Weighted sub-Bergman Hilbert spaces

Abstract. We consider Hilbert spaces which are counterparts of the de Branges–Rovnyak spaces in the context of the weighted Bergman spaces $A^2_\alpha$, $-1 < \alpha < \infty$. These spaces have already been studied in [8], [7], [5] and [1]. We extend some results from these papers.

1. Introduction. Let $\mathbb{D}$ denote the unit disk in the complex plane. For $-1 < \alpha < \infty$, the weighted Bergman space $A^2_\alpha$ is the space of holomorphic functions $f$ in $\mathbb{D}$ such that

$$\int_\mathbb{D} |f(z)|^2 dA_\alpha(z) < \infty,$$

where

$$dA_\alpha(z) = (\alpha + 1)(1 - |z|^2)^\alpha \frac{dxdy}{\pi} = (\alpha + 1)(1 - |z|^2)^\alpha dA(z), \quad z = x + iy.$$ 

The space $A^2_\alpha$ is a Hilbert space with the inner product $\langle f, g \rangle_\alpha$ inherited from $L^2(\mathbb{D}, dA_\alpha)$. It then follows that if

$$f(z) = \sum_{n=0}^{\infty} \hat{f}(n) z^n \quad \text{and} \quad g(z) = \sum_{n=0}^{\infty} \hat{g}(n) z^n$$

are functions in $A^2_\alpha$, then

$$\langle f, g \rangle_\alpha = \sum_{n=0}^{\infty} \frac{n! \Gamma(2 + \alpha)}{\Gamma(n + 2 + \alpha)} \hat{f}(n) \overline{\hat{g}(n)}.$$ 

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Clearly, $A_0^2 = A^2$ is the Bergman space on the unit disk.

For $\varphi \in L^\infty(D)$ the Toeplitz operator $T_\varphi^\alpha$ on $A_\alpha^2$ is defined by
\[
T_\varphi^\alpha(f) = P_\alpha(\varphi f), \quad f \in A_\alpha^2,
\]
where $P_\alpha : L^2(D, dA_\alpha) \to A_\alpha^2$ is the projection operator
\[
P_\alpha(f)(z) = \int_D \frac{f(w)}{(1 - \overline{w}z)^{\alpha+2}} dA_\alpha(w).
\]
Suppose that $T$ is a contraction on a Hilbert space $H$. Following [4], we define the space $H(T)$ to be the range of the operator $(I - TT^*)^{1/2}$ with the inner product given by
\[
\langle (I - TT^*)^{1/2}f, (I - TT^*)^{1/2}g \rangle_{H(T)} = \langle f, g \rangle_{(\ker(I - TT^*)^{1/2})^\perp},
\]
for $f, g \in (\ker(I - TT^*)^{1/2})^\perp$.

For $\varphi$ in the closed unit ball of $H^\infty$, the spaces $H(T_\varphi^\alpha)$ and $H(T_\varphi^\alpha)$ are denoted by $H_\alpha(\varphi)$ and $H_\alpha(\varphi)$, respectively. For the case when $\alpha = 0$ these spaces were studied by Kehe Zhu in [7], [8]. He proved that the spaces $H_0(\varphi)$ and $H_0(\varphi)$ coincide as sets and both the spaces contain $H^\infty$. Zhu also proved that if $\varphi$ is a finite Blaschke product $B$, then, as sets, $H_0(B) = H_0(\overline{B}) = H^2$, the Hardy space on the unit disk. These results were extended to positive $\alpha$ in [5], where the author proved that
\[
H_\alpha(B) = H_\alpha(\overline{B}) = A_{\alpha-1}^2.
\]
For $\alpha$ as above, we define the space $D(\alpha)$ to be the set of holomorphic functions in $D$ and such that $f' \in L^2(D, dA_\alpha)$. Here we further extend the above-mentioned result and show that for $-1 < \alpha < \infty$,
\[
H_\alpha(B) = H_\alpha(\overline{B}) = D(\alpha + 1) \quad \text{as sets}.
\]

After sending this paper for publication we found that a different proof of these equalities was given by F. Symsak in [6].

