E. A. OYEKAN and T. O. OPOOLA

On a subordination result for analytic functions defined by convolution

Abstract. In this paper we discuss some subordination results for a subclass of functions analytic in the unit disk $U$.

1. Introduction. Let $A$ be the class of functions $f(z)$ analytic in the unit disk $U = \{z : |z| < 1\}$ and normalized by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n.$$  

We denote by $K(\alpha)$ the class of convex functions of order $\alpha$, i.e.,

$$K(\alpha) = \left\{ f \in A : \Re \left( 1 + \frac{zf''(z)}{f'(z)} \right) > \alpha, \ z \in U \right\}.$$ 

Definition 1 (Hadamard product or convolution). Given two functions $f(z)$ and $g(z)$, where $f(z)$ is defined in (1.1) and $g(z)$ is given by

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n,$$

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the Hadamard product (or convolution) $f * g$ of $f(z)$ and $g(z)$ is defined by

\[(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n = (g * f)(z).\]  

**Definition 2** (Subordination). Let $f(z)$ and $g(z)$ be analytic in the unit disk $U$. Then $f(z)$ is said to be subordinate to $g(z)$ in $U$ and we write

\[f(z) \prec g(z), \quad z \in U,\]

if there exists a Schwarz function $w(z)$, analytic in $U$ with $w(0) = 0$, $|w(z)| < 1$ such that

\[(f(z) = g(w(z)), \quad z \in U.\]

In particular, if the function $g(z)$ is univalent in $U$, then $f(z)$ is subordinate to $g(z)$ if

\[f(0) = g(0), \quad f(U) \subseteq g(U).\]

**Definition 3** (Subordinating factor sequence). A sequence $\{b_n\}_{n=1}^{\infty}$ of complex numbers is said to be a subordinating factor sequence if whenever $f(z)$ of the form (1.1) is analytic, univalent and convex in $U$, the subordination is given by

\[\sum_{n=1}^{\infty} a_n b_n z^n \prec f(z), \quad z \in U, a_1 = 1.\]

We have the following theorem.

**Theorem 1.1** (Wilf [5]). The sequence $\{b_k\}_{k=1}^{\infty}$ is a subordinating factor sequence if and only if

\[\text{Re} \left\{ 1 + 2 \sum_{k=1}^{\infty} b_k z^k \right\} > 0, \quad z \in U.\]

Let

\[M(\alpha) = \left\{ f \in A : \text{Re} \left( \frac{zf'(z)}{f(z)} \right) < \alpha, \quad z \in U \right\}\]

and let

\[M^{\delta}(b, \delta) = \left\{ f \in A : \text{Re} \left\{ \frac{2}{5} + \frac{2D^{\delta+2}f(z)}{bD^{\delta+1}f(z)} \right\} < \alpha, \quad \alpha > 0, \quad z \in U \right\}.\]
Here $D^\delta f(z)$ is the Ruschewey’s derivative defined as
\[
D^\delta f(z) = \frac{z}{(1 - z)^{\delta+1}} * f(z)
\]
\[
= \left( z + \sum_{n=2}^{\infty} \frac{\Gamma(n + \delta)}{(n - 1)! \Gamma(1 + \delta)} \right) \ast \left( z + \sum_{n=2}^{\infty} a_n z^n \right)
\]
\[
= z + \sum_{n=2}^{\infty} \frac{\Gamma(n + \delta)}{(n - 1)! \Gamma(1 + \delta)} a_n z^n, \quad \delta \geq -1.
\]

**Theorem 1.2 ([3]).** If $f(z) \in A$ satisfies
\[
\sum_{n=2}^{\infty} \left\{ |b(1 - k)(\delta + 2) + 2(n - 1)| + |b(1 - 2\alpha + k)(\delta + 2)
\right. \\
\left. + 2(n - 1)| \right\} \frac{\Gamma(n + \delta + 1)}{(n - 1)! \Gamma(3 + \delta)} |a_n| \leq 2|b(1 - \alpha)|
\]
where $b$ is a non-zero complex number, $\delta \geq -1$, $0 \leq k \leq 1$ and $\alpha > 1$, then $f(z) \in M^\delta(b, \alpha)$.

