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Inclusion and neighborhood properties of certain subclasses of $p$-valent functions of complex order defined by convolution

Abstract. In this paper we introduce and investigate three new subclasses of $p$-valent analytic functions by using the linear operator $D^m_{\lambda,p}(f * g)(z)$. The various results obtained here for each of these function classes include coefficient bounds, distortion inequalities and associated inclusion relations for $(n,\theta)$-neighborhoods of subclasses of analytic and multivalent functions with negative coefficients, which are defined by means of a non-homogenous differential equation.

1. Introduction. Let $A_p(n)$ denote the class of functions of the form

\begin{equation}
\label{eqn1.1}
f(z) = z^p + \sum_{k=n}^{\infty} a_k z^k \quad (n > p; \; p, n \in \mathbb{N} = \{1, 2, \ldots\}),
\end{equation}

which are analytic and $p$-valent in the open unit disk $U = \{z : |z| < 1\}$. The Hadamard product (or convolution) of the functions $f(z)$ given by (1.1), and $g(z) \in A_p(n)$ given by

\begin{equation}
\label{eqn1.2}
g(z) = z^p + \sum_{k=n}^{\infty} b_k z^k \quad (n > p; \; p, n \in \mathbb{N})
\end{equation}

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is defined by

\[ (f \ast g)(z) = z^p + \sum_{k=n}^{\infty} a_k b_k z^k = (g \ast f)(z). \]  

For functions \( f, g \in A_p(n) \), we define the linear operator \( D_{\lambda,p}^m : A_p(n) \rightarrow A_p(n) (\lambda \geq 0; p, n \in \mathbb{N}; m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}) \) by

\[ D_{\lambda,p}^0 (f \ast g)(z) = (f \ast g)(z), \]

(1.4) 

\[ D_{\lambda,p}^1 (f \ast g)(z) = D_{\lambda,p} (f \ast g)(z) = (1 - \lambda)(f \ast g)(z) + \frac{\lambda}{p} (f \ast g)'(z) \]

and (in general)

\[ D_{\lambda,p}^m (f \ast g)(z) = D_{\lambda,p} (D_{\lambda,p}^{m-1} (f \ast g)(z)) \]

\[ = (1 - \lambda) D_{\lambda,p}^{m-1} (f \ast g)(z) + \frac{\lambda z}{p} \left( D_{\lambda,p}^{m-1} (f \ast g) \right)'(z) \]

(1.6) 

\[ = z^p + \sum_{k=n}^{\infty} \left[ \frac{p + \lambda (k-p)}{p} \right]^m a_k b_k z^k \]

(\( \lambda \geq 0; p, n \in \mathbb{N}; m \in \mathbb{N}_0; z \in U \)).

The operator \( D_{\lambda,1}^m (f \ast g)(z) = D_{\lambda}^m (f \ast g)(z) \) was introduced by Aouf and Seoudy [6].

We note that

(i) for \( \lambda = 1 \) and \( b_k = 1 \) (or \( g(z) = \frac{z^n}{1-z} \)), \( D_{\lambda,p}^m (f \ast g)(z) = D_{\lambda}^m f(z) \), where the operator \( D_{\lambda}^m \) is the \( p \)-valent Salagean operator introduced and studied by Aouf and Mostafa [5], Kamali and Orhan [11] and Orhan and Kiziltunc [13];

(ii) for \( b_k = 1 \) (or \( g(z) = \frac{z^n}{1-z} \)), \( D_{\lambda,p}^m (f \ast g)(z) = D_{\lambda,p}^m f(z) \), where the operator \( D_{\lambda,p}^m \) was introduced and studied by El-Ashwah and Aouf [8].

For a function \( f(z) \in A_p(n) \), we have

\[ (D_{\lambda,p}^m (f \ast g)(z))^{(q)} = \delta(p, q) z^{p-q} + \sum_{k=n}^{\infty} \delta(k, q) \left[ \frac{p + \lambda (k-q)}{p} \right]^m a_k b_k z^{k-q}, \]

(1.7) 

(\( \lambda \geq 0; p, n \in \mathbb{N}; q, m \in \mathbb{N}_0; p > q; z \in U \)), where

\[ \delta(p, q) = \begin{cases} 1, & (q = 0), \\ p(p-1) \ldots (p-q+1), & (q \neq 0). \end{cases} \]

(1.8) 

We denote by \( T_p(n) \) the subclass of \( A_p(n) \) consisting of functions of the form

\[ f(z) = z^p - \sum_{k=n}^{\infty} a_k z^k \quad (n > p; a_k \geq 0; p, n \in \mathbb{N}). \]

(1.9)
For a given function $g(z) \in A_{p}(n)$ defined by

\begin{equation}
(1.10) \quad g(z) = z^p + \sum_{k=n}^{\infty} b_k z^k \quad (b_k > 0; \ n > p; \ p, n \in \mathbb{N}),
\end{equation}

we now introduce a new subclass $C_p^q(g(z); n, m, p, \lambda, \beta, b)$ of the class $T_{p}(n)$ of $p$-valently analytic functions, which consists of functions $f(z) \in T_{p}(n)$ satisfying the inequality

\begin{equation}
(1.11) \quad \left| \frac{1}{b} \left[ \sum_{k=n+p}^{\infty} \left( \frac{k-p}{p+b} \right)^{\mu} z^k; n + p, 0, p, \lambda, 1, b \right] \right| < \beta
\end{equation}

($\lambda \geq 0; \ p, n \in \mathbb{N}; \ q, m \in \mathbb{N}_0; \ 0 \leq \gamma \leq 1; \ p > q; \ 0 < \beta \leq 1; \ b \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}; \ z \in U$).

