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Boundedness and compactness
of weighted composition operators
between weighted Bergman spaces

Abstract. We study when a weighted composition operator acting between
different weighted Bergman spaces is bounded, resp. compact.

1. Introduction. Let φ be an analytic self-map of the open unit disk \( \mathbb{D} \) and
ψ be an analytic function on \( \mathbb{D} \). Such maps induce the weighted composition
operator
\[ C_{\phi, \psi} : H(\mathbb{D}) \to H(\mathbb{D}), \quad f \mapsto \psi(f \circ \phi), \]
where \( H(\mathbb{D}) \) denotes the space of all analytic functions endowed with the
compact-open topology \( co \). The study of (weighted) composition operators
acting on various spaces of analytic functions has quite a long and rich
history since they appear naturally in a variety of problems, see the ex-
cellent monographs [5] and [15]. For a deep insight in the recent research
on (weighted) composition operators we refer the reader to the following
sample of papers as well as the references therein: [12], [10], [1], [2], [3], [4],
[13], [14], [11].

We say that a function \( v : \mathbb{D} \to (0, \infty) \) is a weight if it is bounded and
continuous. For a weight \( v \) we consider the space
\[ A_{v,2} := \left\{ f \in H(\mathbb{D}); \quad \| f \|_{v,2} := \left( \int_{\mathbb{D}} |f(z)|^2 v(z) \, dA(z) \right)^{\frac{1}{2}} < \infty \right\}, \]

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where $dA(z)$ is the normalized area measure such that area of $D$ is 1. Endowed with norm $\| \cdot \|_{v,2}$ this is a Banach space. Thus, $A_{1,2}$ denotes the usual Bergman space. An introduction to the concept of Bergman spaces is given in [9] and [7].

In [16] we characterized the boundedness of weighted composition operators acting between weighted Bergman spaces generated by weights given as the absolute value of holomorphic functions using a method by Čučković and Zhao [6]. In this paper we study boundedness and compactness of weighted composition operators acting between different weighted Bergman spaces generated by a quite general class of radial weights.

2. Preliminaries. In this section we collect some geometrical data of the open unit disk as well as some well-known basic facts we will need to treat the problem mentioned above. For $a, z \in D$ let $\sigma_a(z)$ be the M"obius transformation of $D$ which interchanges 0 and $a$, that is

$$\sigma_a(z) = \frac{a - z}{1 - az}.$$  

Obviously

$$\sigma_a'(z) = - \frac{1 - |a|^2}{(1 - az)^2}$$ for every $z \in D$.

It turned out that the Carleson measure is a very useful tool when studying (weighted) composition operators on weighted Bergman spaces, see [6] and [16]. Recall that a positive Borel measure $\mu$ on $D$ is said to be a Carleson measure on the Bergman space if there is a constant $C > 0$ such that, for any $f \in A_{1,2}$

$$\int_D |f(z)|^2 \, d\mu(z) \leq C \|f\|_{1,2}^2.$$  

For an arc $I$ in the unit circle $\partial D$ let $S(I)$ be the Carleson square defined by

$$S(I) = \left\{ z \in D; \ 1 - |I| \leq |z| < 1, \ \frac{z}{|z|} \in I \right\}.$$  

The following result is well known. In its present form it is taken from [6] (see there Theorem A) and [8].

**Theorem 1** ([6] Theorem A). Let $\mu$ be a positive Borel measure on $D$. Then the following statements are equivalent.

(i) There is a constant $C_1 > 0$ such that, for any positive subharmonic function $f$ we have that

$$\int_D f^2(z) \, d\mu(z) \leq C_1 \int_D f^2(z) \, dA(z).$$  

(ii) There is a constant $C_2 > 0$ such that, for any arc $I \subset \partial D$,

$$\mu(S(I)) \leq C_2 |I|^2.$$
(iii) There is a constant $C_3 > 0$ such that, for every $a \in \mathbb{D}$,

$$\int_{\mathbb{D}} |\sigma'_a(z)|^2 \, d\mu(z) \leq C_3.$$

The study of the compactness of the operator $C_{\phi, \psi}$ requires the following proposition which can be found in the book of Cowen and MacCluer, see [5].

**Proposition 2** (Cowen–MacCluer [5], Proposition 3.11). The operator $C_{\phi, \psi} : A_{v,2} \to A_{w,2}$ is compact if and only if for every bounded sequence $(f_n)_{n \in \mathbb{N}}$ in $A_{v,2}$ such that $f_n \to 0$ uniformly on the compact subsets of $\mathbb{D}$, then $C_{\phi, \psi}f_n \to 0$ in $A_{w,2}$.