It is natural to consider the class $M^{\delta^*}(b, \alpha) \subset M^\delta(b, \alpha)$ such that
\[
M^{\delta^*}(b, \alpha) = \left\{ f \in A : \sum_{n=2}^{\infty} \left\{ |b(1 - k)(\delta + 2) + 2(n - 1)|
\right. \\
\left. + |b(1 - 2\alpha + k)(\delta + 2) + 2(n - 1)| \right\} \frac{\Gamma(n + \delta + 1)}{(n - 1)! \Gamma(3 + \delta)} |a_n|
\right. \\
\left. \leq |b(1 - \alpha)| \right\}.
\]

Our main result in this paper is the following theorem.

**Theorem 1.3.** Let $f \in M^{\delta^*}(b, \alpha)$, then
\[
\frac{B}{C} (f \ast g)(z) \prec g(z)
\]
where
\[
B = |b(1 - k)(\delta + 2) + 2| + |b(1 - 2\alpha + k)(\delta + 2) + 2|
\]
\[
C = 2|2|b(1 - \alpha)| + |b(1 - k)(\delta + 2) + 2| + |b(1 - 2\alpha + k)(\delta + 2) + 2|,\]
\[
\delta \geq -1, \quad 0 \leq k \leq 1, \quad b \text{ is a non-zero complex number and } g(z) \in K(\alpha),
\]
z $\in U$. Moreover,
\[
\text{Re}(f(z)) > -\frac{C}{2B}.
\]

The constant factor
\[
\frac{B}{C} = \frac{|b(1 - k)(\delta + 2) + 2| + |b(1 - 2\alpha + k)(\delta + 2) + 2|}{2|2|b(1 - \alpha)| + |b(1 - k)(\delta + 2) + 2| + |b(1 - 2\alpha + k)(\delta + 2) + 2|}
\]
cannot be replaced by a larger one.

2. Proof of the main result. Let \( f(z) \in M^{\delta^*}(b, \alpha) \) and suppose that
\[
g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in K(\alpha).
\]
Then by definition,
\[
\frac{B}{C}(f \ast g)(z) = \frac{B}{C}\left(z + \sum_{n=2}^{\infty} a_n b_n z^n\right).
\]
Hence, by Definition 3, to show the subordination (1.10) it is enough to prove that
\[
\left\{\frac{B}{C}a_n\right\}_{n=1}^{\infty}
\]
is a subordinating factor sequence with \( a_1 = 1 \). Therefore, by Theorem 1.1 it is sufficient to show that
\[
\text{Re}\left\{1 + 2 \sum_{n=1}^{\infty} \frac{B}{C} a_n z^n\right\} > 0, \quad z \in U.
\]
Now,
\[
\text{Re}\left\{1 + 2 \sum_{n=1}^{\infty} \frac{B}{C} a_n z^n\right\} = \text{Re}\left\{1 + 2 \frac{B}{C} a_1 z + 2 \frac{B}{C} \sum_{n=2}^{\infty} B a_n z^n\right\}
\]
\[
\geq 1 - 2 \frac{B}{C} r - 2 \frac{B}{C} \sum_{n=2}^{\infty} |B| a_n |r|^n.
\]
Since \( \frac{\Gamma(n + \delta + 1)}{(n - 1)! \Gamma(3 + \delta)} \) is a monotone non-decreasing function of \( n = 2, 3, \ldots \), we have
\[
\text{Re}\left\{1 + 2 \sum_{n=1}^{\infty} \frac{B}{C} a_n z^n\right\} > 1 - 2 \frac{B}{C} r
\]
\[
- \frac{2}{C} \sum_{n=2}^{\infty} \left[|b(1-k)(\delta+2)+2(n-1)| + |b(1-2\alpha+k)(\delta+2)+2(n-1)|\right]
\]
\[
\times \frac{\Gamma(n + \delta + 1)}{(n - 1)! \Gamma(3 + \delta)} |a_n| r, \quad 0 < r < 1.
\]
By (1.8)
\[
\sum_{n=2}^{\infty} \left[|b(1-k)(\delta+2)+2(n-1)| + |b(1-2\alpha+k)(\delta+2)+2(n-1)|\right]
\]
\[
\times \frac{\Gamma(n + \delta + 1)}{(n - 1)! \Gamma(3 + \delta)} |a_n| \leq 2 |b(1-\alpha)|.
Hence,
\[
\text{Re}\left\{1 + 2 \sum_{n=1}^{\infty} \frac{B}{C} a_n z^n\right\} = \text{Re}\left\{1 + 2 \frac{B}{C} a_1 z + \frac{2}{C} \sum_{n=2}^{\infty} B a_n z^n\right\}
\]
\[
> 1 - 2 \frac{B}{C} r - \frac{4|b(1 - \alpha)|}{C} r
\]
\[
= 1 - \frac{2B + 4|b(\alpha - 1)|}{C} r
\]
\[
= 1 - r > 0
\]

