On certain general integral operators of analytic functions

Abstract. In this paper, we obtain new sufficient conditions for the operators $F_{\alpha_1, \alpha_2, \ldots, \alpha_n, \beta}(z)$ and $G_{\alpha_1, \alpha_2, \ldots, \alpha_n, \beta}(z)$ to be univalent in the open unit disc $U$, where the functions $f_1, f_2, \ldots, f_n$ belong to the classes $S^*(a, b)$ and $K(a, b)$. The order of convexity for the operators $F_{\alpha_1, \alpha_2, \ldots, \alpha_n, \beta}(z)$ and $G_{\alpha_1, \alpha_2, \ldots, \alpha_n, \beta}(z)$ is also determined. Furthermore, and for $\beta = 1$, we obtain sufficient conditions for the operators $F_n(z)$ and $G_n(z)$ to be in the class $K(a, b)$. Several corollaries and consequences of the main results are also considered.

1. Introduction and definitions. Let $A$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are analytic in the open unit disc $U = \{z : |z| < 1\}$. Further, by $S$ we shall denote the class of all functions in $A$ which are univalent in $U$. A function $f(z) \in A$ is said to be starlike of order $\gamma$ ($0 \leq \gamma < 1$) if it satisfies

$$\text{Re} \left( \frac{zf'(z)}{f(z)} \right) > \gamma \quad (z \in U).$$

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Also, we say that a function \( f(z) \in \mathcal{A} \) is said to be convex of order \( \gamma \) \((0 \leq \gamma < 1)\) if it satisfies
\[
\text{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \gamma \quad (z \in \mathcal{U}).
\]
We denote by \( \mathcal{S}^*(\gamma) \) and \( \mathcal{K}(\gamma) \), respectively, the usual classes of starlike and convex functions of order \( \gamma \) \((0 \leq \gamma < 1)\) in \( \mathcal{U} \).

A function \( f \in \mathcal{A} \) is said to be in the class \( \mathcal{S}^*(a,b) \) if
\[
\left|\frac{zf'(z)}{f(z)} - a\right| < b \quad (z \in \mathcal{U}; \ |a - 1| < b \leq a)
\]
and a function \( f \in \mathcal{A} \) is said to be in the class \( \mathcal{K}(a,b) \) if
\[
\left|1 + \frac{zf''(z)}{f'(z)} - a\right| < b \quad (z \in \mathcal{U}; \ |a - 1| < b \leq a).
\]
From (1.3) and (1.4), we have
\[
\text{Re}\left(\frac{zf'(z)}{f(z)}\right) > a - b \quad (z \in \mathcal{U}; \ |a - 1| < b \leq a)
\]
and
\[
\text{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > a - b \quad (z \in \mathcal{U}; \ |a - 1| < b \leq a).
\]
The class \( \mathcal{S}^*(a,b) \) was introduced by Jakubowski [12]. It is clear that \( a > \frac{1}{2} \), \( \mathcal{S}^*(a,b) \subset \mathcal{S}^*(a-b) \subset \mathcal{S}^*(0) \equiv \mathcal{S}^* \) and \( \mathcal{K}(a,b) \subset \mathcal{K}(a-b) \subset \mathcal{K}(0) \equiv \mathcal{K} \). Further, applying the Briot-Bouquet differential subordination [9], we can easily see that \( \mathcal{K}(a,b) \subset \mathcal{S}^*(a,b) \).

Several authors (e.g., see [4, 5, 6, 8, 10, 11, 15, 16]), obtained many sufficient conditions for the univalency of the integral operators
\[
F_{\alpha_1,\alpha_2,\ldots,\alpha_n,\beta}(z) = \left\{\beta \int_0^z t^{\beta-1} \prod_{i=1}^{n} \left(\frac{f_i(t)}{t}\right)^{\alpha_i} dt\right\}^{\frac{1}{\beta}},
\]
and
\[
G_{\alpha_1,\alpha_2,\ldots,\alpha_n,\beta}(z) = \left\{\beta \int_0^z t^{\beta-1} \prod_{i=1}^{n} \left(f'_i(t)\right)^{\alpha_i} dt\right\}^{\frac{1}{\beta}},
\]
where the functions \( f_1, f_2, \ldots, f_n \) belong to the class \( \mathcal{A} \) and the parameters \( \alpha_1, \alpha_2, \ldots, \alpha_n \), and \( \beta \) are complex numbers such that the integrals in (1.5) and (1.6) exist. Here and throughout in the sequel every many-valued function is taken with the principal branch.
For $\beta = 1$, we obtain the integral operators
\begin{equation}
F_n(z) = \int_0^z \left( \frac{f_1(t)}{t} \right)^{\alpha_1} \cdots \left( \frac{f_n(t)}{t} \right)^{\alpha_n} dt
\end{equation}
and
\begin{equation}
G_n(z) = \int_0^z (f'_1(t))^{\alpha_1} \cdots (f'_n(t))^{\alpha_n} dt
\end{equation}
introduced and studied by Breaz and Breaz [5] and Breaz et al. [7], respectively.

