

Rotation indices related to Poncelet’s closure theorem

Abstract. Let $C_R, C_r$ denote an annulus formed by two non-concentric circles $C_R, C_r$ in the Euclidean plane. We prove that if Poncelet’s closure theorem holds for $k$-gons circumscribed to $C_R, C_r$, then there exist circles inside this annulus which satisfy Poncelet’s closure theorem together with $C_r$, with $n$-gons for any $n > k$.

1. Introduction. Poncelet’s closure theorem, going back to the 19th century, has various interesting forms and applications; cf. [2], [7], [4], [9], and the excellent survey [3] as well as [4]. The rich history of this theorem is presented in [1, Ch. 16], [8, § 2.4], and [7], and our paper refers to circular versions of it. Let $C_R, C_r$ be two circles with radii $R > r > 0$ and $C_r$ lying inside $C_R$. From any point on $C_R$, draw a tangent to $C_r$ and extend it to $C_R$ again, using the obtained new intersection point with $C_R$ for starting with a new tangent to $C_r$, etc.; the system of tangential segments obtained in this way inside $C_R$ is called a Poncelet transverse (or bar billiard). We say that the annulus $C_R, C_r$ has Poncelet’s porism property if there is a starting point on $C_R$ for which a Poncelet traverse is a closed polygon. Poncelet’s closure theorem (for circles) says that then the transverse will also close for any other starting point from $C_R$. It is known that such closing polygons (with or without self-intersections) correspond to rational rotations; e.g.,
the rotation number or index $\frac{1}{3}$ is related to a triangle “between” $C_R$ and $C_r$, and the index $\frac{2}{5}$ to a (self-intersecting) pentagram.

In [6] it was proved that “close” to a pair of circles, which have Poncelet’s porism property for index $\frac{1}{3}$, there exist unique pairs of circles having this property with respect to indices $\frac{1}{4}$ and $\frac{1}{6}$, and it was conjectured there that this holds true for arbitrary indices.

In the present paper we show that this conjecture is true in the following sense: for a pair of circles having Poncelet’s porism property for index $\frac{1}{3}$, with $k \geq 3$ as natural number, we prove that there exists a circle lying between the starting circles such that this circle together with the smaller given circle has Poncelet’s porism property for any given index $\frac{1}{n}$, where $n$ is an arbitrary natural number with $n > k$.

2. Basic notions and tools. Let us consider a circular annulus $C_rC_{a,R}$ formed by two circles $C_r$ and $C_{a,R}$. The circles $C_r$ and $C_{a,R}$ are given by the equations $x^2 + y^2 = r^2$ and $(x - a)^2 + y^2 = R^2$, respectively, with

\begin{equation}
0 < a < R - r.
\end{equation}

Recall the following form of Poncelet’s closure theorem which is suitable for our purpose; see [1].

If there exists a one circuminscribed (i.e., simultaneously inscribed in the outer circle and circumscribed about the inner circle) $n$-gon in a circular annulus, then any point of the outer circle is the vertex of some circuminscribed $n$-gon.

If Poncelet’s closure theorem holds for $n = 3$, then Euler’s condition

\begin{equation}
R^2 - 2Rr - a^2 = 0
\end{equation}

is satisfied. We will denote this condition by $\text{Pct}(C_rC_{a,R}, 3)$. There is no elementary formula for the analogously defined condition $\text{Pct}(C_rC_{a,R}, n)$, but we note that $\text{Pct}(C_rC_{a,R}, 4)$ and $\text{Pct}(C_rC_{a,R}, 6)$ have the forms

\begin{equation}
(R^2 - a^2)^2 = 2r^2 (R^2 + a^2)
\end{equation}

and

\begin{equation}
3 (R^2 - a^2)^4 = 4r^2 (R^2 + a^2) (R^2 - a^2)^2 + 16r^2 a^2 R^2,
\end{equation}

respectively; see [3].

It is amazing that for particular natural numbers we have elementary conditions involving also radicals, while for an arbitrary natural number $n \geq 3$ only the Jacobi formula (cf. formula (7) in [10]), using elliptic functions, is involved.

For further use we introduce a convenient parametrization of the annulus $C_rC_{a,R}$. Namely, we take the parametrization $z(t) = re^{it}$ for $C_r$, and for $C_{a,R}$ we use

\begin{equation}
w(t) = z(t) + \lambda(t) ie^{it}, \quad t \in [0, 2\pi],
\end{equation}
where $\lambda(t) = \sqrt{R^2 - (r - a \cos t)^2} - a \sin t$.