For $a \in D$, set
\[
\varphi_a(z) = \frac{a - z}{1 - \overline{a}z}.
\]
Let $K_\alpha^a(z) = \frac{1}{(1-az)^{\alpha+2}}$ be a reproducing kernel for $A_\alpha^2$ and let
\[
k_\alpha^a(z) = \frac{(1 - |a|^2)^{1+\frac{\alpha}{2}}}{(1 - \overline{a}z)^{\alpha+2}}
\]
be the normalized kernel. Since the linear operator $A : A_\alpha^2 \to A_\alpha^2$ defined by
\[
Af(z) = k_\alpha^a f \circ \varphi_a
\]
is a surjective isometry, the functions
\[
e_{a,n} = \frac{k_\alpha^a \varphi_a^n}{\sqrt{(\alpha + 1)\beta(n + 1, \alpha + 1)}}
\]
form an orthonormal basis for $A_\alpha^2$. 
The following formula for the operator \((I - T^\alpha_{\varphi_a} T^\alpha_{\varphi_a})^{1/2} = (T^\alpha_{1-|\varphi_a|^2})^{1/2}\) has been derived in [5]:

\[
(T^\alpha_{1-|\varphi_a|^2})^{1/2} = \sum_{n=0}^{\infty} \frac{\sqrt{\alpha + 1}}{\sqrt{n + \alpha + 2}} e_{a,n} \otimes e_{a,n},
\]

where \(e_{a,n} \otimes e_{a,n}(f) = \langle f, e_{a,n} \rangle_a e_{a,n}\) for \(f \in A^2_\alpha\).

In this paper we obtain the analogous formula for the operator \((I - T^\alpha_{\varphi_a} T^\alpha_{\varphi_a})^{1/2}\). We also find the formulas for the inner products in \(H_\alpha(\varphi_a)\) and \(H_\alpha(\varphi_a)\) in terms of the Fourier coefficients with respect to the orthonormal basis \(\{e_{a,n}\}\).

We note that since

\[
\varphi^a_n(z) = \sum_{k=0}^{n} \binom{n}{k} (-1)^k a^{n-k} \frac{(1 - |a|^2)k z^k}{(1 - \bar{a}z)^k},
\]

(see [5]), we have

\[
\langle f, \varphi^a_n K^\alpha_a \rangle_\alpha = \sum_{k=0}^{n} \binom{n}{k} (-1)^k \bar{a}^{n-k} (1 - |a|^2)^k \left\langle f, \frac{z^k}{(1 - \bar{a}z)^{k+\alpha+2}} \right\rangle_\alpha
\]

\[
= \bar{a}^n f(a) + \sum_{k=1}^{n} \binom{n}{k} \frac{(-1)^k \bar{a}^{n-k} (1 - |a|^2)^k f(k)(a)}{(\alpha + 2)(\alpha + 3) \ldots (\alpha + k + 1)}.
\]

So, in particular, the constant function \(f_1 \equiv 1\) can be written as follows

\[
1 \equiv f_1 = \sum_{n=0}^{\infty} \frac{\bar{a}^n}{||\varphi^a_n K^\alpha_a||} e_{a,n}(z) = \sum_{n=0}^{\infty} \frac{\bar{a}^n (1 - |a|^2)^{\frac{\alpha}{2} + 1}}{\sqrt{(\alpha + 1)\beta(n + 1, \alpha + 1)}} e_{a,n}
\]

\[
= (1 - |a|^2)^{\alpha+2} \sum_{n=0}^{\infty} \frac{\Gamma(n + 2 + \alpha)}{n!\Gamma(\alpha + 2)} \bar{a}^n \left(\frac{z - a}{1 - \bar{a}z}\right)^n.
\]

2. The spaces \(H_\alpha(\varphi_a)\) and \(H_\alpha(\varphi_a)\). The following theorem describes the operator \((I - T^\alpha_{\varphi_a} T^\alpha_{\varphi_a})^{1/2}\).