In this paper, we obtain new sufficient conditions for the operators $F_{\alpha_1,\alpha_2,\ldots,\alpha_n,\beta}(z)$ and $G_{\alpha_1,\alpha_2,\ldots,\alpha_n,\beta}(z)$ defined by (1.5) and (1.6) to be univalent in the open unit disc $U$, where the functions $f_1, f_2, \ldots, f_n$ belong to the above classes $S^*(a,b)$ and $K(a,b)$. The order of convexity for the operators $F_{\alpha_1,\alpha_2,\ldots,\alpha_n,\beta}(z)$ and $G_{\alpha_1,\alpha_2,\ldots,\alpha_n,\beta}(z)$ is also determined. Furthermore, we obtain sufficient conditions for the operators $F_n(z)$ and $G_n(z)$ defined by (1.5) and (1.6) to be in the class $K(a,b)$.

In the proofs of our main results we need the following univalence criteria. The first result, i.e. Lemma 1.1 is a generalization of the well-known univalence criterion of Becker [2] (which in fact corresponds to the case $\beta = \delta = 1$), while the second, i.e. Lemma 1.2 is a generalization of Ahlfors’ and Becker’s univalence criterion [1, 3] (which corresponds to the case $\beta = 1$).

**Lemma 1.1** ([13]). Let $\delta$ be a complex number with $\Re(\delta) > 0$. If $f \in \mathcal{A}$ satisfies
\begin{equation}
\frac{1 - |z|^{2\Re(\delta)}}{\Re(\delta)} \left| \frac{zf''(z)}{f'(z)} \right| \leq 1,
\end{equation}
for all $z \in \mathcal{U}$, then, for any complex number $\beta$ with $\Re(\beta) \geq \Re(\delta)$, the integral operator
\begin{equation}
F_\beta(z) = \left\{ \beta \int_0^z t^{\beta-1} f'(t) dt \right\}^{\frac{1}{\beta}}
\end{equation}
is in the class $S$.

**Lemma 1.2** ([14]). Let $\beta$ be a complex number with $\Re(\beta) > 0$ and $c$ be a complex number with $|c| \leq 1$, $c \neq -1$. If $f \in \mathcal{A}$ satisfies
\begin{equation}
|c| z^{2\beta} + \left( 1 - |z|^{2\beta} \right) \left| \frac{zf''(z)}{\beta f'(z)} \right| \leq 1
\end{equation}
for all \( z \in U \), then the integral operator

\[
F_\beta(z) = \left\{ \beta \int_0^z t^{\beta-1} f(t) dt \right\}^{\frac{1}{\beta}}
\]

is in the class \( S \).

2. Univalence conditions for \( F_{\alpha_1,\alpha_2,\ldots,\alpha_n,\beta}(z) \). We first prove

**Theorem 2.1.** Let \( f_i(z) \in S^\star(a_i,b_i); |a_i - 1| < b_i \leq a_i, \alpha_i \in \mathbb{C} \) for all \( i = 1, \ldots, n \), and \( \delta \in \mathbb{C} \) with

\[
\text{Re}(\delta) \geq 2 \sum_{i=1}^n |\alpha_i| b_i.
\]

Then for any \( \beta \in \mathbb{C} \) with \( \text{Re}(\beta) \geq \text{Re}(\delta) \), the integral operator \( F_{\alpha_1,\alpha_2,\ldots,\alpha_n,\beta}(z) \) defined by (1.5) is analytic and univalent in \( U \).