The line which is tangent to the circle $C_r$ at a point $z(t)$ intersects the circle $C_R$ at a point $w(t) = z(t) + \lambda(t)ie^{it}$. Let us draw a second tangent line to $C_r$, passing at $w(t)$. It intersects $C_r$ at a point $z(\varphi(t))$, where $\varphi(t)$ satisfies the condition

$$\tan \frac{\varphi(t) - t}{2} = \frac{\lambda(t)}{r}.$$  

In [5] it is proved that

$$\varphi' = \frac{\sqrt{1 - (\sigma \circ \varphi)^2}}{\sqrt{1 - \sigma^2}},$$

where

$$\sigma(t) = \frac{r - a \cos t}{R}.$$  

It is routine to check that the solution of this differential equation with initial condition $\varphi(0) = m$ is given by the formula

$$\varphi(t) = B^{-1}(B(t) + B(m)),$$

where

$$B(t) = \int_0^t \frac{ds}{\sqrt{1 - \sigma^2(s)}}.$$  

3. Results and proofs.

**Theorem 1.** Poncelet’s closure theorem holds in the annulus $C_rC_{a,R}$ for $n$-gons, $n \geq 3$, if and only if the following identity holds:

$$B\left(t + 2 \arctan \frac{\lambda(t)}{r}\right) \equiv B(t) + \frac{1}{n}B(2\pi).$$

**Proof.** $\Rightarrow$) From the assumption it follows that Poncelet’s transverse closes after $n$ reflections, forming a circuminscribed convex $n$-gon. This is equivalent to the condition

$$\varphi^{[n]}(t) = t + 2\pi \quad \text{for all } t \in \mathbb{R},$$

where

$$\varphi^{[1]} = \varphi \quad \text{and} \quad \varphi^{[n+1]} = \varphi^{[n]} \circ \varphi \quad \text{for } n = 1, 2, 3, \ldots$$

Note that formula (9) implies

$$\varphi^{[n]}(t) = B^{-1}(B(t) + nB(m)).$$

From (12) and (14) it follows immediately that

$$B(2\pi) = nB(m).$$
Finally, the function $\varphi$ is given by the formula
\begin{equation}
\varphi(t) = B^{-1}\left(B(t) + \frac{1}{n}B(2\pi)\right),
\end{equation}
and
\begin{equation}
\varphi(0) = m = B^{-1}\left(\frac{1}{n}B(2\pi)\right).
\end{equation}

From (6) we get
\begin{equation}
\varphi(t) = t + 2\arctan\frac{\lambda(t)}{r}.
\end{equation}

The formulas (17) and (18) imply the identity (11).

$\Leftarrow$) Assume that in the annulus $C_r C_{a,R}$ the identity (11) holds for some natural number $n \geq 3$. From the formulas (10) and (16) we get
\begin{equation}
\varphi^{[n]}(t) = B^{-1}(B(t) + B(2\pi)) = B^{-1}(B(t + 2\pi)) = t + 2\pi.
\end{equation}

Now, using (10), we can rewrite the identity (11) in the form
\begin{equation}
t + 2\arctan\frac{\lambda(t)}{r} \int_0^t \frac{1}{\sqrt{1 - \sigma^2(s)}} ds \equiv \int_0^t \frac{1}{\sqrt{1 - \sigma^2(s)}} ds + \frac{1}{n} \int_0^{2\pi} \frac{1}{\sqrt{1 - \sigma^2(s)}} ds.
\end{equation}

Hence we have
\begin{equation}
2\arctan\frac{\lambda(t)}{r} \int_t^0 \frac{1}{\sqrt{1 - \sigma^2(s)}} ds \equiv \frac{2\pi}{n} \int_0^t \frac{1}{\sqrt{1 - \sigma^2(s)}} ds.
\end{equation}

In the particular case $t = 0$ we have
\begin{equation}
2\arctan\frac{1}{r} \sqrt{R^2 -(r-a)^2} \int_0^t \frac{1}{\sqrt{1 - \sigma^2(s)}} ds = \frac{2\pi}{n} \int_0^{2\pi} \frac{1}{\sqrt{1 - \sigma^2(s)}} ds.
\end{equation}

This is exactly the formula (5.6) from [5], and we note that it implies Poncelet’s porism property for $n$-gons.