We note that

1. $C_{0}^{q}(g(z); n, 0, p, \lambda, 1, b) = S_{q}(p, n, b, q)$ (Prajapat et al. [14]);
2. $C_{1}^{q}(z^p + \sum_{k=n+p}^{\infty} \left( \frac{k-p}{p+b} \right)^{\mu} z^k; n + p, 0, p, \lambda, 1, b) = S_{n,q}^{p}(\mu, r, \gamma, b)$ ($\mu \geq 0$ and $r \in \mathbb{N}_0$) (Srivastava et al. [18]);
3. $C_{0}^{q}(z^p + \sum_{k=n+p}^{\infty} \left\{ 1 + \frac{\gamma(k-p)}{p^{r+1}} \right\} z^k; n + p, 0, p, \lambda, 1, b) = H_{n,q}^{p}(b, \zeta, \eta)$ ($\zeta, \eta, \gamma, r \in \mathbb{R}; \ \zeta \geq 0, \ \eta \geq 0, \ r \geq 0$) (Mahzoon and Latha [12]);
4. $C_{1}^{q}(z^p + \sum_{k=n+p}^{\infty} \left( \frac{k-p}{p+b} \right) z^k; n + p, 0, p, \lambda, 1, b) = S_{q,n,p}(\gamma, \beta, b)$ (Altintaş et al. [2]);
5. $C_{0}^{q}(z^p + \sum_{k=n+p}^{\infty} \left( \frac{\mu+k-1}{k-p} \right) z^k; n + p, 0, p, \lambda, 1, b) = H_{n,q}^{p}(\mu, b)$ ($\mu \geq 0$) (Raina and Srivastava [15]);
6. $C_{1}^{q}(z^p + \sum_{k=n+p}^{\infty} \left( \frac{k-p}{p+b} \right) z^k; n + p, 0, p, \lambda, p - q - \alpha, 1) = C_{1}^{q}(z^p + \sum_{k=n+p}^{\infty} \left( \frac{\mu+k-1}{k-p} \right) z^k; n + p, 0, p, \lambda, 1, p - q - \alpha) = T_{n}(p, q, \alpha, \gamma)$ ($0 \leq \alpha < p - q$) (Altintaş [1]);
7. $C_{0}^{q}(g(z); n, 0, p, \lambda, \beta, b) = C_{0}^{q}(g(z); n, p, \beta, b)$ (Srivastava and Orhan [17] and Aouf [4]);
8. $C_{0}^{q}(z^p + \sum_{k=n+p}^{\infty} \left( \frac{k-p}{p+b} \right) z^k; n + p, 0, p, \lambda, 1, \beta, b) = T_{n-p}(p, m, \lambda, b, \beta)$ (El-Ashwah and Aouf [8]).
Also, we note that

\[ C^q_{\gamma} \left( \frac{z^p + \sum_{k=n}^{\infty} \left[ \frac{p+\ell+(k-p)}{p+q} \right]^s z^k}{n,0,p,\lambda,\beta,b} \right) \]

\[ = C^q_{\gamma}(\lambda,\ell,s;n,p,\beta,b) \]

\[ = \left\{ f \in T_p(n) : \left| \frac{\left[ \frac{p+\ell+(k-p)}{p+q} \right]^s z^k}{n,0,p,\lambda,\beta,b} \right| < \beta, \right. \]

\[ p,n \in \mathbb{N}; q,s \in \mathbb{N}_0; 0 \leq \gamma \leq 1; p > q; 0 < \beta \leq 1; \]

\[ \ell,\zeta \geq 0; b \in \mathbb{C}^*; z \in U \}, \]

where \( I^q_{\lambda}(\zeta,\ell) \) is an extended multiplier transformation (see Cătăs [7]), defined by

\[ I^q_{\lambda}(\zeta,\ell)f(z) = z^p - \sum_{k=n}^{\infty} \left[ \frac{p+\ell+(k-p)}{p+q} \right]^s a_k z^k \]

\((\ell,\zeta \geq 0; p \in \mathbb{N} \text{ and } s \in \mathbb{N}_0);\)

\[ C^q_{\gamma} \left( \frac{z^p}{1-z} ; n,m,p,\lambda,\beta,b \right) \]

\[ = C^q_{\gamma}(n,m,p,\lambda,\beta,b) \]

\[ = \left\{ f \in T_p(n) : \left| \frac{\left[ \frac{p+\ell+(k-p)}{p+q} \right]^s z^k}{n,0,p,\lambda,\beta,b} \right| < \beta, \right. \]

\[ p,n \in \mathbb{N}; q,m \in \mathbb{N}_0; b \in \mathbb{C}^*; 0 \leq \gamma \leq 1; p > q; \]

\[ 0 < \beta \leq 1; \lambda \geq 0 \}. \]

Also let \( R^q_{\lambda}(g;z); n,m,p,\lambda,\beta,b) \) denote the subclass \( T_p(n) \) consisting of functions \( f(z) \) of the form (1.9) and the function \( g(z) \) of the form (1.10) which satisfy the following inequality:

\[ \left| f \left( (1 - \gamma) \frac{(D^m_{\lambda,\beta}g(z))^{(q)}}{z^{p-q}} + \gamma \frac{(D^m_{\lambda,\beta}g(z))^{(q)}}{(p-q)z^{p-q-1}} - \delta(p,q) \right) \right| < \beta \]

\((\lambda \geq 0; p,n \in \mathbb{N}; q,m \in \mathbb{N}_0; 0 \leq \gamma \leq 1; p > q; 0 < \beta \leq 1; b \in \mathbb{C}^*; z \in U).\)