**Proof.** Defining

\[
h(z) = \int_0^z \prod_{i=1}^n \left( \frac{f_i(t)}{t} \right)^{\alpha_i} dt,
\]

we observe that \( h(0) = h'(0) - 1 = 0 \), where

\[
h'(z) = \prod_{i=1}^n \left( \frac{f_i(z)}{z} \right)^{\alpha_i}.
\]

Differentiating both sides of (2.2) logarithmically, we obtain

\[
\frac{z h''(z)}{h'(z)} = \sum_{i=1}^n \alpha_i \left( \frac{z f_i'(z)}{f_i(z)} - 1 \right)
\]

which is equivalent to

\[
\frac{z h''(z)}{h'(z)} = \sum_{i=1}^n \alpha_i \left( \frac{z f_i'(z)}{f_i(z)} - a_i \right) + \sum_{i=1}^n \alpha_i a_i - \sum_{i=1}^n \alpha_i.
\]

Since \( f_i(z) \in S^\star(a_i,b_i); |a_i - 1| < b_i \leq a_i \) for all \( i = 1, 2, \ldots, n \), it follows from (2.3) that

\[
\left| \frac{z h''(z)}{h'(z)} \right| \leq \sum_{i=1}^n |\alpha_i| \left| \frac{z f_i'(z)}{f_i(z)} - a_i \right| + \sum_{i=1}^n |\alpha_i| |a_i - 1|
\]

\[
\leq 2 \sum_{i=1}^n |\alpha_i| b_i.
\]
Multiplying both sides of (2.4) by $\frac{1-|z|^{2 \text{Re}(\delta)}}{\text{Re}(\delta)}$ and making use of (2.1), we obtain

$$1 - |z|^{2 \text{Re}(\delta)} \left| \frac{zh''(z)}{h'(z)} \right| \leq 2 \left( \frac{1-|z|^{2 \text{Re}(\delta)}}{\text{Re}(\delta)} \right) \sum_{i=1}^{n} |\alpha_i| b_i$$

$$\leq \frac{2}{\text{Re}(\delta)} \sum_{i=1}^{n} |\alpha_i| b_i \leq 1.$$

Applying Lemma 1.1 for the function $h(z)$, we prove that $F_{\alpha_1, \alpha_2, \ldots, \alpha_n, \beta}(z) \in \mathcal{S}$. □

Letting $n = 1$, $\alpha_1 = \alpha$, $a_1 = a$, $b_1 = b$ and $f_1 = f$ in Theorem 2.1, we have

**Corollary 2.2.** Let $f(z) \in \mathcal{S}^*(a,b)$; $|a - 1| < b \leq a$, $\alpha \in \mathbb{C}$ and $\delta \in \mathbb{C}$ with $\text{Re}(\delta) > 2 |\alpha| b$. Then for any $\beta \in \mathbb{C}$ with $\text{Re}(\beta) \geq \text{Re}(\delta)$, the integral operator

$$F_{\alpha, \beta}(z) = \left\{ \beta \int_{0}^{z} t^{\beta-1} \left( \frac{f(t)}{t} \right)^{\alpha} dt \right\}^{\frac{1}{\beta}}$$

is analytic and univalent in $\mathcal{U}$.

Making use of Lemma 1.2, we prove the following theorem:

**Theorem 2.3.** Let $f_i(z) \in \mathcal{S}^*(a_i,b_i)$; $|a_i - 1| < b_i \leq a_i$, $\alpha_i \in \mathbb{C}$ for all $i = 1, 2, \ldots, n$, and $\beta \in \mathbb{C}$ with

$$\text{Re}(\beta) \geq 2 \sum_{i=1}^{n} |\alpha_i| b_i$$

and

$$|c| \leq 1 - \frac{2}{\text{Re}(\beta)} \sum_{i=1}^{n} |\alpha_i| b_i \quad (c \in \mathbb{C}).$$

Then the integral operator $F_{\alpha_1, \alpha_2, \ldots, \alpha_n, \beta}(z)$ defined by (1.5) is analytic and univalent in $\mathcal{U}$. 
**Proof.** Let \( f_i(z) \in S^*(a_i, b_i); \ |a_i - 1| < b_i \leq a_i \) for all \( i = 1, 2, \ldots, n \), it follows from (2.4) that
\[
|c| |z|^{2\beta} + (1 - |z|^{2\beta}) \frac{zh''(z)}{h'(z)} \leq |c| + \left| \frac{1 - |z|^{2\beta}}{\beta} \right| \frac{|zh''(z)|}{|h'(z)|}
\]
\[
\leq |c| + 2 \left| \frac{1 - |z|^{2\beta}}{\beta} \right| \sum_{i=1}^{n} |\alpha_i| b_i
\]
\[
< |c| + 2 \frac{2}{\beta} \sum_{i=1}^{n} |\alpha_i| b_i
\]
\[
< |c| + 2 \frac{2}{\Re(\beta)} \sum_{i=1}^{n} |\alpha_i| b_i
\]
which, in the light of the hypothesis (2.6), yields
\[
|c| |z|^{2\beta} + (1 - |z|^{2\beta}) \frac{zh''(z)}{h'(z)} \leq 1.
\]

Finally, by applying Lemma 1.2, we conclude that \( F_{\alpha_1, \alpha_2, \ldots, \alpha_n, \beta}(z) \in S \). 

