e + 1$, we have

$$h^1(F_e, \mathcal{O}_{F_e}((a - 3 - c)f + (e(a - c - 1) - e - 2)f)) = 0.$$  

Since $S$ is general and $S_e \subseteq S$, to have $h^1(F_e, \mathcal{I}_{S_e+1}((a - 3 - c)f + (e(a - c - 1) - e - 2)f)) = 0$ (and hence a contradiction), it is sufficient to have

$$\sharp(S_e) \leq h^0(F_e, \mathcal{O}_{F_e}((a - 3 - c)f + (e(a - c - 1) - e - 2)f)).$$

Since $\sharp(S_e) \leq x$, it is sufficient to have $x \leq (a - 3 - c)(e(a - 3 - c) + 2e - 2)/2$. Since $x \leq ae - e - 2$, it is sufficient to have $(a - c - 3)e/2 \geq ae$. Thus it is sufficient to have $c \leq a - 3 - \sqrt{2a}$. Since $c \leq ea/(e + 1)$, it is sufficient to have $a - (e + 1)\sqrt{2a} - 3e - 3 \geq 0$. Hence we may take $A_e = 32(e + 1)^2$.

Notice that we also checked the assumption $a - c - 1 \leq a - 5$. \hfill $\Box$

**Lemma 7.** Take $e \geq 2$, $A_e$, $a \geq A_e$, $0 \leq x \leq ea - e - 2$, $S$, $Y$ and $C$ as in Lemma 5.

(a) We have $s_e(C) = ea - 1 - \min\{1, x\}$.

(b) If $x > 0$, then each $L \in \text{Pic}(C)$ evincing $s_e(C)$ is induced by $|\mathcal{I}_P(h + ef)|$ (after deleting the degree 2 base locus $u^{-1}(P)$) for some $P \in S$. For an arbitrary $x$ any spanned and birationally very ample line bundle $M$ of degree $ea - 1$ is induced by $|\mathcal{I}_P(h + ef)|$ (after deleting the degree 1 base locus $u^{-1}(P)$) for some $P \in Y \setminus (S \cup h)$.

**Proof.** The linear systems described in part (b) shows that $s_e(C) \leq ea - 1 - \min\{1, x\}$. By Lemma 7 any such birationally very ample and spanned complete linear system $|L|$ is induced (after deleting the base locus) from a codimension 1 linear subspace $V$ of $H^0(F_e, \mathcal{O}_{F_e}(h + ef))$. Call $B \subset F_e$ the base scheme of $V$ as a linear system on $F_e$ and $B$ the base locus of $u^*(V)$ on $C$. Since $h^0(C, u^*(\mathcal{O}_Y(h + ef))) \geq e + 2$, we have $B \neq \emptyset$. Obviously $B_{\text{red}} = u^{-1}(B \cap Y)$. Hence $B \cap Y \neq \emptyset$. Since $\mathcal{O}_h(h + ef) \cong \mathcal{O}_h$,

$$h^0(F_e, \mathcal{O}_{F_e}(h + ef)) = 2 + h^0(F_e, \mathcal{O}_{F_e}(ef))$$

and $V$ has codimension 1 in $H^0(F_e, \mathcal{O}_{F_e}(h + ef))$, we have $h \cap B = \emptyset$. Since $|\mathcal{O}_{F_e}(h + ef)|$ induces an embedding of $F_e \setminus h$, the scheme $B$ must be a single point, $P$, with its reduced structure. Since $B \cap Y \neq \emptyset$, we have $P \in Y$. We have $\deg(L) = ae - 1$ if $P \notin S$ and $\deg(L) = ae - 2$ if $P \in S$. \hfill $\Box$

