Some gap power series in multidimensional setting

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Abstract. We study extensions of classical theorems on gap power series of a complex variable to the multidimensional case.

1. Power series with Ostrowski gaps. Let

$$ f(z) = \sum_{0}^{\infty} Q_{j}(z), \quad \text{where} \quad Q_{j}(z) = \sum_{|\alpha|=j} a_{\alpha} z^{\alpha}, \quad \alpha \in \mathbb{Z}_{+}^{N}, $$

be a power series in $\mathbb{C}^{N}$, i.e. a series of homogeneous polynomials $Q_{j}$ of $N$ complex variables of degree $j$.

The set $\mathcal{D}$ given by the formula $\mathcal{D} := \{a \in \mathbb{C}^{N}; \text{the sequence (1.1) is convergent in a neighborhood of } a\}$ is called a domain of convergence of (1.1).

It is known that

$$ \mathcal{D} = \{z \in \mathbb{C}^{N}; \psi^{*}(z) < 1\}, $$

where

$$ \psi(z) := \limsup_{j \to \infty} \sqrt[|\alpha|]{|Q_{j}(z)|}, $$

and $\psi^{*}$ denotes the upper semicontinuous regularization of $\psi$.

2000 Mathematics Subject Classification. 30B10, 30B30, 30B40, 32A05, 32A07, 32A10, 32D15.

Key words and phrases. Plurisubharmonic functions, negligible sets in $\mathbb{C}^{N}$, power series, lacunary power series, multiple power series.
If \( \psi^* \) is finite, then it is plurisubharmonic and absolutely homogeneous (i.e. \( \psi^*(\lambda z) = |\lambda|\psi^*(z), \lambda \in \mathbb{C}, z \in \mathbb{C}^N \)). Therefore, the domain of convergence \( D \) is either empty, or it is a balanced (i.e. \( \lambda z \in D \) for all \( \lambda \in \mathbb{C} \) with \( |\lambda| \leq 1 \) and \( z \in D \)) domain of holomorphy. Every balanced domain of holomorphy is a domain of convergence of a series (1.1).

For every balanced domain \( D \) in \( \mathbb{C}^N \) there is a unique nonnegative function \( h \) (so-called Minkowski functional of \( D \)) such that \( h(\lambda z) = |\lambda|h(z) \) for all \( \lambda \in \mathbb{C} \) and \( z \in \mathbb{C}^N \), and \( D = \{ z \in \mathbb{C}^N; h(z) < 1 \} \). In particular, if \( D \) is a domain of convergence of (1.1), then \( h(z) \equiv \psi^*(z) \).

It is known that a balanced domain in \( \mathbb{C}^N \) is a domain of holomorphy if and only if its Minkowski functional \( h \) is an absolutely homogeneous plurisubharmonic function.

The number

\[
\rho := 1/\limsup_{j \to \infty} \sqrt[Q_j]{\|Q_j\|},
\]

where \( B := \{ z \in \mathbb{C}^N; \|z\| \leq 1 \} \), is called a radius of convergence of series (1.1) (with respect to a given norm \( \| \cdot \| \)).

If \( N = 1 \), then \( \psi(z) = \frac{|z|}{\rho} \) and \( D = \rho B \). If \( N \geq 2 \), then \( \rho B \subset D \) but, in general, \( D \neq \rho B \).

Series (1.1) is normally geometrically convergent in \( D \), i.e.

\[
\limsup_{j \to \infty} \sqrt[Q_j]{\|Q_j\|}_K < 1, \quad \limsup_{n \to \infty} \sqrt[f_n]{\|f - s_n\|}_K < 1,
\]

for all compact sets \( K \subset D \), where \( s_n := Q_0 + \cdots + Q_n \) is the \( n \)th partial sum of (1.1).

**Definition 1.1.** We say that a function \( f \) holomorphic in a neighborhood of a point \( z^0 \in \mathbb{C}^N \) possesses at the point \( z^0 \) Ostrowski’s gaps \( (m_k, n_k) \), if

1. \( m_k, n_k \) are natural numbers such that \( m_k < n_k < m_{k+1} \) \( (k \geq 1) \),

2. \( \lim_{j \to \infty, j \in I} \sqrt[Q_j]{\|Q_j\|} = 0 \), where \( B \) is the unit ball in \( \mathbb{C}^N \),

\[
Q_j(z) \equiv Q_j(f^{(z^0)})(z) := \sum_{|\alpha| = j} \frac{f^{(\alpha)}(z^0)}{\alpha!} z^\alpha = \frac{1}{j!} \left( \frac{d}{d\lambda} \right)^j f(z^0 + \lambda z)\big|_{\lambda=0},
\]

and \( I := \bigcup_{k=1}^{\infty} (m_k, n_k], (m_k, n_k] \) denoting the set of integers \( j \) with \( m_k < j \leq n_k \).