In this paper we shall study some properties of the classes \( C^q_{\gamma}(g;z); n,m, p, \lambda, \beta, b) \) and \( R^q_{\lambda}(g;z); n,m,p,\lambda,\beta,b) \) and derive several results for functions in the subclass \( H^q_{\lambda}(g;z); n,m,p,\lambda,\beta,b,\alpha) \) of the function class \( T_p(n) \), which is defined as follows:

A function \( f(z) \in T_p(n) \) is said to belong to the class \( H^q_{\lambda}(g;z); n,m,p,\lambda, \beta,b,\alpha) \) if \( w = f(z) \) satisfies the following non-homogenous Cauchy–Euler
Theorem 1. Let the function $C$ be defined by

$$d^{n+1}w \over dz^{q+2} + 2(1 + \alpha)z \cdot d^{n+1}w \over dz^{q+1} + \alpha(1 + \alpha) \cdot d^n w \over dz^q = (p - q + \alpha)(p - q + \alpha + 1) \cdot d^k w \over dz^q,$$

where $k(z) \in C^n(g(z); n, m, p, \lambda, \beta, b)$ and $\alpha > q - p, \alpha \in R, p \in N, q \in N_0$.

2. Basic properties of the classes $C^q(g(z); n, m, p, \lambda, \beta, b)$ and $R^q(g(z); n, m, p, \lambda, \beta, b)$. We begin by proving a necessary and sufficient condition for a function belonging to the class $T_p(n)$ to be in the class $C^q(g(z); n, m, p, \lambda, \beta, b)$.

Theorem 1. Let the function $f(z) \in T_p(n)$ be defined by (1.9) and let $g(z)$ be defined by (1.10). Then $f(z)$ is in the class $C^q(g(z); n, m, p, \lambda, \beta, b)$ if and only if

$$\sum_{k=n}^{\infty} |k - p + \beta| [1 + \gamma(k - q - 1)] \left[ \frac{p+\lambda(k-p)}{p} \right]^m \delta(k, q) a_k b_k$$

$$\leq \beta |b| [1 + \gamma(p - q - 1)] \delta(p, q).$$

Proof. If the condition (2.1) holds true, we find from (1.9), (1.10) and (2.1) that

$$\left| z(D^{n+1}_{\lambda,p}(f * g)(z))^{(q+1)} + \gamma z^2(D^n_{\lambda,p}(f * g)(z))^{(q+2)} - (p - q) \left[ (1 - \gamma)(D^{n+1}_{\lambda,p}(f * g)(z))^{(q)} + \gamma z(D^n_{\lambda,p}(f * g)(z))^{(q+1)} \right] \right|$$

$$- \beta |b| \left[ (1 - \gamma)(D^{n+1}_{\lambda,p}(f * g)(z))^{(q)} + \gamma z(D^n_{\lambda,p}(f * g)(z))^{(q+1)} \right]$$

$$= \left| \delta(p, q + 1) z^{p-q} - \sum_{k=n}^{\infty} \left[ \frac{p+\lambda(k-p)}{p} \right]^m \delta(k, q + 1) a_k b_k z^{k-q} + \gamma \delta(p, q + 2) z^{p-q} - \sum_{k=n}^{\infty} \gamma \left[ \frac{p+\lambda(k-p)}{p} \right]^m \delta(k, q + 2) a_k b_k z^{k-q} - (p - q) \left[ (1 - \gamma) \delta(p, q) z^{p-q} - \sum_{k=n}^{\infty} (1 - \gamma) \left[ \frac{p+\lambda(k-p)}{p} \right]^m \delta(k, q) a_k b_k z^{k-q} + \gamma \delta(p, q + 1) z^{p-q} - \sum_{k=n}^{\infty} \gamma \left[ \frac{p+\lambda(k-p)}{p} \right]^m \delta(k, q + 1) a_k b_k z^{k-q} - \beta |b| \left[ (1 - \gamma) \delta(p, q) z^{p-q} - \sum_{k=n}^{\infty} (1 - \gamma) \left[ \frac{p+\lambda(k-p)}{p} \right]^m \delta(k, q) a_k b_k z^{k-q} + \gamma \delta(p, q + 1) z^{p-q} - \sum_{k=n}^{\infty} \gamma \left[ \frac{p+\lambda(k-p)}{p} \right]^m \delta(k, q + 1) a_k b_k z^{k-q} \right] \right|$$
Putting \( z \) and remains so for all \( p \). Theorem 2.

(2.2)

\[
\left[ \sum_{k=n}^{\infty} (k-p) [1 + \gamma(k-q-1)] \left[ \frac{p+\lambda(k-p)}{p} \right]^m \delta(k,q) a_k b_k z^{k-q} \right] - \beta |b| \left[ 1 + \gamma(p-q-1) \right] \delta(p,q) z^{p-q} \leq \leq \sum_{k=n}^{\infty} \left[ 1 + \gamma(k-q-1) \right] \left[ \frac{p+\lambda(k-p)}{p} \right]^m \delta(k,q) a_k b_k |z|^{k-p} \leq \leq \sum_{k=n}^{\infty} [k-p+\beta |b|] [1 + \gamma(k-q-1)] \left[ \frac{p+\lambda(k-p)}{p} \right]^m \delta(k,q) a_k b_k - \beta |b| [1 + \gamma(p-q-1)] \delta(p,q) \leq 0
\]

(\( z \in \partial U = \{ z : z \in \mathbb{C} \text{ and } |z| = 1 \} \)). Hence, by the maximum modulus theorem, \( f(z) \in C^2_{p}(g(z); n, p, \beta, b) \).