Introducing
\begin{equation}
V_\xi = \frac{1}{r} \sqrt{[(1 - \xi)r + \xi R]^2 - (r - \xi a)^2}
\end{equation}
for $\xi \in [0, 1]$, we have
\begin{equation}
V_\xi = \frac{1}{r} \sqrt{(R - r + a)[(R - r + a)\xi^2 + 2r\xi]}.
\end{equation}
Since $0 < a < R - r$, we can write

\begin{equation}
V_\xi = \frac{1}{r} c(\xi) \sqrt{R - r + a} \quad \text{for} \; \xi \in [0, 1],
\end{equation}

where

\begin{equation}
c(\xi) = \sqrt{(R - r - a) \xi^2 + 2r\xi}.
\end{equation}

Note that

\begin{equation}
V_1 = \frac{1}{r} \sqrt{R^2 - (r - a)^2} \quad \text{and} \quad V_0 = 0.
\end{equation}

Similarly, we define

\begin{equation}
\sigma_\xi(t) = \frac{r - \xi a \cos t}{(1 - \xi) r + \xi R} \quad \text{for} \; \xi \in [0, 1],
\end{equation}

and one has $\sigma_1 = \sigma$ and $\sigma_0 = 1$.

Now we will prove our main theorem.

**Theorem 2.** Assume that Poncelet’s closure theorem holds in an annulus $C_r C_{a,R}$ for $k$-gons, $k \geq 3$. Then for any $n > k$ there exists $\gamma \in (0, 1)$ such that Poncelet’s closure theorem holds in the annulus $C_r C_{\gamma a, (1-\gamma)r+\gamma R}$ for $n$-gons.

**Proof.** Using the equality (20) from the proof of Theorem 1, we introduce the function

\begin{equation}
F_n(\xi) = n \int_0^{\frac{2\arctan V_\xi}{\pi}} \frac{1}{\sqrt{1 - \sigma_\xi^2(s)}} ds - \int_0^{\frac{2\pi}{\pi}} \frac{1}{\sqrt{1 - \sigma^2(s)}} ds.
\end{equation}

First we have

\begin{equation}
F_n(1) = n \int_0^{\frac{2\arctan V_1}{\pi}} \frac{1}{\sqrt{1 - \sigma^2(s)}} ds - \int_0^{\frac{2\pi}{\pi}} \frac{1}{\sqrt{1 - \sigma^2(s)}} ds.
\end{equation}

From now on we assume that the starting annulus $C_r C_{a,R}$ has Poncelet’s porism property for a natural number $k \geq 3$, and we consider $n > k$. Then by (20) we have

\begin{equation}
k \int_0^{\frac{2\arctan V_1}{\pi}} \frac{1}{\sqrt{1 - \sigma^2(s)}} ds = \int_0^{\frac{2\pi}{\pi}} \frac{1}{\sqrt{1 - \sigma^2(s)}} ds.
\end{equation}
Using this condition, we get

\[ F_n(1) = (n - k) \int_0^1 \frac{1}{\sqrt{1 - \sigma^2(s)}} ds + k \int_0^{2 \text{arctan} V_1} \frac{1}{\sqrt{1 - \sigma^2(s)}} ds \]

\[ - \int_0^{2 \pi} \frac{1}{\sqrt{1 - \sigma^2(s)}} ds = (n - k) \int_0^{2 \text{arctan} V_1} \frac{1}{\sqrt{1 - \sigma^2(s)}} ds > 0. \]

In order to evaluate \( F_n(0) \), we first calculate the value \( F_n(\varepsilon) \) for \( \varepsilon \in (0, 1) \).

We have

\[ F_n(\varepsilon) = n \int_0^{2 \text{arctan} V_{\varepsilon}} \frac{1}{\sqrt{1 - \sigma^2_{\varepsilon}(s)}} ds - \int_0^{2 \pi} \frac{1}{\sqrt{1 - \sigma^2_{\varepsilon}(s)}} ds \]

\[ = (n - 1) \int_0^{2 \text{arctan} V_{\varepsilon}} \frac{1}{\sqrt{1 - \sigma^2_{\varepsilon}(s)}} ds - \int_0^{2 \text{arctan} V_{\varepsilon}} \frac{1}{\sqrt{1 - \sigma^2_{\varepsilon}(s)}} ds. \]