Letting \( n = 1, \alpha_1 = \alpha, a_1 = a, b_1 = b \) and \( f_1 = f \) in Theorem 2.3, we have

**Corollary 2.4.** Let \( f(z) \in S^*(a, b); \ |a - 1| < b \leq a, \alpha \in \mathbb{C}, \) and \( \beta \in \mathbb{C} \) with
\[
\Re(\beta) \geq 2 |\alpha| b
\]
and
\[
|c| \leq 1 - \frac{2}{\Re(\beta)} |\alpha| b \quad (c \in \mathbb{C}).
\]

Then the integral operator \( F_{\alpha, \beta}(z) \) defined by (2.5) is analytic and univalent in \( U \).

**3. Univalence conditions for** \( G_{\alpha_1, \alpha_2, \ldots, \alpha_n, \beta}(z) \). Now, we prove

**Theorem 3.1.** Let \( f_i(z) \in K(a_i, b_i); \ |a_i - 1| < b_i \leq a_i, \alpha_i \in \mathbb{C} \) for all \( i = 1, \ldots, n \), and \( \delta \in \mathbb{C} \) with
\[
\Re(\delta) \geq 2 \sum_{i=1}^{n} |\alpha_i| b_i.
\]

Then for any \( \beta \in \mathbb{C} \) with \( \Re(\beta) \geq \Re(\delta) \), the integral operator \( G_{\alpha_1, \alpha_2, \ldots, \alpha_n, \beta}(z) \) defined by (1.6) is analytic and univalent in \( U \).
Proof. Defining
\[ h(z) = \int_0^z \prod_{i=1}^n \left( f_i'(t) \right)^{a_i} dt, \]
we observe that \( h(0) = h'(0) - 1 = 0 \). On the other hand, it is easy to see that
\[ (3.1) \quad h'(z) = \prod_{i=1}^n \left( f_i'(z) \right)^{a_i}. \]
Differentiating both sides of (3.1) logarithmically, we obtain
\[ \frac{zh''(z)}{h'(z)} = \sum_{i=1}^n \alpha_i \left( 1 + \frac{zf_i''(z)}{f_i'(z)} \right)\alpha_i - \sum_{i=1}^n a_i (a_i - 1). \]
Thus, we have
\[ (3.2) \quad \frac{zh''(z)}{h'(z)} = \sum_{i=1}^n \alpha_i \left( 1 + \frac{zf_i''(z)}{f_i'(z)} \right) + \sum_{i=1}^n \alpha_i (a_i - 1). \]
Let \( f_i(z) \in K(a_i, b_i); |a_i - 1| < b_i \leq a_i, \) for all \( i = 1, 2, \ldots, n, \) and following the same steps in the proof of Theorem 2.1, we get the required result. □

Letting \( n = 1, a_1 = \alpha, a_1 = a, b_1 = b \) and \( f_1 = f \) in Theorem 3.1, we have

**Corollary 3.2.** Let \( f(z) \in K(a, b); |a - 1| < b \leq a, \alpha \) and \( \delta \in \mathbb{C} \) with \( \text{Re}(\delta) \geq 2|\alpha|b. \) Then for any \( \beta \in \mathbb{C} \) with \( \text{Re}(\beta) \geq \text{Re}(\delta), \) the integral operator
\[
G_{\alpha, \beta}(z) = \left\{ \beta \int_0^z t^{\beta-1} \left( f'(t) \right)^{\alpha} dt \right\}^{\frac{1}{\beta}}
\]
is analytic and univalent in \( \mathcal{U}. \)

Using (3.1), (1.4) and applying Lemma 1.2, we prove the following theorem:

**Theorem 3.3.** Let \( f_i(z) \in K(a_i, b_i); |a_i - 1| < b_i \leq a_i, \alpha_i \in \mathbb{C} \) for all \( i = 1, \ldots, n \) and \( \beta \in \mathbb{C} \) with
\[
\text{Re}(\beta) \geq 2 \sum_{i=1}^n |\alpha_i|b_i
\]
and
\[
|c| \leq 1 - \frac{2}{\text{Re}(\beta)} \sum_{i=1}^n |\alpha_i|b_i \quad (c \in \mathbb{C}).
\]
Then the integral operator \( G_{\alpha_1, \alpha_2, \ldots, \alpha_n, \beta}(z) \) defined by (1.6) is analytic and univalent in \( \mathcal{U}. \)
Letting $n = 1$, $\alpha_1 = \alpha$, $a_1 = a$, $b_1 = b$ and $f_1 = f$ in Theorem 3.3, we have