\(|z| = r < 1\). Therefore, we obtain
\[
\text{Re}\left\{1 + 2 \sum_{n=1}^{\infty} \frac{B}{C} a_n z^n\right\} > 0
\]
which is (2.3) that was to be established.

We now show that
\[
\text{Re}(f(z)) > -\frac{C}{2B}.
\]

Taking
\[
g(z) = \frac{z}{1 - z} \in K(\alpha),
\]
(1.10) becomes
\[
\frac{B}{C} f(z) \prec \frac{z}{1 - z}.
\]
Therefore,
\[
(2.5) \quad \text{Re}\left(\frac{B}{C} f(z)\right) > \text{Re}\left(\frac{z}{1 - z}\right).
\]
Since
\[
(2.6) \quad \text{Re}\left(\frac{z}{1 - z}\right) > -\frac{1}{2}, \quad |z| < r,
\]
this implies that
\[
(2.7) \quad \frac{B}{C} \text{Re}(f(z)) > -\frac{1}{2}.
\]
Hence, we have
\[
\text{Re}(f(z)) > -\frac{C}{2B}
\]
which is (1.11).

To show the sharpness of the constant factor
\[
\frac{B}{C} = \frac{|b(1 - k)(\delta + 2) + 2| + |b(1 - 2\alpha + k)(\delta + 2) + 2|}{2[2|b(1 - \alpha)| + |b(1 - k)(\delta + 2) + 2| + |b(1 - 2\alpha + k)(\delta + 2) + 2]|},
\]
we consider the function:

\[(2.8) \quad f_1(z) = z - \frac{2|b(1-\alpha)|}{B} z^2 = \frac{Bz - 2|b(1-\alpha)|z^2}{B}\]

\((z \in U; \delta \geq -1; 0 \leq k \leq 1; b \in \mathbb{C} \setminus \{0\})\). Applying (1.10) with \(g(z) = \frac{z}{1-z}\) and \(f(z) = f_1(z)\) we have

\[(2.9) \quad \frac{Bz - 2b(\alpha - 1)z^2}{C} \prec \frac{z}{1-z}.
\]

Using the fact that

\[(2.10) \quad |\text{Re } z| \leq |z|,
\]

we now show that

\[(2.11) \quad \min \left\{ \text{Re} \frac{Bz - 2b(\alpha - 1)z^2}{C} : z \in U \right\} = -\frac{1}{2}.
\]

Now,

\[(2.12) \quad \left| \text{Re} \frac{Bz - 2|b(1-\alpha)|z^2}{C} \right| \leq \left| \frac{Bz - 2|b(1-\alpha)|z^2}{C} \right|
\]

\[= \frac{|Bz - 2|b(1-\alpha)|z^2|}{|C|}
\]

\[\leq \frac{B|z| + 2|b(1-\alpha)||z^2|}{C} = \frac{B + 2|b(1-\alpha)|}{C} = \frac{1}{2}
\]

