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Majorization for certain classes
of meromorphic functions defined
by integral operator

Abstract. Here we investigate a majorization problem involving starlike
meromorphic functions of complex order belonging to a certain subclass of
meromorphic univalent functions defined by an integral operator introduced
recently by Lashin.

1. Introduction and preliminaries. Let \( f(z) \) and \( g(z) \) be analytic in
the open unit disk
\[
\Delta = \{ z \in \mathbb{C} \text{ and } |z| < 1 \}.
\]

For analytic functions \( f(z) \) and \( g(z) \) in \( \Delta \), we say that \( f(z) \) is majorized
by \( g(z) \) in \( \Delta \) (see [9]) and write
\[
f(z) \ll g(z) \quad (z \in \Delta),
\]
if there exists a function \( \phi(z) \), analytic in \( \Delta \) such that \( |\phi(z)| \leq 1 \), and
\[
f(z) = \phi(z)g(z) \quad (z \in \Delta).
\]

Let \( \Sigma \) denote the class of meromorphic functions of the form
\[
f(z) = \frac{1}{z} + \sum_{k=1}^{\infty} a_k z^k,
\]

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which are analytic and univalent in the punctured unit disk
\( \Delta^* := \{ z \in \mathbb{C} : 0 < |z| < 1 \} := \Delta \setminus \{0\} \)
with a simple pole at the origin.

For functions \( f_j \in \Sigma \) given by

\[
f_j(z) = \frac{1}{z} + \sum_{k=1}^{\infty} a_{k,j} z^k \quad (j = 1, 2; z \in \Delta^*),
\]
we define the Hadamard product (or convolution) of \( f_1 \) and \( f_2 \) by

\[
(f_1 \ast f_2)(z) = \frac{1}{z} + \sum_{k=1}^{\infty} a_{k,1} a_{k,2} z^k = (f_2 \ast f_1)(z).
\]

Analogously to the operators defined by Jung, Kim and Srivastava [7] on the normalized analytic functions, Lashin [8] introduced the following integral operators
\[
P_{\beta}^\alpha : \Sigma \rightarrow \Sigma
\]
defined by

\[
P_{\beta}^\alpha = P_{\beta}^\alpha f(z) = \frac{\beta^\alpha}{\Gamma(\alpha)} \frac{1}{z^{\beta+1}} \int_0^z \left( \log \frac{z}{t} \right)^{\alpha-1} f(t) dt
\]
\((\alpha > 0, \beta > 0; z \in \Delta^*)\), where \( \Gamma(\alpha) \) is the familiar Gamma function.

Using the integral representation of the Gamma function and (1.4), it can be easily shown that

\[
P_{\beta}^\alpha f(z) = \frac{1}{z} + \sum_{k=1}^{\infty} \left( \frac{\beta}{\beta+k+1} \right)^\alpha a_k z^k, \quad (\alpha > 0, \beta > 0; z \in \Delta^*).
\]

Obviously

\[
P_{\beta}^1 f(z) := J_{\beta}.
\]

The operator
\[
J_{\beta} : \Sigma \rightarrow \Sigma
\]
has also been studied by Lashin [8].

It is easy to verify that (see [8]),

\[
z(P_{\beta}^\alpha f(z))' = \beta P_{\beta}^{\alpha-1} f(z) - (\beta + 1) P_{\beta}^\alpha f(z).
\]

**Definition 1.1.** A function \( f(z) \in \Sigma \) is said to be in the class \( S_{\beta}^{\alpha,j}(\gamma) \) of meromorphic functions of complex order \( \gamma \neq 0 \) in \( \Delta \) if and only if

\[
\Re \left\{ 1 - \frac{1}{\gamma} \left( \frac{z(P_{\beta}^\alpha f(z))^{(j+1)}}{(P_{\beta}^\alpha f(z))^{(j)}} + j + 1 \right) \right\} > 0
\]
\((z \in \Delta, j \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \alpha > 0, \beta > 0, \gamma \in \mathbb{C}\setminus\{0\})\).
Clearly, we have the following relationships:

(i) \( S_{0}^{0,0}(\gamma) = S(\gamma) \quad (\gamma \in \mathbb{C} \setminus \{0\}) \),

(ii) \( S_{0}^{0,0}(1 - \eta) = S^{*}(\eta) \quad (0 \leq \eta < 1) \).

The classes \( S(\gamma) \) and \( S^{*}(\eta) \) are said to be classes of meromorphic star-like univalent functions of complex order \( \gamma \neq 0 \) and meromorphic starlike univalent functions of order \( \eta \in \mathbb{R} \) such that \( 0 \leq \eta < 1 \) in \( \Delta^{*} \).