In the sequel we consider the following class of weights. Let $\nu$ be a holomorphic function on $\mathbb{D}$, non-vanishing, strictly positive on $[0,1]$ and satisfying $\lim_{r \to 1} \nu(r) = 0$. Then we define the weight $v$ by

$$v(z) := \nu(|z|^2)$$

for every $z \in \mathbb{D}$.

Next, we give some illustrating examples of weights of this type:

(i) Consider $\nu(z) = (1 - z)^\alpha$, $\alpha \geq 1$. Then the corresponding weight is the so-called standard weight $v(z) = (1 - |z|^2)^\alpha$.

(ii) Select $\nu(z) = e^{-(1 - |z|^2)^\alpha}$, $\alpha \geq 1$. Then we obtain the weight $v(z) = e^{-(1 - |z|^2)^\alpha}$.

(iii) Choose $\nu(z) = \sin(1 - z)$ and the corresponding weight is given by $v(z) = \sin(1 - |z|^2)$.

(iv) Let $\nu(z) = (1 - \log(1 - z))^q$, $q \leq -1$, for every $z \in \mathbb{D}$. Hence we obtain the weight $v(z) = (1 - \log(1 - |z|^2))^q$, $q \leq -1$, for every $z \in \mathbb{D}$.

For a fixed point $a \in \mathbb{D}$ we introduce a function $\nu_a(z) := \nu(\overline{a}z)$ for every $z \in \mathbb{D}$. Since $\nu$ is holomorphic on $\mathbb{D}$, so is the function $\nu_a$.

It can be easily seen that each weight, which is defined as above, is subharmonic.

3. Boundedness. This section is devoted to the study of the boundedness of $C_{\phi, \psi} : A_{v,2} \to A_{w,2}$. In fact, the following result corresponds to the results obtained in [6] and [16]. Actually, the idea to use Carleson measures is due to [6].

**Theorem 3.** Let $v$ be a weight as defined above such that

$$M := \sup_{a \in \mathbb{D}} \sup_{z \in \mathbb{D}} \frac{v(z)}{|\nu_a(z)|} < \infty.$$

Then the weighted composition operator $C_{\phi, \psi} : A_{v,2} \to A_{w,2}$ is bounded if and only if

$$\sup_{a \in \mathbb{D}} \int_{\mathbb{D}} \frac{|\sigma'_a(\phi(z))|^2}{|\nu_a(\phi(z))|} w(z)|\psi(z)|^2 \, dA(z) < \infty.$$
for every positive subharmonic function $g$ for every $z \in \mathbb{D}$. Then

$$\|f_a\|^2_{\nu,2} = \int_{\mathbb{D}} \left| \frac{\sigma_a'(z)}{\nu_a(z)} \right|^2 v(z) \, dA(z) \leq M$$

for every $a \in \mathbb{D}$ and the constant $M$ is independent of the choice of the point $a$. The boundedness of the operator $C_{\phi,\psi}$ yields that

$$\|C_{\phi,\psi} f_a\|^2_{\nu,2} = \int_{\mathbb{D}} \left| \frac{\sigma_a'(z)}{\nu_a(z)} \right|^2 w(z) |\psi(z)|^2 \, dA(z) \leq C \|f_a\|^2_{\nu,2} \leq CM$$

for every $a \in \mathbb{D}$. Finally,

$$\sup_{a \in \mathbb{D}} \int_{\mathbb{D}} \left| \frac{\sigma_a'(z)}{\nu_a(z)} \right|^2 w(z) |\psi(z)|^2 \, dA(z) < \infty,$$

as desired.

Conversely, we assume that

$$K := \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} \left| \frac{\sigma_a'(z)}{\nu_a(z)} \right|^2 w(z) |\psi(z)|^2 \, dA(z) < \infty.$$  

Obviously, this yields that $\sup_{a \in \mathbb{D}} \int_{\mathbb{D}} \sigma_a'(z) w(z) \frac{v(z)\psi(z)^2}{\nu_a(z)} \, dA(z) \leq K < \infty$. Putting $d\nu_{v,w,\psi} \circ \phi^{-1}$ and changing variable $s = \phi(z)$, this is equivalent with

$$\sup_{a \in \mathbb{D}} \int_{\mathbb{D}} \sigma_a'(s)^2 \, d\mu_{v,w,\psi}(s) < \infty.$$  

By Theorem 1 this holds if and only if there is a constant $C > 0$ such that

$$\int_{\mathbb{D}} g^2(s) \, d\mu_{v,w,\psi}(s) \leq C \int_{\mathbb{D}} g^2(s) \, dA(s)$$

for every positive subharmonic function $g$. Since

$$\int_{\mathbb{D}} g^2(\phi(z)) |\psi(z)|^2 \frac{w(z)}{v(\phi(z))} \, dA(z) = \int_{\mathbb{D}} g^2(\phi(z)) \, d\nu_{v,w,\psi}(z) = \int_{\mathbb{D}} g^2(s) \, d\mu_{v,w,\psi}(s),$$