Conversely, let \( f(z) \in C^2_{p}(g(z); n, p, \beta, b) \) be given by (1.9) and \( g(z) \) be given by (1.10). Then from (1.7) and (1.11), we have

\[
\left( 1 \right) \left( 2.2 \right) \left( 3 \right) \left( 4 \right)
\]

(2.2)

Putting \( z = r \) (0 \( r < 1 \)) on the right-hand side of (2.2) and noting the fact that for \( r = 0 \), the resulting expression in the denominator is positive and remains so for all \( r \in (0, 1) \), the desired inequality (2.1) follows upon letting \( r \to 1^- \).

\[\square\]

**Theorem 2.** Let the function \( f(z) \in T_p(n) \) be defined by (1.9) and \( g(z) \) be defined by (1.10). Then \( f(z) \) is in the class \( R^2_p(g(z); n, m, p, \lambda, \beta, b) \) if and only if

\[
(2.3) \sum_{k=n}^{\infty} [p-q+\gamma(k-p)] \left[ \frac{p+\lambda(k-p)}{p} \right]^m \delta(k,q) a_k b_k \leq \beta |b| (p-q).
\]
Corollary 1. Let the function \( f(z) \in T_p(n) \) be given by (1.9) and \( g(z) \) be defined by (1.10). If \( f(z) \in \mathcal{C}_q^\delta(g(z); n, m, p, \lambda, \beta, b) \), then

\[
(2.4) \quad a_k \leq \frac{\beta |b| [1 + \gamma(p - q - 1)] \delta(p, q)}{[k - p + \beta |b|][1 + \gamma(k - q - 1)] \left[ \frac{p + \lambda(k - p)}{p} \right]^m} \delta(k, q)b_k
\]

\((k \geq n; \lambda \geq 0; 0 \leq \gamma \leq 1; 0 < \beta \leq 1; b \in \mathbb{C}^*; p, n \in \mathbb{N}; q, m \in \mathbb{N}_0)\).

The result is sharp for the function \( f(z) \) given by

\[
(2.5) \quad f(z) = z^p - \frac{\beta |b| [1 + \gamma(p - q - 1)] \delta(p, q)}{\delta(k, q)[k - p + \beta |b|][1 + \gamma(k - q - 1)] \left[ \frac{p + \lambda(k - p)}{p} \right]^m} b_k z^k
\]

\((k \geq n; \lambda \geq 0; 0 \leq \gamma \leq 1; 0 < \beta \leq 1; b \in \mathbb{C}^*; p, n \in \mathbb{N}; q, m \in \mathbb{N}_0)\).

We next prove the following growth and distortion property for the functions of the form (1.9) belonging to the class \( \mathcal{C}_q^\delta(g(z); n, m, p, \lambda, \beta, b) \).

Theorem 3. If a function \( f(z) \) defined by (1.9) is in the class \( \mathcal{C}_q^\delta(g(z); n, m, p, \lambda, \beta, b) \) and \( g(z) \) defined by (1.10). Then

\[
||f(z)|| - |z|^p
\]

\[
(2.6) \quad \leq \frac{\beta |b| [1 + \gamma(p - q - 1)] \delta(p, q)}{(n - p + \beta |b|)[1 + \gamma(n - q - 1)] \left[ \frac{p + \lambda(n - p)}{p} \right]^m} |z|^n
\]

\((\lambda \geq 0; p, n \in \mathbb{N}; q, m \in \mathbb{N}_0; 0 \leq \gamma \leq 1; n > p > q; 0 < \beta \leq 1; b \in \mathbb{C}^*; z \in U)\) and (in general)

\[
\left| \left| f^{(r)}(z) \right| - \delta(p, r) |z|^{p-r} \right|
\]

\[
(2.7) \quad \leq \frac{\beta |b| [1 + \gamma(p - q - 1)] (n - q)! \delta(p, q)}{(n - p + \beta |b|)(n - r)! [1 + \gamma(n - q - 1)] \left[ \frac{p + \lambda(n - p)}{p} \right]^m} |z|^{n-r}
\]

\((z \in U; p, n \in \mathbb{N}; n > p; m, q \in \mathbb{N}_0; r \leq q < r; p > \max(r, q); \lambda \geq 0)\). The result is sharp for the function \( f(z) \) given by

\[
(2.8) \quad f(z) = z^p \frac{\beta |b| [1 + \gamma(p - q - 1)] \delta(p, q)}{(n - p + \beta |b|)[1 + \gamma(n - q - 1)] \left[ \frac{p + \lambda(n - p)}{p} \right]^m} \delta(n, q)b_n z^n
\]

\((n > p; p, n \in \mathbb{N})\).

Proof. In view of Theorem 1, we have

\[
(n - p + \beta |b|)[1 + \gamma(n - q - 1)] \left[ \frac{p + \lambda(n - p)}{p} \right]^m \delta(n, q)b_n \sum_{k=n}^{\infty} a_k
\]

\[
\leq \sum_{k=n}^{\infty} \left[ k - p + \beta |b|\right][1 + \gamma(k - q - 1)] \left[ \frac{p + \lambda(k - p)}{p} \right]^m \delta(k, q)a_kb_k
\]

\[
\leq \beta |b| [1 + \gamma(p - q - 1)] \delta(p, q),
\]
which readily yields
\[
(2.9) \quad \sum_{k=n}^{\infty} a_k \leq \frac{\beta |b| [1 + \gamma(p - q - 1)] \delta(p, q)}{(n - p + \beta |b|) [1 + \gamma(n - q - 1)]} \left[ \frac{p + \lambda(n-p)}{p} \right]^m \delta(n, q)b_n.
\]

Also, (2.1) yields
\[
(2.10) \quad \sum_{k=n}^{\infty} k!a_k \leq \frac{\beta |b| [1 + \gamma(p - q - 1)] (n - q)!\delta(p, q)}{(n - p + \beta |b|) [1 + \gamma(n - q - 1)]} \left[ \frac{p + \lambda(n-p)}{p} \right]^m b_n.
\]

Now, by differentiating \( r \) times both sides of (1.9), we have
\[
(2.11) \quad f^{(r)}(z) = \delta(p, r)z^{p-r} - \sum_{k=n}^{\infty} \delta(k, r)a_k z^{k-r}
\]
for \((p, n \in \mathbb{N}; r \in \mathbb{N}_0; p > r)\).