**Theorem 2.1.** For \(a \in \mathbb{D}\),

\[
(I - T^\alpha_{\varphi_a} T^\alpha_{\varphi_a})^{1/2} = \sum_{n=0}^{\infty} \sqrt{\frac{\alpha + 1}{n + \alpha + 1}} e_{a,n} \otimes e_{a,n}.
\]

**Proof.** Our aim is to prove that the functions \(\varphi^a_n K^\alpha_a\), \(n = 0, 1 \ldots\), are eigenvectors of the operator \((I - T^\alpha_{\varphi_a} T^\alpha_{\varphi_a})^{1/2}\) with corresponding eigenvalues
\[
\sqrt{\frac{\alpha+1}{n+\alpha+1}}. \text{ We have }
\]
\[
T_{\alpha \varphi a}^{\alpha} (\varphi_a^n K_\alpha^\alpha)(z) = \int_\mathbb{D} \frac{\varphi_a(w) \varphi_a^n(w)}{(1 - \bar{w}w)^{\alpha+2}(1 - z\bar{w})^{\alpha+2}} dA_\alpha(w)
\]
\[
= \int_\mathbb{D} \frac{\bar{u}u^n}{(1 - \bar{u}z + z\bar{u})^{2+\alpha}} dA_\alpha(u)
\]
\[
= K_\alpha^\alpha(z) \int_\mathbb{D} \frac{\bar{u}u^n}{(1 - \bar{u}\varphi_a(z))^{2+\alpha}} dA_\alpha(u)
\]
\[
= K_\alpha^\alpha(z) \int_\mathbb{D} \sum_{k=0}^{\infty} \frac{\Gamma(k + 2 + \alpha)}{k!\Gamma(2 + \alpha)} (\bar{u}\varphi_a(z))^k \bar{u}u^n dA_\alpha(u)
\]
\[
= \frac{\Gamma(n + 1 + \alpha)}{(n-1)!\Gamma(2 + \alpha)} K_\alpha^\alpha(z) \varphi_a^{n-1}(z) \int_\mathbb{D} |u|^{2n} dA_\alpha(u)
\]
\[
= \frac{n}{n + 1 + \alpha} K_\alpha^\alpha(z) \varphi_a^{n-1}(z).
\]

Hence
\[
(I - T_{\alpha \varphi a}^{\alpha} T_{\alpha \varphi a}^{\alpha})(\varphi_a^n K_\alpha^\alpha)(z) = \frac{\alpha + 1}{n + \alpha + 1} \varphi_a^n K_\alpha^\alpha,
\]
and consequently,
\[
(I - T_{\alpha \varphi a}^{\alpha} T_{\alpha \varphi a}^{\alpha})^{\frac{1}{2}} (\varphi_a^n K_\alpha^\alpha)(z) = \sqrt{\frac{\alpha + 1}{n + \alpha + 1}} \varphi_a^n K_\alpha^\alpha.
\]

Expanding \( f \in A_\alpha^2 \) in the Fourier series with respect to the basis \( \{ e_{a,n} \} \)
\[
f = \sum_{n=0}^{\infty} \langle f, e_{a,n} \rangle e_{a,n},
\]
we find that
\[
(I - T_{\alpha \varphi a}^{\alpha} T_{\alpha \varphi a}^{\alpha})^{\frac{1}{2}} f = \sum_{n=0}^{\infty} \langle f, e_{a,n} \rangle (I - T_{\alpha \varphi a}^{\alpha} T_{\alpha \varphi a}^{\alpha})^{\frac{1}{2}} e_{a,n}
\]
\[
= \sum_{n=0}^{\infty} \langle f, e_{a,n} \rangle \sqrt{\frac{\alpha + 1}{n + \alpha + 1}} e_{a,n}
\]
\[
= \sum_{n=0}^{\infty} \sqrt{\frac{\alpha + 1}{n + \alpha + 1}} (e_{a,n} \otimes e_{a,n}) f.
\]

By Proposition 1.3.10 in [9] we also get

**Corollary 2.1.** \((I - T_{\alpha \varphi a}^{\alpha} T_{\alpha \varphi a}^{\alpha})^{\frac{1}{2}}\) is a compact operator on \(A_\alpha^2\).
In our next result we give formulas for inner products \( \langle f, g \rangle_{\mathcal{H}_\alpha(\varphi_a)} \) and \( \langle f, g \rangle_{\mathcal{H}_\alpha(\overline{\varphi_a})} \) in terms of the Fourier coefficients \( \hat{f}_a(n) = \langle f, e_{a,n} \rangle_\alpha \) and \( \hat{g}_a(n) = \langle f, e_{a,n} \rangle_\alpha \).