Observe that \( f_o(z) := \sum_{j \in I} Q_j(z - z^0) \) is an entire function such that the function \( g := f - f_o \) possesses Ostrowski’s gaps \( (m_k, n_k) \) at \( z^0 \) with \( Q_j^{(g,z^0)} = 0 \) for \( m_k < j \leq n_k, k \geq 1 \). Hence, a holomorphic function \( f \) possesses Ostrowski’s gaps \( (m_k, n_k) \) at a point \( z^0 \) if and only if there exists an entire function \( f_o \) such that \( Q_j^{(f-f_o,z^0)} = 0 \) for \( m_k < j \leq n_k, k \geq 1 \).
Moreover, the maximal domain of existence $G = G_f$ of $f$ is identical with the maximal domain of existence of $f - f_0$.

**Definition 1.2.** We say that a function $f$ holomorphic in a neighborhood of a point $z^0$ possesses Ostrowski’s gaps relative to a sequence of positive integers $\{n_k\}$, if $\{n_k\}$ is increasing and there exists a sequence of positive real numbers $\{q_k\}$ such that $q_k \to 0$ as $k \to \infty$ and $\lim_{j \to \infty, j \in I} \sqrt[1/n_k]{\|Q_j\|} = 0$, where $I := \bigcup_{k=1}^{\infty} ([n_k/n_{k+1}], n_k]$.

A function $f$ possesses Ostrowski’s gaps according to Definition 1.1 if and only if $f$ possesses Ostrowski’s gaps according to Definition 1.2.

Indeed, if the conditions of Definition 1.1 are satisfied, then it is sufficient to put $q_k := m_k/n_k$.

If the conditions of Definition 1.2 are satisfied, consider two cases. If $m_k := \liminf_{k \to \infty} q_k n_k$ is finite, then the function $f$ is entire, so that $f$ has Ostrowski’s gaps $([q_k n_k], n_k]$ for a suitable chosen increasing subsequence $k_l$ of positive integers.

We say that a compact subset $K$ of $\mathbb{C}^N$ is polynomially convex if $K$ is identical with its polynomially convex hull $\hat{K} := \{a \in \mathbb{C}^N; |P(a)| \leq \|P\|_K \text{ for every polynomial } P \text{ of } N \text{ complex variables}\}$. We say that an open set $\Omega$ in $\mathbb{C}^N$ is polynomially convex, if for every compact subset $K$ of $\Omega$ the polynomially convex hull $\hat{K}$ of $K$ is contained in $\Omega$.

The following theorem is known (see [7]). It is a multidimensional version of the classical Ostrowski’s Theorem (see Theorem 3.1.1 in [1]).

**Theorem 1.** If a holomorphic function $f$ possesses Ostrowski’s gaps $(m_k, n_k]$ at a point $z^o \in \mathbb{C}^N$, then the maximal domain of existence $G = G_f$ of $f$ is one-sheeted and polynomially convex. Moreover, for every compact subset $K$ of $G$ we have

$$\limsup_{k \to \infty} \|f - s_{n_k}\|_K^{1/n_k} < 1,$$

where

$$s_n(z) \equiv s_n(f(z^o))(z) = \sum_{j=0}^{n} Q_j(f,z^o)(z - z^o)$$

is the $n$th partial sum of the Taylor series development of $f$ around $z^o$.

**Corollary 1.1.** If

$$f(z^o + z) = \sum_{k=1}^{\infty} Q_k(f,z^o)(z),$$

where $m_k/m_k+1 \to 0$ as $k \to \infty$, then $Q_j(f,z^o) = 0$ for $j \notin \{m_k\}$ so that $f$ has Ostrowski’s gaps $(m_k, n_k]$ with $n_k := m_{k+1} - 1$. Therefore, the maximal
domain of existence $G_f$ of $f$ is identical with the domain of convergence $D_f$ of the Taylor series development of $f$ around $z^0$, i.e.

$$G_f = D_f := \{ z \in \mathbb{C}^N : \psi^*(z - z^0) < 1 \},$$

where $\psi(z) := \limsup_{k \to \infty} \frac{\sqrt{m_k}}{Q(f(z^0))}$. 

The following result gives an $N$-dimensional version of W. Luh’s Theorem 1 in [4]. In particular, it says that if a function $f$ holomorphic in a domain $G$ in $\mathbb{C}^N$ possesses Ostrowski’s gaps at some point $z^0 \in G$, then $f$ possesses the same property at every other point $a$ of the maximal domain of existence of $f$.