First we prove that

\[ \lim_{\varepsilon \to 0^+} \int_0^{2 \text{arctan} V_{\varepsilon}} \frac{1}{\sqrt{1 - \sigma^2_{\varepsilon}(s)}} ds \leq C, \]

for some positive constant \( C \). We calculate

\[ \int_0^{2 \text{arctan} V_{\varepsilon}} \frac{1}{\sqrt{1 - \sigma^2_{\varepsilon}(s)}} ds \]

\[ = \int_0^{2 \text{arctan} \frac{1}{r} \varepsilon \sqrt{R - r + a}} \left[ 1 - \left( \frac{r - a\varepsilon \cos t}{(1 - \varepsilon) r + \varepsilon R} \right)^2 \right]^{-\frac{1}{2}} dt \]

\[ = \int_0^{2 \text{arctan} \frac{1}{r} \varepsilon \sqrt{R - r + a}} \left( \frac{(1 - \varepsilon) r + \varepsilon R}{(1 - \varepsilon) r + \varepsilon R} \right)^2 - (r - \varepsilon a \cos t)^2 \right)^{-\frac{1}{2}} dt \]

\[ = \int_0^{2 \text{arctan} \frac{1}{r} \varepsilon \sqrt{R - r + a}} \frac{(1 - \varepsilon) r + \varepsilon R}{\sqrt{(R - r + a \cos t) [(R - r - a \cos t) \varepsilon^2 + 2r \varepsilon]}} dt \]
\[ \begin{align*}
2 \arctan \frac{1}{r} \frac{(1 - \varepsilon) r + \varepsilon R}{\sqrt{(R - r - a) \left( R - r - a \right)^2 + 2r \varepsilon}} dt
\leq \int_0^2 \arctan \frac{1}{r} \frac{1}{c(\varepsilon) \sqrt{R - r - a}} dt
\end{align*} \]

\[ = \left[(1 - \varepsilon) r + \varepsilon R\right] \frac{2 \arctan \frac{1}{r} c(\varepsilon) \sqrt{R - r - a}}{c(\varepsilon) \sqrt{R - r - a}}. \]

Since \( \arctan x < x \) for \( x > 0 \), then

\[ \int_0^2 \arctan V_\varepsilon \frac{1}{\sqrt{1 - \sigma_\varepsilon^2(s)}} ds \leq \frac{2}{r} \left[(1 - \varepsilon) r + \varepsilon R\right] \frac{\sqrt{R - r + a}}{\sqrt{R - r - a}}. \]

Thus

\[ \lim_{\varepsilon \to 0^+} \int_0^2 \arctan V_\varepsilon \frac{1}{\sqrt{1 - \sigma_\varepsilon^2(s)}} ds \leq C = \frac{2 \sqrt{R - r + a}}{r \sqrt{R - r - a}}. \]

Next, we claim that

\[ \lim_{\varepsilon \to 0^+} \int_0^{2\pi} \frac{1}{\sqrt{1 - \sigma_\varepsilon^2(s)}} ds = +\infty. \]

We have

\[ \int_{2 \arctan V_\varepsilon}^{2\pi} \frac{1}{\sqrt{1 - \sigma_\varepsilon^2(s)}} ds \]

\[ = \int_{2 \arctan V_\varepsilon}^{2\pi} \frac{(1 - \varepsilon) r + \varepsilon R}{\sqrt{R - r + a \cos t} \cdot \sqrt{(R - r - a \cos t) \varepsilon^2 + 2r \varepsilon}} dt \]

and, furthermore,

\[ \left[(1 - \varepsilon) r + \varepsilon R\right] \int_{2 \arctan V_\varepsilon}^{2\pi} \frac{1}{\sqrt{R - r + a \cdot \sqrt{(R - r + a) \varepsilon^2 + 2r \varepsilon}}} dt \]

\[ = \frac{(1 - \varepsilon) r + \varepsilon R}{\sqrt{R - r + a}} \cdot \frac{2\pi - 2 \arctan \frac{1}{r} \sqrt{R - r + a \cdot c(\varepsilon)}}{\sqrt{(R - r + a) \varepsilon^2 + 2r \varepsilon}} \rightarrow +\infty, \]

when \( \varepsilon \to 0 \). Hence

\[ \lim_{\varepsilon \to 0^+} \int_{2 \arctan V_\varepsilon}^{2\pi} \frac{1}{\sqrt{1 - \sigma_\varepsilon^2(s)}} ds = +\infty. \]
Thus, we have

\[ F_n(0^+) = \lim_{\varepsilon \to 0^+} F_n(\varepsilon) = -\infty \]

and

\[ F_n(1) > 0. \]

These conditions imply that there exists a number \( \gamma \in (0, 1) \) such that

\[ F_n(\gamma) = 0. \]

Thus, with Theorem 1 the proof is finished. \( \square \)

REFERENCES