**Proposition 2.1.** For \( a \in \mathbb{D} \),
\[
\langle f, g \rangle_{\mathcal{H}_\alpha(\varphi_a)} = \langle f, g \rangle_\alpha + \sum_{n=1}^{\infty} \frac{n}{\alpha + 1} \hat{f}_a(n) \overline{\hat{g}_a(n)}
\]
and
\[
\langle f, g \rangle_{\mathcal{H}_\alpha(\overline{\varphi_a})} = \langle f, g \rangle_\alpha + \sum_{n=0}^{\infty} \frac{n + 1}{\alpha + 1} \hat{f}_a(n) \overline{\hat{g}_a(n)}.
\]

**Proof.** We shall prove the first formula. The other can be proved analogously. By Sarason ([4], p. 3) we know that \( f, g \in \mathcal{H}_\alpha(\varphi_a) \) if and only if \( T^{\alpha}_{\varphi_a}f \in \mathcal{H}_\alpha(\varphi_a) \) and
\[
\langle f, g \rangle_{\mathcal{H}_\alpha(\varphi_a)} = \langle f, g \rangle_\alpha + \langle T^{\alpha}_{\varphi_a}f, T^{\alpha}_{\varphi_a}g \rangle_{\mathcal{H}_\alpha(\varphi_a)}.
\]
It follows from the proof of Theorem 2.1 that
\[
T^{\alpha}_{\varphi_a}(\varphi_a^n K_a^n)(z) = \frac{n}{n + 1 + \alpha} K_a^\alpha(z) \varphi_a^{n-1}(z)
\]
and consequently,
\[
T^{\alpha}_{\varphi_a}(e_{a,n}) = \sqrt{\frac{n}{n + 1 + \alpha}} e_{a,n-1}.
\]

Hence
\[
\langle T^{\alpha}_{\varphi_a}f, T^{\alpha}_{\varphi_a}g \rangle_{\mathcal{H}_\alpha(\varphi_a)} = \sum_{n=1}^{\infty} \frac{n}{n + 1 + \alpha} \hat{f}_a(n) \overline{\hat{g}_a(n)} \| e_{a,n-1} \|^2_{\mathcal{H}_\alpha(\varphi_a)}.
\]
Since
\[
(I - T^{\alpha}_{\overline{\varphi_a}})^{\frac{1}{2}}(e_{a,n}) = \sqrt{\frac{\alpha + 1}{n + \alpha + 2}} e_{a,n},
\]
we have
\[
\| e_{a,n-1} \|^2_{\mathcal{H}_\alpha(\varphi_a)} = \frac{n + 1 + \alpha}{\alpha + 1}.
\]
\[\square\]

**3. Finite Blaschke products.** Throughout this section \( B \) will stand for a finite Blaschke product. The spaces \( \mathcal{H}_\alpha(B) \) and \( \mathcal{H}_\alpha(\overline{B}) \) have been described for \( \alpha \geq 0 \) in [8] and [1]. We will use the methods developed in these papers to extend the result for \(-1 < \alpha < 0\).

For \(-1 < \alpha < \infty \) let \( \mathcal{D}(\alpha) \) denote the Hilbert space consisting of analytic functions in \( \mathbb{D} \) whose derivatives are in \( L^2(\mathbb{D}, dA_\alpha) \) with the inner product
\[
\langle f, g \rangle_{\mathcal{D}(\alpha)} = \hat{f}(0) \overline{\hat{g}(0)} + \int_{\mathbb{D}} f'(z) \overline{g'(z)} dA_\alpha(z).
\]
We shall show the following

**Theorem 3.1.** For $-1 < \alpha < \infty$,

$$\mathcal{H}_\alpha(\overline{B}) = \mathcal{D}(\alpha + 1)$$

as sets.