**Theorem 2.** Let $f$ possess Ostrowski’s gaps $(m_k, n_k)$ at a point $z^0 \in \mathbb{C}^N$. Then

1°. $f$ possesses Ostrowski’s gaps $(m_k, \left[ \frac{n_k}{l} \right])$ at every point $a \in G_f$, where the sequence of natural numbers $\{k_l\}$ (independent of $a$) is chosen in such a way that $n_{k_l} \geq m_{k_l}l^2$ and $\left[ \frac{n_{k_l}}{l} \right] < m_{k_l+1}$ for $l \geq 1$;

2°. If $Q_j^{(f,z^0)} = 0$ for $m_k < j \leq n_k$, $k \geq 1$, then the sequence $\{ (s_{m_k}^{(f,z^0)})_j - (s_{m_k}^{(f,a)})_j \}$ converges to zero normally with order $n_k$ on $\mathbb{C}^N$, i.e.

$$\limsup_{k \to \infty} \left\| s_{m_k}^{(f,z^0)} - s_{m_k}^{(f,a)} \right\|_K^{1/n_k} < 1$$

for every compact set $K \subset \mathbb{C}^N$.

By 2° and Theorem 1 we get the following:

**Corollary 1.2.** If $f$ possesses ordinary Ostrowski’s gaps $(m_k, n_k)$ at a fixed point $z^0 \in G$, then

$$\limsup_{k \to \infty} \left\| f - s_{m_k}^{(f,a)} \right\|_K^{1/n_k} < 1$$

for every point $a \in G_f$ and every compact subset $K$ of $G_f$.

**Proof of Theorem 2.** 1°. Without loss of generality we may assume that $z^0 = 0$ and

$$Q_j^{(f,z^0)} = 0, \quad m_k < j \leq n_k, \quad k \geq 1.$$

Given a fixed point $a \in G_f$, we have

$$Q_j^{(f,a)}(z) = \frac{1}{2\pi i} \int_{|\lambda| = r} \frac{f(a + \lambda z) - s_{n_k}(a + \lambda z)}{\lambda^{j+1}} d\lambda,$$

1° In such a case we say that $f$ possesses ordinary Ostrowski’s gaps at $z^0$. 
By Theorem 1 there exist 
\[ M > (1.7) \]
\[ \| f - s_{m_{k}} \|_{\mathbb{B}(a,r)} \leq M \theta^{m_{k}}, \quad k \geq 1. \]

Therefore, by Cauchy inequalities,
\[ (1.8) \]
\[ \left\| Q_{j}^{(f,a)} \right\|_{B} \leq \frac{M}{r^{j}} \theta^{m_{k}}, \quad j > m_{k}, \quad k \geq 1. \]

Let \( \{ k_{l} \} \) be an increasing sequence of natural numbers such that
\[ m_{k_{l+1}} > \left[ \frac{n_{k_{l}}}{l} \right], \quad \frac{n_{k_{l}}}{m_{k_{l}}} \geq l^{2}, \quad l \geq 1. \]

By (1.8) we get
\[ (1.9) \]
\[ \left\| Q_{j}^{(f,a)} \right\|_{B}^{1/j} \leq \frac{M}{r} \theta^{n_{k_{l}}/j} \leq \frac{M}{l} \theta^{j}, \quad m_{k_{l}} < j \leq \left[ \frac{n_{k_{l}}}{l} \right], \quad l \geq 1. \]

The choice of the sequence \( \{ k_{l} \} \) does not depend on \( a \in G_{f} \). Therefore, \( f \) possesses Ostrowski’s gaps \( \left\{ m_{k_{l}}, \left[ \frac{n_{k_{l}}}{l} \right] \right\} \) at every point \( a \) of \( G_{f} \) (according to Definition 1.1). The proof of the case 1° is ended.