A majorization problem for the normalized classes of starlike functions has been investigated by Altinas et al. [1] and MacGregor [9]. In the recent paper Goyal and Goswami [2] generalized these results for the class of multivalent functions, using fractional derivatives operators. Further, Goyal et al. [3], Goswami and Wang [4], Goswami [5], Goswami et al. [6] studied majorization property for different classes. In this paper, we will study majorization properties for the class of meromorphic functions using integral operator \( P^{\beta}_{\alpha} \).

2. Majorization problems for the class \( S_{\beta}^{0,j}(\gamma) \).

**Theorem 2.1.** Let the function \( f \in \Sigma \) and suppose that \( g \in S_{\beta}^{0,j}(\gamma) \). If \( (P^{\beta}_{\alpha}f(z))^{(j)} \) is majorized by \( (P^{\beta}_{\alpha}g(z))^{(j)} \) in \( \Delta^{*} \), then

\[
(|(P^{\beta}_{\alpha}^{-1}f(z))^{(j)}| \leq |(P^{\beta}_{\alpha}^{-1}g(z))^{(j)}| \text{ for } |z| \leq r_{1}(\beta, \gamma),}
\]

where

\[
r_{1}(\beta, \gamma) = \frac{k_{1} - \sqrt{k_{1}^{2} - 4\beta|\beta + 2\gamma|}}{2|\beta + 2\gamma|}
\]

and

\[
k_{1} = \beta + 2 + |\beta + 2\gamma|, (\beta > 0, j \in \mathbb{N}_{0}, \gamma \in \mathbb{C} \setminus \{0\}).
\]

**Proof.** Since \( g \in S_{\beta}^{0,j}(\gamma) \), we find from (2.1) that if

\[
h_{1}(z) = 1 - \frac{1}{\gamma} \left( z(P^{\beta}_{\alpha}g(z))^{(j+1)}(P^{\beta}_{\alpha}g(z))^{(j)} + j + 1 \right)
\]

(\( \alpha, \beta > 0, \gamma \in \mathbb{C} \setminus \{0\}, j \in \mathbb{N}_{0} \)), then \( \Re\{h_{1}(z)\} > 0 \) (\( z \in \Delta \)) and

\[
h_{1}(z) = \frac{1 + w(z)}{1 - w(z)} \quad (w \in \mathcal{P}),
\]

where \( \mathcal{P} \) denotes the well-known class of bounded analytic functions in \( \Delta \) and \( w(z) = c_{1}z + c_{2}z^{2} + \ldots \) satisfies the conditions

\[
w(0) = 0 \text{ and } |w(z)| \leq |z| \text{ (} z \in \Delta \).
\]
Making use of (2.3) and (2.4), we get

\[
(2.5) \quad \frac{z(P_\beta^\alpha g(z))^{(j+1)}}{P_\beta^\alpha g(z)^{(j)}} = \frac{(1+j-2\gamma)w(z) - (1+j)}{1-w(z)}.
\]

By the principle of mathematical induction, and (1.11), we easily get

\[
(2.6) \quad z(P_\beta^\alpha g(z))^{(j+1)} = \beta(P_\beta^{\alpha-1} g(z))^{(j)} - (\beta + j + 1)(P_\beta^\alpha g(z))^{(j)},
\]

\((\alpha > 1, \beta > 0; z \in \Delta^*)\). Now using (2.6) in (2.5), we find that

\[
\frac{\beta(P_\beta^{\alpha-1} g(z))^{(j)}}{P_\beta^\alpha g(z)^{(j)}} = (\beta + j + 1) + \frac{(1+j-2\gamma)w(z) - (1+j)}{1-w(z)}
\]

\[
= \frac{\beta - (\beta + 2\gamma)w(z)}{1-w(z)}
\]

or

\[
(2.7) \quad (P_\beta^\alpha g(z))^{(j)} = \frac{\beta(1-w(z))}{\beta - (\beta + 2\gamma)w(z)} (P_\beta^{\alpha-1} g(z))^{(j)}.
\]

Since \(|w(z)| \leq |z| (z \in \Delta)\), the formula (2.6) yields

\[
(2.8) \quad \left| (P_\beta^\alpha g(z))^{(j)} \right| \leq \frac{\beta[1 + |z|]}{\beta - |z|z} \left| (P_\beta^{\alpha-1} g(z))^{(j)} \right|.
\]