Theorem 3 follows from (2.9), (2.10) and (2.11). Finally, it is easy to see that the bounds in Theorem 1 are attained for the function \( f(z) \) given by (2.8).

\[ \square \]

3. Properties of the class \( H^q_\gamma(g(z); n, m, p, \lambda, \beta, b, \alpha) \). Applying the results of Section 2, which are obtained for the function \( f(z) \) of the form (1.9) belonging to the class \( C^q_\gamma(g(z); n, m, p, \lambda, \beta, b) \), we now derive the corresponding results for the function \( f(z) \) belonging to the class \( H^q_\gamma(g(z); n, m, p, \lambda, \beta, b, \alpha) \).

**Theorem 4.** If a function \( f(z) \) is defined by (1.9) and \( g(z) \) is defined by (1.10), and \( f(z) \) is in the class \( H^q_\gamma(g(z); n, m, p, \lambda, \beta, b, \alpha) \). Then
\[
||f(z)|| - |z|^p
\]
\[
(3.1) \quad \leq \frac{\beta |b| [1 + \gamma(p - q - 1)] (p - q +\alpha)(p - q + \alpha + 1)\delta(p, q)}{(n - p + \beta |b|)(1 + \gamma(n - q - 1))} \left[ \frac{p + \lambda(n-p)}{p} \right]^m \delta(n, q)b_n |z|^n
\]
and (in general)
\[
||f^{(r)}(z)|| - |\delta(p, r)z^{p-r}|
\]
\[
(3.2) \quad \leq \frac{\beta |b| [1 + \gamma(p - q - 1)] (p - q +\alpha)(p - q + \alpha + 1)(n - q)!\delta(p, q)}{(n - p + \beta |b|)(1 + \gamma(n - q - 1))} \left[ \frac{p + \lambda(n-p)}{p} \right]^m \delta(n, q)(n-r)b_n |z|^{n-r}
\]
for \((p, n \in \mathbb{N}; m, q \in \mathbb{N}_0; r \leq q < p; p > \max(r, q); 0 \leq \gamma \leq 1; 0 < \beta \leq 1; b \in \mathbb{C}; \lambda > 0; z \in U)\). The results in (3.1) and (3.2) are sharp for the function \( f(z) \) given by
\[
(3.3) \quad f(z) = z^p - \frac{\beta |b| \delta(p, q) [1 + \gamma(p - q - 1)] (p - q +\alpha)(p - q + \alpha + 1)}{(n + \beta |b|)\delta(n+p, q)(1 + \gamma(n + p - q - 1))} (n + p - q + \alpha) b_{n+p} |z|^n.
\]
Assume that \( f(z) \in T_p(n) \) is given by (1.9) and \( g(z) \) given by (1.10). Also, let function \( k(z) \in C_n^q(g(z); n, m, p, \lambda, \beta, b) \), occurring in the non-homogenous differential equation (1.13) be of the form:

(3.4) \[ k(z) = z^p - \sum_{k=n}^\infty c_k z^k \]

(\( c_k \geq 0; \ n > p; \ p, n \in \mathbb{N} \)). Then, we readily find from (1.13) that

(3.5) \[ a_k = \frac{(p - q + \alpha)(p - q + \alpha + 1)}{(k - q + \alpha)(k - q + \alpha + 1)} c_k \]

(\( k \geq n; p, n \in \mathbb{N} \)), so that

(3.6) \[ f(z) = z^p - \sum_{k=n}^\infty a_k z^k = z^p - \sum_{k=n}^\infty \frac{(p - q + \alpha)(p - q + \alpha + 1)}{(k - q + \alpha)(k - q + \alpha + 1)} c_k z^k \]

(\( z \in U \)), and

(3.7) \[ |f(z)| - |z|^p \leq |z|^n \sum_{k=n}^\infty \frac{(p - q + \alpha)(p - q + \alpha + 1)}{(k - q + \alpha)(k - q + \alpha + 1)} c_k \]

(\( z \in U \)). Next, since \( k(z) \in C_n^q(g(z); n, m, p, \lambda, \beta, b) \), therefore, on using the assertion (2.4) of Corollary 1, we get the following coefficient inequality:

(3.8) \[ c_k \leq \frac{\beta |b| [1 + \gamma(p - q - 1)] \delta(p, q)}{(n - p + \beta |b|) [1 + \gamma(n - q - 1)] \left[ \frac{p + \lambda(n - p)}{p} \right]^m \delta(n, q) b_n} \]

(\( k \geq n; n > p > q; \lambda \geq 0; 0 \leq \gamma \leq 1; 0 < \beta \leq 1; p, n \in \mathbb{N}; q, m \in \mathbb{N}_0; \ b \in \mathbb{C}^* \)), which in conjunction with (3.6) and (3.7) yields

(3.9) \[ |f(z)| - |z|^p \leq \frac{\beta |b| [1 + \gamma(p - q - 1)] \delta(p, q)}{(n - p + \beta |b|) [1 + \gamma(n - q - 1)] \left[ \frac{p + \lambda(n - p)}{p} \right]^m \delta(n, q) b_n} |z|^n \]

\[ \times \sum_{k=n}^\infty \frac{1}{(k - q + \alpha)(k - q + \alpha + 1)} \]

(\( z \in U \)). Note that the following summation result holds

(3.10) \[ \sum_{k=n}^\infty \frac{1}{(k - q + \alpha)(k - q + \alpha + 1)} = \sum_{k=n}^\infty \left( \frac{1}{(k - q + \alpha)} - \frac{1}{(k - q + \alpha + 1)} \right) \]

\[ = \frac{1}{(n - q + \alpha)} \]

where \( \alpha \in \mathbb{R}^* = \mathbb{R} \setminus \{-n, -n - 1, \ldots\} \). The assertion (3.1) of Theorem 4 follows from (3.9) and (3.10), respectively. The assertion (3.2) of Theorem 4
can be established similarly by applying (2.10), (2.11), (3.5) and (3.10), respectively.