\((|z| = 1)\). This implies that

\[(2.13) \quad \left| \text{Re} \frac{Bz - 2|b(1-\alpha)|z^2}{C} \right| \leq \frac{1}{2},
\]

i.e.,

\[-\frac{1}{2} \leq \text{Re} \frac{Bz - 2|b(1-\alpha)|z^2}{C} \leq \frac{1}{2}.
\]

Hence,

\[\min \left\{ \text{Re} \left( \frac{B}{C} f_1(z) \right) : z \in U \right\} = -\frac{1}{2}.
\]

which completes the proof of Theorem 1.3.
3. Some applications. Taking $\delta = 1$ and $b = 1$ in Theorem 1.3, we obtain the following:

**Corollary 1.** If the function $f(z)$ defined by (1.1) is in $M^{\delta^*}(b, \alpha)$, then

$$(3.1) \quad \frac{|5 - 3\alpha|}{2|6 - 4\alpha|} (f * g)(z) < g(z)$$

$(z \in U; \alpha > 1, g \in K(\alpha))$. In particular,

$$(3.2) \quad \text{Re}(f(z)) > -\frac{|6 - 4\alpha|}{|5 - 3\alpha|}.$$  

The constant factor

$$\frac{|5 - 3\alpha|}{2|6 - 4\alpha|}$$

cannot be replaced by any larger one.

**Remark 1.** By taking $\alpha = \frac{71}{45} > 1$ in Corollary 1, we obtain the result of Aouf et al. [1].

Taking $b = 1$, $\delta = 0$ in Theorem 1.3, we obtain the following:

**Corollary 2.** If the function $f(z)$ defined by (1.1) is in $M^{\delta^*}(b, \alpha)$, then

$$(3.3) \quad \frac{|2 - \alpha|}{|5 - 3\alpha|} (f * g)(z) < g(z)$$

$(z \in U; \alpha > 1, g \in K(\alpha))$. In particular,

$$(3.4) \quad \text{Re}(f(z)) > -\frac{|5 - 3\alpha|}{2|2 - \alpha|}, \quad z \in U.$$  

The constant factor

$$\frac{|2 - \alpha|}{|5 - 3\alpha|}$$

cannot be replaced by any larger one.

**Remark 2.** By taking $\alpha = \frac{11}{6}$ and $\alpha = \frac{20}{11}$ in Corollary 2, we obtain the results of Selvaraj and Karthikeyan [4].

Taking $b = 1$, $\delta = -1$ and $k = 0$ in Theorem 1.3, we obtain the following:

**Corollary 3.** If the function $f(z)$ defined by (1.1) is in $M^{\delta^*}(b, \alpha)$, then

$$(3.5) \quad \frac{|3 - \alpha|}{|8 - 4\alpha|} (f * g)(z) < g(z)$$

$(z \in U; \alpha > 1, g \in K(\alpha))$. In particular,

$$(3.6) \quad \text{Re}(f(z)) > -\frac{|4 - 2\alpha|}{|3 - \alpha|}, \quad z \in U.$$
The constant factor
\[ \frac{|3 - \alpha|}{|8 - 4\alpha|} \]
cannot be replaced by any larger one.

Remark 3. If we take \( \alpha = \frac{7 + 3m}{3 + m} \) in Corollary 3, \((m > 0)\) and in particular \( m = 1 \) (i.e., \( \alpha = \frac{5}{2} > 1 \)), we obtain the result of Attiya et al. [2].

References


E. A. Oyekan
Department of Mathematics and Statistics
Bowen University
Iwo, Osun State
Nigeria
e-mail: shalomfa@yahoo.com

T. O. Opoola
Department of Mathematics
University of Ilorin
Ilorin
Nigeria
e-mail: opoolato@unilorin.edu.ng

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