Next since \((P_\beta^\alpha f(z))^{(j)}\) is majorized by \((P_\beta^\alpha g(z))^{(j)}\) in the unit disk \(\Delta^*\), from (1.3), we have

\[
(P_\beta^\alpha f(z))^{(j)} = \phi(z)(P_\beta^\alpha g(z))^{(j)}.
\]

Differentiating it with respect to \(z\) and multiplying by \(z\), we get

\[
z(P_\beta^\alpha f(z))^{(j+1)} = z\varphi'(z)(P_\beta^\alpha g(z))^{(j)} + z\varphi(z)(P_\beta^\alpha g(z))^{(j+1)}.
\]

Using (2.7), in the above equality, it yields

\[
(2.9) \quad (P_\beta^{\alpha-1} f(z))^{(j)} = \frac{z\varphi'(z)}{\beta}(P_\beta^\alpha g(z))^{(j)} + \varphi(z)(P_\beta^{\alpha-1} g(z))^{(j)}.
\]

Thus, nothing that \(\varphi \in \mathcal{P}\) satisfies the inequality (see, e.g. Nehari [6])

\[
(2.10) \quad |\varphi'(z)| \leq \frac{1-|\varphi(z)|^2}{1-|z|^2}
\]

and making use of (2.8) and (2.10) in (2.9), we get

\[
\left| (P_\beta^{\alpha-1} f(z))^{(j)} \right|
\]

\[
\leq \left( |\varphi(z)| + \frac{1-|\varphi(z)|^2}{1-|z|^2} \frac{|z|}{|\beta - |2\gamma + \beta||z||z|} \right) \left| (P_\beta^{\alpha-1} g(z))^{(j)} \right|,
\]

which upon setting

\[
|z| = r \text{ and } |\varphi(z)| = \rho \quad (0 \leq \rho \leq 1),
\]
Corollary 2.2. Let the function $f \in S_{\beta}^{\alpha - 1}$, where $\alpha = 2$ and $\beta = 2$. The function $\Theta(\rho) = -r\rho^2 + (1-r)(\beta - |2\gamma + \beta|\rho + r$ takes its maximum value at $\rho = 1$, with $r_2 = r_2(\beta, \gamma)$, where $r_2(\beta, \gamma)$ is given by equation (2.2). Furthermore, if $0 \leq \rho \leq r_2(\beta, \gamma)$, then the function $\theta(\rho)$ defined by

$$\theta(\rho) = -\sigma\rho^2 + (1 - \sigma)(\beta - |2\gamma + \beta|\rho + \sigma$$

is an increasing function on the interval $0 \leq \rho \leq 1$, so that $0 \leq \rho \leq 1$; $0 \leq \sigma \leq r_1(\beta, \gamma)$. Hence upon setting $\rho = 1$ in (2.14), we conclude that (2.1) of Theorem 2.1 holds true for $|z| \leq r_1(\beta, \gamma)$, where $r_1(\beta, \gamma)$ is given by (2.2). This completes the proof of Theorem 2.1. □

Setting $\alpha = 1$ in Theorem 2.1, we get

**Corollary 2.1.** Let the function $f \in \Sigma$ and suppose that $g \in S_{\beta}^{1,1}(\gamma)$. If $(J_\beta f(z))^{(j)}$ is majorized by $(J_\beta g(z))^{(j)}$ in $\Delta^*$, then

$$|(f(z))^{(j)}| \leq |(g(z))^{(j)}| \text{ for } |z| \leq r_2(\beta, \gamma),$$

where

$$r_2(\beta, \gamma) = \frac{k_2 - \sqrt{k_2^2 - 4\beta|\beta + 2\gamma|}}{2|\beta + 2\gamma|}$$

and

$$k_2 = \beta + 2 + |\beta + 2\gamma|, \text{ (}\beta > 0, j \in \mathbb{N}_0, \gamma \in \mathbb{C}\{0\}).$$

Further putting $\beta = 1$ and $\gamma = 1 - \eta, j = 0$ in Corollary 2.1, we get

**Corollary 2.2.** Let the function $f \in \Sigma$ and suppose that $g \in S_1^{1,0}(1 - \eta)$. If $(J_1 f(z))$ is majorized by $(J_1 g(z))$ in $\Delta^*$, then

$$|f(z)| \leq |g(z)| \text{ for } |z| \leq r_3,$$

where

$$r_3 = \frac{3 - \eta - \sqrt{\eta^2 - 4\eta + 6}}{3 - \eta}.$$
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