3. **Proof of Theorem 1.** We fix the integer $r \geq 3$ for which we want to prove Theorem 1 and set $e := r - 1$. Hence $e \geq 2$. Fix $A_e$ as in Lemma 6 and any integer $g \geq eA_e^2/2 - eA_e + e + 2$. Let $a$ be the minimal integer such that $g \leq ga_e$. Since $ga_e - ga_{e-1, e} = ae - e - 1$, we have $a \geq A_e$ and there is a unique integer $x$ such that $0 \leq x \leq ae - e - 2$ and $g = ga_e - x$. Take $C$ as in Lemmas 6 and 7. Lemma 6 gives $s_{e+1}(C) = ae$. Hence it is sufficient to prove that $s_{e+2}(C) > (e + 2)ea/(e + 1)$. Assume $z := s_{e+2}(C) \leq (e + 2)ea/(e + 1)$ and fix $L \in \text{Pic}^2(C)$ evincing $s_{e+2}(C)$. The line bundle $L$ is spanned, $h^0(C, L) = e + 3$ and $|L|$ induces a morphism
$v : C \to \mathbb{P}^{e+2}$ birationally onto its image and with $v(C)$ a degree $z$ non-degenerate curve with arithmetic genus $\geq g$. Set $m_1 := [(z-1)/(e+2)]$, $c_1 = z - 1 - m_1(e+2)$, $\mu_1 := 0$ if $c_1 \neq e+1$ and $\mu_1 := 1$ if $c_1 = e+1$. Set $
abla_1(z, e + 2) = (e + 2)m_1(m_1 - 1)/2 + m_1(e_1 + 1) + \mu_1$. Notice that

$$
\pi_1(z, e + 2) \leq z(z+2)/(e+2) \leq ea(e+2)(ea(e+2)+2e+2)/(2(e+2)(e+1)^2).
$$

Notice that $e^2(e+2)^2/(2(e+2)(e+1)^2) < e/2$. Since $g > g_{a-1,e} = 1 + (a-1)(ae - 2 - 2e)/2$, we have $g > \pi(z, e + 2)$ if $a \gg 0$, say if $a \geq A_e$. Hence [3], Theorem 3.15, gives that $v(C)$ is contained in a degree $e + 1$ surface $T \subset \mathbb{P}^{e+2}$. By the classification of all minimal degree surfaces ([2]), either $T$ is a cone over a rational normal curve or $T \cong F_m$ embedded by the complete linear system $|O_{F_m}(h + (e + 1 + m)f)|$ for some integer $m \equiv e + 1 \pmod{2}$ with $0 \leq m \leq e - 1$. In the latter case we set $E := v(C)$. In the former case $T$ is the image of $F_{e+1}$ by the complete linear system $|O_{F_{e+1}}(h + (e + 1)f)|$; in this case set $m := e + 1$ and call $E$ the strict transform of $v(C)$ in $F_{e+1}$. In both cases $E$ is a curve contained in $F_m$ with $C$ as its normalization. Call $u' : C \to E$ the normalization map. Hence there are integers $c, y$ such that $E \in |O_{F_m}(ch + yf)|$ with $y \geq mc$ and $c > 0$. Lemma 4 gives $c \geq a$; if $m = 0$ it also gives $y \geq a$.

(a) Here we assume $m \leq e - 1$. Let $T' \subset \mathbb{P}^e$ be the image of $F_m$ by the complete linear system $|O_{F_m}(h + (e + m)f)|$. Since either $T' \cong F_m$ (case $m \neq e - 1$) or $T'$ is the blowing down of $h$ (case $m = e - 1$), the image of $E$ in $T'$ gives $s(E) \leq O_{F_m}(h + (e + m)f) \cdot O_{F_m}(ch + yf) = z - c$. Since $c \geq a$, Lemma 7 gives $z \geq c + ae - 2 \geq a(e + 1) - 2$, contradicting the assumption $z \leq ea(e+2)/(e+1)$ (with $a > 2(e+1)^2$).

(b) Now assume $m = e + 1$. Since $y \geq mc = (e + 1)c$ and $c \geq a$ (Lemma 6), this case is impossible.

The proof of Theorem 1 is complete.

References


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