**Proof.** As in [7] and [1] we define the Hilbert space $A^2_{\alpha,B}$ consisting of functions $f$ analytic in $\mathbb{D}$ and such that

$$\int_{\mathbb{D}} |f(z)|^2(1 - |B(z)|^2) \, dA_\alpha(z) < \infty$$

with the inner product

$$\langle f, g \rangle_{A^2_{\alpha,B}} = \int_{\mathbb{D}} f(z) \overline{g(z)}(1 - |B(z)|^2) \, dA_\alpha(z).$$

Since, for $z \in \mathbb{D},$

$$1 - |B(z)|^2 \sim 1 - |z|^2$$

(see, e.g., Lemma 1 of [8]),

the function $g \in A^2_{\alpha,B}$ if and only $g \in A^2_{\alpha+1}$ and the norms in these spaces are equivalent.

It was proved in [8] and [1] that the space $\mathcal{H}_\alpha(\overline{B})$ consists of analytic functions of the form

$$f(z) = S_\alpha(g)(z) = \int_{\mathbb{D}} \frac{1 - |B(w)|^2}{(1 - z\overline{w})^{\alpha+2}} g(w) \, dA_\alpha(w),$$

where $g \in A^2_{\alpha,B}$. It then follows that if $f \in \mathcal{H}_\alpha(\overline{B})$, then

$$f'(z) = (\alpha + 2) \int_{\mathbb{D}} \frac{\overline{w}(1 - |B(w)|^2)}{(1 - z\overline{w})^{\alpha+3}} g(w) \, dA_\alpha(w).$$

By Theorem 1.9 of [3] the operator

$$\Lambda g(z) = \int_{\mathbb{D}} \frac{(1 - |w|^2)^{\alpha+1}}{|1 - z\overline{w}|^{\alpha+3}} |g(w)| \, dA(w)$$

is bounded on $L^2(\mathbb{D}, dA^2_{\alpha+1})$. Therefore, there is a constant $C > 0$ such that

$$\int_{\mathbb{D}} |f'(z)|^2 \, dA_{\alpha+1}(z) \leq \|\Lambda g\|_{L^2(\mathbb{D}, dA^2_{\alpha+1})} \leq C\|g\|_{A^2_{\alpha+1}},$$

which proves the inclusion $\mathcal{H}_\alpha(\overline{B}) \subset \mathcal{D}(\alpha + 1)$. To prove that $\mathcal{D}(\alpha + 1) \subset \mathcal{H}_\alpha(\overline{B})$ we consider the operator $R_\alpha : \mathcal{D}(\alpha + 1) \rightarrow A^2_{\alpha,B}$ given by

$$R_\alpha f(z) = (\alpha + 2)zf'(z) + f(0).$$
Using the Fubini Theorem, one can easily check that $R_{\alpha} = S_{\alpha}^*$, where $S_{\alpha} : A_{\alpha,B}^2 \to D(\alpha + 1)$ is given by (3.1). Indeed, for $f \in D(\alpha + 1)$,

$$
\langle f, S_{\alpha} g \rangle_{D(\alpha+1)} = \hat{f}(0) \overline{S_{\alpha} g(0)}
+ (\alpha + 2) \int_{D} f'(z) \int_{D} \frac{(1 - |B(w)|^2) w g(w) \overline{dA_\alpha(w)} \overline{dA_{\alpha+1}(z)}}{(1 - \overline{z} w)^{\alpha+3}}
= \hat{f}(0) \langle 1, g \rangle_{A_{\alpha,B}^2}
+ \int_{D} (1 - |B(w)|^2 w g(w)) (\alpha + 2) f'(w) dA_\alpha(w)
= \langle R_{\alpha} f, g \rangle_{A_{\alpha,B}^2}.
$$

Since $R_{\alpha}$ is invertible, the image of the unit ball of $D(\alpha + 1)$ under $R_{\alpha}$ contains a ball of radius $r > 0$ centered at zero. As in [8], [1], for every unit vector $g \in A_{\alpha,B}^2$ we have

$$
\| S_{\alpha} g \|_{D(\alpha+1)} = \sup \left\{ |\langle S_{\alpha} g, f \rangle_{D(\alpha+1)}| : \| f \|_{D(\alpha+1)} \leq 1 \right\}
= \sup \left\{ \| (g, R_{\alpha} f)_{A_{\alpha,B}^2} \| : \| f \|_{D(\alpha+1)} \leq 1 \right\}
= \sup \left\{ \left| \int_{D} g(w) \overline{R_{\alpha} f(w)} (1 - |B(w)|^2) dA_\alpha(w) \right| : \| f \|_{D(\alpha+1)} \leq 1 \right\}
\geq \sup \left\{ \left| \int_{D} g(w) \overline{h(w)} (1 - |B(w)|^2) dA_\alpha(w) \right| : \| h \|_{A_{\alpha,B}^2} \leq r \right\}
= r \| g \|_{A_{\alpha,B}^2} = r.
$$