4. Inclusion relations involving \((n, \theta)\)-neighborhood for the classes \(C^\gamma_{n}(g(z); n, m, p, \lambda, \beta, b)\), \(R^\delta_{n}(g(z); n, m, p, \lambda, \beta, b)\) and \(H^\gamma_{n}(g(z); n, m, p, \lambda, \beta, b, \alpha)\). Following the works of Goodman [10], Ruscheweyh [16] and Altinta¸s [1] (see also [2], [3] and [9]), we define the \((n, \theta)\)-neighborhood of a function \(f(q)(z)\) when \(f \in T_p(n)\) by

\[
N^\theta_{n,p}(f(q), k(q)) = \left\{ k \in T_p(n) : k(z) = z^p - \sum_{k=n}^{\infty} c_k z^k \quad \text{and} \quad \sum_{k=n}^{\infty} k \delta(k, q) |a_k - c_k| \leq \theta \right\}.
\]

It follows from (4.1) that, if \(h(z) = z^p\) \((p \in \mathbb{N})\), then

\[
N^\theta_{n,p}(h(q)) = \left\{ k \in T_p(n) : k(z) = z^p - \sum_{k=n}^{\infty} c_k z^k \quad \text{and} \quad \sum_{k=n+p}^{\infty} k \delta(k, q) |c_k| \leq \theta \right\}.
\]

Next, we establish inclusion relationships for the function classes \(C^\gamma_{n}(g(z); n, m, p, \lambda, \beta, b)\) and \(R^\delta_{n}(g(z); n, m, p, \lambda, \beta, b)\), involving the \((n, \theta)\)-neighborhood \(N^\theta_{n,p}(h(q))\) defined by (4.3).

**Theorem 5.** If \(b_k \geq b_n \quad (k \geq n)\) and

\[
\theta = \frac{n \beta |b| [1 + \gamma(p - q - 1)] \delta(p, q)}{(n - p + \beta |b|) [1 + \gamma(n - q - 1)] \left[ \frac{p + \lambda(n - p)}{p} \right]^m b_n}
\]

\((p > |b|)\), then

\[
C^\gamma_{n}(g(z); n, m, p, \lambda, \beta, b) \subset N^\theta_{n,p}(h(q)).
\]

**Proof.** Let \(f \in C^\gamma_{n}(g(z); n, m, p, \lambda, \beta, b)\). Then, in view of the assertion (2.1) of Theorem 1, and the given condition that \(b_k \geq b_n \quad (k \geq n)\), we have

\[
(n - p + \beta |b|) [1 + \gamma(n - q - 1)] \left[ \frac{p + \lambda(n - p)}{p} \right]^m b_n \sum_{k=n}^{\infty} \delta(k, q) a_k \leq \beta |b| [1 + \gamma(p - q - 1)] \delta(p, q)
\]

so that

\[
\sum_{k=n}^{\infty} \delta(k, q) a_k \leq \frac{\beta |b| [1 + \gamma(p - q - 1)] \delta(p, q)}{(n - p + \beta |b|) [1 + \gamma(n - q - 1)] \left[ \frac{p + \lambda(n - p)}{p} \right]^m b_n}.
\]
On the other hand, we also find from (2.1) and (4.7) that
\[
\sum_{k=n}^{\infty} k\delta(k,q)a_k \leq \frac{\beta \, |b| \, [1 + \gamma(p - q - 1)] \delta(p,q)}{1 + \gamma(n - q - 1)} \left[ \frac{p + \lambda(n-p)}{p} \right]^m b_n + (p - \beta \, |b|) \sum_{k=n}^{\infty} \delta(k,q)a_k
\]
\[
\leq \frac{\beta \, |b| \, [1 + \gamma(p - q - 1)] \delta(p,q)}{1 + \gamma(n - q - 1)} \left[ \frac{p + \lambda(n-p)}{p} \right]^m b_n + \frac{(p - \beta \, |b|) \beta \, |b| \, [1 + \gamma(p - q - 1)] \delta(p,q)}{(n - p + \beta \, |b|) \, [1 + \gamma(n - q - 1)] \left[ \frac{p + \lambda(n-p)}{p} \right]^m b_n},
\]
that is
\[
(4.8) \sum_{k=n}^{\infty} \delta(k,q)ka_k \leq \frac{n\beta \, |b| \, \delta(p,q) \, [1 + \gamma(p - q - 1)]}{(n - p + \beta \, |b|) \, [1 + \gamma(n - q - 1)] \left[ \frac{p + \lambda(n-p)}{p} \right]^m b_n} = \theta.
\]
This evidently completes the proof of Theorem 5. \qed

**Remark 1.** (i) Taking \( g(z) = \frac{z^p}{1 - z} \), \( b = \gamma, \, m = 0 \) and \( \gamma = \lambda \) in Theorem 5, we obtain the result obtained by Altintas et al. [2, Theorem 2];
(ii) Taking \( g(z) = \frac{z^p}{1 - z} \), \( b = 1, \, \beta = p - \alpha \) \((0 \leq \alpha < p)\) and \( \gamma = \lambda \) in Theorem 5, we obtain the result obtained by Altintas [1, Theorem 2].