(3.1) is equivalent with

$$\int_{\mathbb{D}} g^2(\phi(z)) \frac{w(z)}{v(\phi(z))} |\psi(z)|^2 \, dA(z) \leq C \int_{\mathbb{D}} g^2(z) \, dA(z).$$

Next, put $f(z) := \frac{g(z)}{v(z)^2}$ for every $z \in \mathbb{D}$. Now, if $\int_{\mathbb{D}} g^2(z) \, dA(z) \leq K_1 < \infty$, then, obviously we can find a constant $L > 0$ such that

$$\int_{\mathbb{D}} v(z) f^2(z) \, dA(z) \leq L.$$
Hence
\[ \int_{\mathbb{D}} f^2(\phi(z))|\psi(z)|^2 w(z) \, dA(z) \leq C \int_{\mathbb{D}} f^2(z)v(z) \, dA(z) \]
for every positive subharmonic function \( f \) on \( \mathbb{D} \) as defined above. Then obviously
\[ \int_{\mathbb{D}} |f(\phi(z))|^2|\psi(z)|^2 w(z) \, dA(z) \leq C \int_{\mathbb{D}} |f(z)|^2 v(z) \, dA(z). \]
for every \( f \in A_{v,2} \). \( \square \)

4. Compactness.

**Proposition 4.** Let \( v \) be a weight and \( K := \sup_{z \in \mathbb{D}} w(z)|\psi(z)|^2 < \infty \). Moreover, let the weighted composition operator \( C_{\phi,\psi} : A_{v,2} \to A_{w,2} \) be bounded. If for every \( K \subset D \) there is \( \varepsilon > 0 \) such that \( \frac{w(z)}{v(\phi(z))}|\psi(z)|^2 < \varepsilon \) for every \( z \in \mathbb{D} \setminus K \), then the operator \( C_{\phi,\psi} : A_{v,2} \to A_{w,2} \) is compact.

**Proof.** The idea is to use Proposition 2. Thus, fix a bounded sequence \( (f_n)_n \subset A_{v,2} \) such that \( (f_n)_n \) converges to zero uniformly on the compact subsets of \( \mathbb{D} \). We have to show that \( \|C_{\phi,\psi}f_n\|_{w,2} \to 0 \) if \( n \to \infty \). However,
\[ \|C_{\phi,\psi}f_n\|_{w,2}^2 = \int_{\mathbb{D}} |f_n(\phi(z))|^2|\psi(z)|^2 w(z) \, dA(z) \]
\[ \leq \int_{\mathbb{D}_r} |f_n(\phi(z))|^2|\psi(z)|^2 w(z) \, dA(z) \]
\[ + \int_{\mathbb{D} \setminus \mathbb{D}_r} |f_n(\phi(z))|^2 \frac{w(z)}{v(\phi(z))} |\psi(z)|^2 \, v(\phi(z)) \, dA(z) \]
\[ \leq K \sup_{|z| \leq r} |f_n(\phi(z))| + \sup_{|z| > r} \frac{w(z)}{v(\phi(z))} \|f_n\|_{v,2}^2, \]
where \( \mathbb{D}_r = \{ z \in \mathbb{D} ; |z| \leq r \} \). Finally, the claim follows. \( \square \)

**Proposition 5.** Let \( v \) be a weight as defined above such that
\[ M := \sup_{z \in \mathbb{D}} \sup_{a \in \mathbb{D}} \frac{v(z)}{|\nu_a(z)|} < \infty. \]
If the operator \( C_{\phi,\psi} : A_{v,2} \to A_{w,2} \) is compact, then
\[ \lim_{|a| \to 1} \int_{\mathbb{D}} |\sigma'_a(\phi(z))|^2 \frac{|\psi(z)|^2}{w(z)} \, dA(z) = 0. \]

**Proof.** Consider the function
\[ f_a(z) = \frac{-\sigma'_a(z)}{\nu(a \overline{z})^2} \]
for every \( z \in \mathbb{D} \).
Then $\|f_a\|_{2,2}^2 \leq M$ for every $a \in \mathbb{D}$ and $f_a \to 0$ uniformly on the compact subsets of $\mathbb{D}$. Hence, by Proposition 2

$$\|C_{\phi,\psi}f_a\|_{w,2}^2 = \int_{\mathbb{D}} \left| \frac{\sigma'_a(\phi(z))}{\nu_a(\phi(z))} w(z) |\psi(z)| \right|^2 dA(z) \to 0$$

if $|a| \to 1$. Hence the claim follows. \hfill \Box

**References**


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