**Corollary 3.4.** Let $f(z) \in K(a, b)$; $|a - 1| < b \leq a$, $\alpha$ and $\beta \in \mathbb{C}$ with \[ \text{Re}(\beta) \geq 2 |\alpha| b \]
and
\[ |c| \leq 1 - \frac{2}{\text{Re}(\beta)} |\alpha| b. \]
Then the integral operator $G_{\alpha, \beta}(z)$ defined by (3.3) is analytic and univalent in $U$.

4. Order of convexity. Now, we prove

**Theorem 4.1.** Let $f_i(z) \in S^*(a_i, b_i)$; $|a_i - 1| < b_i \leq a_i$, and $\alpha_i > 0$ for all $i = 1, \ldots, n$, with
\[ 0 \leq 1 - \sum_{i=1}^{n} \alpha_i \left( b_i + \frac{1}{2} \right) < 1 \quad \text{and} \quad \sum_{i=1}^{n} \alpha_i \left( b_i + \frac{1}{2} \right) \leq 1. \]
Then the integral operator $F_n(z)$ defined by (1.7) is in the class
\[ K \left( 1 - \sum_{i=1}^{n} \alpha_i \left( b_i + \frac{1}{2} \right) \right). \]

**Proof.** From (1.7), it follows that
\[ F_n(z) = \prod_{i=1}^{n} \left( \frac{f_i(z)}{z} \right)^{\alpha_i}. \]
Differentiating both sides of (4.1) logarithmically, we obtain
\[ 1 + \frac{zF_n''(z)}{F_n'(z)} = \sum_{i=1}^{n} \alpha_i \left( \frac{zf_i'(z)}{f_i(z)} \right) - \sum_{i=1}^{n} \alpha_i + 1. \]
Since $f_i(z) \in S^*(a_i, b_i)$; $|a_i - 1| < b_i \leq a_i$ and $a_i > \frac{1}{2}$ for all $i = 1, 2, \ldots, n$, we have
\[ \text{Re} \left( 1 + \frac{zF_n''(z)}{F_n'(z)} \right) = \sum_{i=1}^{n} \alpha_i \text{Re} \left( \frac{zf_i'(z)}{f_i(z)} \right) - \sum_{i=1}^{n} \alpha_i + 1 \]
\[ \geq \sum_{i=1}^{n} \alpha_i (a_i - b_i - 1) + 1 \]
\[ > 1 - \sum_{i=1}^{n} \alpha_i \left( b_i + \frac{1}{2} \right). \]
Therefore, $F_n(z)$ is convex of order $1 - \sum_{i=1}^{n} \alpha_i \left( b_i + \frac{1}{2} \right)$ in $U$. □
Letting $n = 1$, $\alpha_1 = \alpha$, $a_1 = a$, $b_1 = b$ and $f_1 = f$ in Theorem 4.1, we have

**Corollary 4.2.** Let $f(z) \in S^*(a,b)$; $|a - 1| < b \leq a$, and $\alpha > 0$ with $0 \leq 1 - \alpha(b + \frac{1}{2}) < 1$ and $\alpha(b + \frac{1}{2}) \leq 1$. Then $\int_0^z \left( \frac{f'(t)}{t^\alpha} \right) dt \in K(1-\alpha(b+\frac{1}{2})).$

Next, we prove

**Theorem 4.3.** Let $f_i(z) \in K(a_i, b_i)$; $|a_i - 1| < b_i \leq a_i$, and $\alpha_i > 0$ for all $i = 1, \ldots, n$, with

$$0 \leq 1 - \sum_{i=1}^n \alpha_i \left( b_i + \frac{1}{2} \right) < 1 \quad \text{and} \quad \sum_{i=1}^n \alpha_i \left( b_i + \frac{1}{2} \right) \leq 1.$$  