Putting \( g(z) = z^p + \sum_{k=n}^{\infty} \left[ \frac{p + \ell + \zeta(k-p)}{p + \ell + \zeta(n-p)} \right]^s z^k \, (\ell, \, \zeta \geq 0; \, s \in \mathbb{N}_0) \) and \( m = 0 \) in Theorem 5, we obtain the following corollary.

**Corollary 2.** If \( f(z) \in T_p(n) \) is in the class \( C^g_{\gamma}(\zeta, \ell, s; n, p, \beta, b) \), then
\[
C^g_{\gamma}(\zeta, \ell, s; n, p, \beta, b) \subset N^h_{n,p}(h^{(q)}),
\]
where \( h(z) \) is given by (4.2) and
\[
\theta = \frac{n\beta \, |b| \, [1 + \gamma(p - q - 1)] \delta(p,q)}{(n - p + \beta \, |b|) \, [1 + \gamma(n - q - 1)] \left[ \frac{p + \ell}{p + \ell + \zeta(n-p)} \right]^s}.
\]

Putting \( g(z) = z^p + \sum_{k=n}^{\infty} \left[ \frac{p + \zeta(k-p)}{p + \zeta(n-p)} \right]^s z^k \, (\zeta \geq 0; \, s \in \mathbb{N}_0) \) and \( m = 0 \) in Theorem 5, we obtain the following corollary.

**Corollary 3.** If \( f(z) \in T_p(n) \) is in the class \( C^g_{\beta}(\zeta, s; n, p, \beta, b) \), then
\[
C^g_{\beta}(\zeta, s; n, p, \beta, b) \subset N^h_{n,p}(h^{(q)}),
\]
where \( h(z) \) is given by (4.2) and
\[
\theta = \frac{n\beta \, |b| \, [1 + \gamma(p - q - 1)] \delta(p,q)}{(n - p + \beta \, |b|) \, [1 + \gamma(n - q - 1)] \left[ \frac{p}{p + \zeta(n-p)} \right]^s}.
\]
Theorem 6. If
\begin{equation}
\theta = \frac{n\beta |b| (p - q)}{|p - q + \gamma(n - p)| \left[p + \lambda(n-p)\right]^m b_n},
\end{equation}
then
\begin{equation}
R_q^\gamma(g(z); n, m, p, \lambda, \beta, b) \subset N^\theta_{n,p}(h^{(q)}).
\end{equation}

Proof. Let \( f \in R_q^\gamma(g(z); n, m, p, \lambda, \beta, b) \). Then, in view of the assertion (2.3) of Theorem 2, we have
\[
\sum_{k=n}^\infty |p - q + \gamma(k - p)| \left[p + \lambda(k-p)\right]^m \delta(k,q) a_k b_k 
\leq \beta |b| (p - q),
\]
so that
\begin{equation}
\sum_{k=n}^\infty \delta(k,q) a_k \leq \frac{n\beta |b| (p - q)}{|p - q + \gamma(n - p)| \left[p + \lambda(n-p)\right]^m b_n} = \theta,
\end{equation}
which by means of the definition (4.1), establishes the inclusion (4.10) asserted by Theorem 6.

Theorem 7. If \( f(z) \in T_p(n) \) is in the class \( H^\gamma_q(g(z); n, m, p, \lambda, \beta, b, \alpha) \), then
\begin{equation}
H^\gamma_q(g(z); n, m, p, \lambda, \beta, b, \alpha) \subset N^\theta_{n,p}(f^{(q)}, k^{(q)}),
\end{equation}
where \( k(z) \) is given by (1.13) and
\begin{equation}
\theta = \frac{n\beta \max(|b|, |c|)}{(n - p + \beta |b|) \left[1 + \gamma(n - q - 1)\right] \left[p + \lambda(n-p)\right]^m (n - q + \alpha)}.
\end{equation}

Proof. Suppose that \( f(z) \in H^\gamma_q(g(z); n, m, p, \lambda, \beta, b, \alpha) \). Then upon substituting from (3.5) into the following coefficient inequality
\begin{equation}
\sum_{k=n}^\infty k \delta(k,q) |a_k - c_k| \leq \sum_{k=n}^\infty k \delta(k,q) |c_k| + \sum_{k=n}^\infty k \delta(k,q) |a_k|
\end{equation}
\((a_k; c_k \geq 0)\), we readily obtain
Inclusion and neighborhood properties of certain subclasses...

\[ \sum_{k=n}^{\infty} k \delta(k, q) |a_k - c_k| \leq \sum_{k=n}^{\infty} k \delta(k, q) |c_k| \]

(4.15)

\[ + \sum_{k=n}^{\infty} k \delta(k, q) \frac{(p-q+\alpha)(p-q+\alpha+1)}{(k-q+\alpha)(k-q+\alpha+1)} |c_k|. \]

Now, since \( k(z) \in C^q_\gamma(g(z); n, m, p, \lambda, \beta, b) \) the second assertion (4.8) yields

(4.16)

\[ k \delta(k, q)c_k \leq \frac{n \beta |b| [1 + \gamma(p-q-1)]}{(n-p + \beta |b|) [1 + \gamma(n-q-1)] \left[ \frac{p+\lambda(n-p)}{p} \right]^m b_n}. \]