This means that $S_{\alpha}$ is bounded from below, so that its range is closed in $D(\alpha + 1)$. Since polynomials are dense in the space $D(\alpha + 1)$, it is enough to prove that $S_{\alpha}(A_{\alpha,B}^2)$ contains all polynomials. To show that $z^n$ is in $S_{\alpha}(A_{\alpha,B}^2)$ consider the closed subspace $M$ of $A_{\alpha,B}^2$ spanned by functions $z^m$, $m \neq n$, $m \in \mathbb{N}$. Let $g$ be a unit vector in $A_{\alpha,B}^2 \ominus M$. Then

$$
S_{\alpha}(g)(z) = \int_{D} \frac{1 - |B(u)|^2}{(1 - z \overline{u})^{\alpha+2}} g(u) dA_\alpha(u) = \frac{\Gamma(n + 2 + \alpha)}{n! \Gamma(2 + \alpha)} z^n \langle g, u^n \rangle_{A_{\alpha,B}^2}
$$

for every $z \in D$. If $\langle g, u^n \rangle_{A_{\alpha,B}^2} = 0$ for every unit vector $g$ in $A_{\alpha,B}^2 \ominus M$, then it will follow that $z^n \in M$, which is clearly impossible. So, there is $c_n \neq 0$ such that $c_n z^n \in S_{\alpha}(A_{\alpha,B}^2)$. □
We remark that also in the case when \(-1 < \alpha < 0\), \(\mathcal{H}_\alpha(B) = H_\alpha(B)\). It follows from Douglas criterion that \(H_\alpha(B) \subset H_\alpha(B)\) (see [4]). Moreover, it was showed in [5] that for \(-1 < \alpha < 0\), \(\mathcal{H}_\alpha(B)\) is equal to a Hilbert space with the reproducing kernel \(K_\alpha^w(z) = (1 - \bar{w}z)^{-(1 + \alpha)}\). It is easy to see that the norm in such a space is given by

\[
\|f\|_\alpha^2 = \frac{1}{(\alpha + 1)(\alpha + 2)} \|f'\|_{A^{\alpha+1}}^2 + \|f\|_{A^\alpha}^2.
\]

Indeed, for \(z, w \in \mathbb{D}\) we have

\[
K_\alpha^w(z) = k^\alpha(\bar{w}z)
\]

where

\[
k^\alpha(z) = \sum_{k=0}^{\infty} \frac{\Gamma(k + 1 + \alpha)}{k!\Gamma(1 + \alpha)}(\bar{w}z)^k.
\]

This means that this space is the weighted Hardy space introduced in [2] with the generating function \(k^\alpha\). Hence

\[
\|z^k\|^2 = \frac{k!\Gamma(\alpha + 2)}{\Gamma(k + \alpha + 2)}
\]

and formula (3.2) follows. Thus, also for \(-1 < \alpha < 0\), \(\mathcal{H}_\alpha(B) = D(\alpha + 1) = H_\alpha(B)\). Finally, we note that in this case \(H^\infty\) is not contained in \(\mathcal{H}_\alpha(B) = H_\alpha(B)\). This follows, for example, from the result proved in [10] that \(H^\infty\) is contained in the weighted Hardy space \(H^2(\beta)\) if and only if \(\beta\) is bounded.

**REFERENCES**

Maria Nowak
Instytut Matematyki UMCS
pl. M. Curie-Skłodowskiej 1
20-031 Lublin
Poland
e-mail: mt.nowak@poczta.umcs.lublin.pl

Renata Rososzczuk
Politechnika Lubelska
Katedra Matematyki Stosowanej
ul. Nadbystrzycka 38
20-618 Lublin
Poland
e-mail: renata.rososzczuk@gmail.com

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