Then the integral operator $G_\alpha(z)$ defined by (1.8) is in the class

$$K \left( 1 - \sum_{i=1}^n \alpha_i \left( b_i + \frac{1}{2} \right) \right).$$

**Proof.** From (1.8), we have

$$\left( 1 + \frac{z G''_\alpha(z)}{G'_\alpha(z)} \right) = \sum_{i=1}^n \alpha_i \left( 1 + \frac{z f''_i(z)}{f'_i(z)} \right) - \sum_{i=1}^n \alpha_i + 1. \quad (4.3)$$

Let $f_i(z) \in K(a_i, b_i)$; $|a_i - 1| < b_i \leq a_i$; $a_i > \frac{1}{2}$ for all $i = 1, 2, \ldots, n$, and following the same steps in the proof of Theorem 4.1, we get the required result. \(\square\)

Letting $n = 1$, $\alpha_1 = \alpha$, $a_1 = a$, $b_1 = b$ and $f_1 = f$ in Theorem 4.3, we have

**Corollary 4.4.** Let $f(z) \in K(a,b)$; $|a - 1| < b \leq a$, and $\alpha > 0$ with $0 \leq 1 - \alpha(b + \frac{1}{2}) < 1$ and $\alpha(b + \frac{1}{2}) \leq 1$. Then $\int_0^z \left( f'(t) \right)^\alpha dt \in K(1-\alpha(b+\frac{1}{2})).$

**5. Sufficient conditions for the operators $F_n(z)$ and $G_n(z)$.**

**Theorem 5.1.** Let $f_i(z) \in S^*(\gamma_i)$; $0 \leq \gamma_i < 1$, for all $i = 1, 2, \ldots, n$. Then the integral operator $F_n(z)$ defined by (1.7) is in the class $K(a_i, b_i)$, where $a_i = \sum_{i=1}^n \alpha_i \gamma_i + 1$, $b_i = \sum_{i=1}^n \alpha_i$ and $\sum_{i=1}^n \alpha_i (1 - \gamma_i) \leq 1$ for all $i = 1, 2, \ldots, n$.

**Proof.** Let $f_i(z) \in S^*(\gamma_i)$; $0 \leq \gamma_i < 1$, for all $i = 1, 2, \ldots, n$. Then it follows from (4.2) that

$$\Re \left( 1 + \frac{z F''_n(z)}{F'_n(z)} \right) = \sum_{i=1}^n \alpha_i \Re \left( \frac{z f'_i(z)}{f_i(z)} \right) + 1 - \sum_{i=1}^n \alpha_i$$

$$> \sum_{i=1}^n \alpha_i \gamma_i + 1 - \sum_{i=1}^n \alpha_i,$$
which proves that \( F_n(z) \in \mathcal{K}(a_i, b_i) \), where \( a_i = \sum_{i=1}^{n} \alpha_i \gamma_i + 1 \) and \( b_i = \sum_{i=1}^{n} \alpha_i \) for all \( i = 1, 2, \ldots, n \).

Letting \( n = 1, \alpha_1 = \alpha, \gamma_1 = \gamma, a_1 = a, b_1 = b \) and \( f_1 = f \) in Theorem 5.1, we have

**Corollary 5.2.** Let \( f(z) \in S^*(\gamma); 0 \leq \gamma < 1. \) Then \( \int_0^z \left( \frac{f(t)}{t} \right)^\alpha \, dt \in \mathcal{K}(\alpha \gamma + 1, \alpha) \), where \( 0 < \alpha(1 - \gamma) \leq 1 \).

Using (4.3), we can prove the following theorem:

**Theorem 5.3.** Let \( f_i(z) \in \mathcal{K}(\gamma_i); 0 \leq \gamma_i < 1, \) for all \( i = 1, 2, \ldots, n. \) Then the integral operator \( G_n(z) \) defined by (1.8) is in the class \( \mathcal{K}(a_i, b_i) \), where \( a_i = \sum_{i=1}^{n} \alpha_i \gamma_i + 1, b_i = \sum_{i=1}^{n} \alpha_i \) and \( \sum_{i=1}^{n} \alpha_i (1 - \gamma_i) \leq 1 \) for all \( i = 1, 2, \ldots, n. \)

Letting \( n = 1, \alpha_1 = \alpha, \gamma_1 = \gamma, a_1 = a, b_1 = b \) and \( f_1 = f \) in Theorem 5.3, we have

**Corollary 5.4.** Let \( f(z) \in \mathcal{K}(\gamma); 0 \leq \gamma < 1. \) Then \( \int_0^z (f'(t))^\alpha \, dt \in \mathcal{K}(\alpha \gamma + 1, \alpha) \), where \( 0 < \alpha(1 - \gamma) \leq 1 \).

**References**


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