Finally, by making use of (4.8) as well as (4.16) on the right-hand side of (4.15), we find that

\[ \sum_{k=n+p}^{\infty} \delta(k, q)k |a_k - c_k| \]

\[ \leq \frac{n \beta |b| [1 + \gamma(p-q-1)] \delta(p, q)}{(n-p + \beta |b|) [1 + \gamma(n-q-1)] \left[ \frac{p+\lambda(n-p)}{p} \right]^m b_n} \]

\[ \times \left( 1 + \sum_{k=n+p}^{\infty} \frac{(p-q+\alpha)(p-q+\alpha+1)}{(k-q+\alpha)(k-q+\alpha+1)} \right) \]

\[ = \frac{n \beta |b| [1 + \gamma(p-q-1)] [n + (p-q+\alpha)(p-q+\alpha+2)] \delta(p, q)}{(n-p + \beta |b|) [1 + \gamma(n-q-1)] (n+p-q+\alpha) \left[ \frac{p+\lambda(n-p)}{p} \right]^m b_n} \]

\[ = \theta, \]

we conclude that \( f \in N_{n,p}^0 (f^{(q)}, k^{(a)}) \). This evidently completes the proof of Theorem 7. \( \square \)

5. Neighborhood for the classes \( C^q_\gamma(g(z); n, m, p, \lambda, \beta, b) \) and \( R^q_\gamma(g(z); n, m, p, \lambda, \beta, b) \). In this section we determine the neighborhood for the classes \( C^q_\gamma(g(z); n, m, p, \lambda, \beta, b) \) and \( R^q_\gamma(g(z); n, m, p, \lambda, \beta, b) \) which we define as follows. A function \( f \in T_p(n) \) is said to be in the class \( C^q_\gamma(g(z); n, m, p, \lambda, \beta, b) \) if there exists a function \( k \in C^q_\gamma(g(z); n, m, p, \lambda, \beta, b) \) such that

\[ \left| \frac{f(z)}{k(z)} - 1 \right| < p - \zeta \]

(5.1)

\( (z \in U; \ 0 \leq \zeta < p) \).
Theorem 8. If \( k(z) \in C^q_\gamma (g(z); n, m, p, \lambda, \beta, b) \) and
\[
\zeta = p - \frac{\theta(n-p+\beta |b|)[1+\gamma(n-q-1)]}{n} \left[ \frac{p+\lambda(n-p)}{p} \right]^m b_n
\]
then
\[
N_{n,p}^\theta (k^{(q)}) \subset C^q_\gamma (g(z); n, m, p, \lambda, \beta, b),
\]
where
\[
\theta \leq np \left[ \delta(n,q) - \beta |b| [1 + \gamma(p-q-1)] \delta(p,q) \right. \times \left. \{ (n - p + \beta |b|) [1 + \gamma(n-q-1)] \left[ \frac{p+\lambda(n-p)}{p} \right]^m b_n \}^{-1} \right].
\]

Proof. Suppose that \( f \in N_{n,p}^\theta (k^{(q)}) \), then we find from the definition (4.1) that
\[
\sum_{k=n}^{\infty} \delta(k,q) k |a_k - c_k| \leq \theta,
\]
which implies the coefficient inequality
\[
\sum_{k=n}^{\infty} |a_k - c_k| \leq \frac{\theta}{n \delta(n,q)}
\]
\((p > q; n, p \in \mathbb{N}, q \in \mathbb{N}_0)\). Next, since \( k(z) \in C^q_\gamma (g(z); n, m, p, \lambda, \beta, b) \), we have
\[
\sum_{k=n}^{\infty} c_k \leq \frac{\beta |b| [1 + \gamma(p-q-1)] \delta(p,q)}{(n - p + \beta |b|) [1 + \gamma(n-q-1)] \left[ \frac{p+\lambda(n-p)}{p} \right]^m \delta(n,q)b_n},
\]
so that
\[
\left| \frac{f(z)}{k(z)} - 1 \right| \leq \frac{\sum_{k=n}^{\infty} |a_k - c_k|}{1 - \sum_{k=n}^{\infty} |c_k|}
\]
\[
\leq \frac{\theta}{n \delta(n,q)} \frac{\beta |b|[1+(p-q-1)]\delta(p,q)}{(n-p+\beta |b|)[1+\gamma(n-q-1)] \left[ \frac{p+\lambda(n-p)}{p} \right]^m \delta(n,q)b_n}
\]
\[
= \frac{\theta(n-p+\beta |b|)[1+\gamma(n-q-1)] \left[ \frac{p+\lambda(n-p)}{p} \right]^m \delta(n,q)b_n - \beta |b| \delta(p,q)[1+\gamma(p-q-1)]}{n \left[ (n-p+\beta |b|)[1+\gamma(n-q-1)] \left[ \frac{p+\lambda(n-p)}{p} \right]^m \delta(n,q)b_n - \beta |b| \delta(p,q)[1+\gamma(p-q-1)] \right]}
\]
\[
= p - \zeta,
\]
because by the assumption
\[
\zeta = p - \frac{\theta(n-p+\beta|b|)[1+\gamma(n-q-1)]}{n}\left\{\frac{p+\lambda(n-p)}{p}\right\}^m b_n
\]
This implies that \( f \in C^\theta_{n,p}(g(z); n, m, p, \lambda, \beta, b) \).

Similarly, we can prove the following theorem.

**Theorem 9.** If \( k(z) \in R^\theta_{n,p}(g(z); n, m, p, \lambda, \beta, b) \) and
\[
(5.5) \quad \zeta = p - \frac{\theta[p - q + \gamma(n-p)]}{n}\left\{\frac{p+\lambda(n-p)}{p}\right\}^m \delta(n,q)b_n - \beta|b|(p-q),
\]
then
\[
(5.6) \quad N^\theta_{n,p}(k(z)) \subset R^\theta_{n,p}(g(z); n, m, p, \lambda, \beta, b),
\]
where
\[
\theta \leq np\left\{\delta(n,q) - \beta|b|(p-q)\right\}\left\{\frac{p+\lambda(n-p)}{p}\right\}^m b_n^{-1}.
\]

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