2°. Observe that for \( \| z - a \| \leq \frac{r}{2} \) we have
\[ (1.10) \]
\[ \left\| s_{m_{k}}^{(f,z^{o})} - s_{m_{k}}^{(f,a)} \right\|_{B(a,1/4)} \leq 2 M \theta^{n_{k}}, \quad k \geq 1. \]

By (1.7) and (1.9) we get
\[ (1.11) \]
\[ \left\| s_{m_{k}}^{(f,z^{o})} \right\|_{B(z^{o},r)} \leq \frac{2}{r} \log \left( 1 + \frac{\| z^{o} \|}{r} \right). \]

Put \( M := \| f \|_{B(z^{o},r),\mathbb{B}(a,r)} \) and \( c := \max\{\| z^{o} \|, \| a \|\} \). Then for \( z \in \mathbb{C}^{N} \)
\[ (1.12) \]
\[ u_{k}(z) := \frac{1}{n_{k}} \log \left| s_{m_{k}}^{(f,z^{o})} (z) - s_{m_{k}}^{(f,a)} (z) \right| \leq \frac{1}{n_{k}} \log [2 M (m_{k} + 1) + \frac{m_{k}}{n_{k}} \log \left( 1 + \frac{\| z \| + \| c \|}{r} \right)]. \]
It follows that the sequence of plurisubharmonic functions \( \{u_k\} \) is locally uniformly upper bounded in \( \mathbb{C}^N \), and

\[
    u(z) := \limsup_{k \to \infty} u_k(z) \leq 0, \quad z \in \mathbb{C}^n.
\]

Therefore, the plurisubharmonic function \( u^* = \text{const} \).

By (1.10) \( u_k(z) \leq \frac{1}{m_k} \log 2M + \log \theta \) for \( z \in \mathbb{B}(a, r), \ k \geq 1 \). Hence \( u^* \leq \log \theta \) in \( \mathbb{C}^N \) which ends the proof of 2º.

\[ \square \]

2. E. Fabry's Theorem. Now we shall present a multidimensional version of E. Fabry's Theorem (Theorem 2.2.1 in [1]). Let \( f \) be a function of \( N \) complex variables holomorphic in a neighborhood of 0 with a gap Taylor series development

(2.1)

\[
    f(z) = \sum_{k=1}^{\infty} Q_{mk}(z), \quad m_k < m_{k+1}.
\]

Put \( \psi(z) := \limsup_{k \to \infty} m_k \sqrt{|Q_{mk}(z)|}, \ h(z) := \psi^*(z) \). It is known that \( \mathcal{D} := \{z \in \mathbb{C}^N; \ h(z) < 1\} = \{a \in \mathbb{C}^N; \text{series (2.1) is convergent in a neighborhood of } a\} \) is a domain of convergence of (2.1).

**Theorem 3.** If \( \lim_{k \to \infty} \frac{k}{m_k} = 0 \), then the domain of convergence \( \mathcal{D} \) of the series (2.1) is identical with the maximal domain of existence \( G_f \) of \( f \).

**Proof.** Without loss of generality we may assume that \( \mathcal{D} \neq \mathbb{C}^N \).

Due to Fabry we know that Theorem 3 is true for \( N = 1 \). It is also well known (by Bedford–Taylor Theorem on negligible sets) that the set \( E := \{z \in \mathbb{C}^N; \ h(z) < \psi^*(z)\} \) is pluripolar. Therefore, in particular, the set \( E \) is of 2N-dimensional Lebesgue measure zero.

Suppose Theorem 3 is not true for some \( N > 1 \). Then there is a function \( g \) holomorphic in a ball \( B(z_o, R) \) with \( z_o \in \mathcal{D}, \ R > r := \text{dist}(z_o, \partial \mathcal{D}) \) such that \( g(z) = f(z) \) for \( z \in B(z_o, r) \).

Let \( b_o \) be a fixed point of \( \partial \mathcal{D} \) such that \( \|b_o - z_o\| = r \).

Since the ball \( B(z_o, r) \) is non-thin at the point \( b_o \), we have

\[
    \limsup_{z \to b_o, z \in B(z_o, r)} \psi^*(z) = \psi^*(b_o).
\]

Therefore, there is a sequence \( \{z'_k\} \subset B(z_o, r) \) such that \( z'_k \to b_o \), and \( \psi^*(z'_k) \to \psi^*(b_o) \) as \( k \to \infty \). It follows that \( \psi^*(b_o) \leq 1 \). Since \( b_o \in \partial \mathcal{D} \), we have \( \psi^*(b_o) \geq 1 \). Therefore, \( \psi^*(b_o) = 1 \).

We know that the 2N-dimensional Lebesgue measure \( \nu_{2N}(E) = 0 \). Therefore, by the sub-mean-value property, for every \( k \geq 1 \) there is a point \( z_k \in B(z'_k, \frac{1}{k}) \cap B(z_o, r) \setminus E \) such that \( \psi(z_k) = \psi^*(z_k), \ |\psi^*(z'_k) - \psi(z_k)| < \frac{1}{k} \). It is clear that the sequence \( \{z_k\} \) satisfies the following properties:

\[
    z_k \in B(z_o, r), \ z_k \to b_o, \ \psi(z_k) = \psi^*(z_k), \ \psi(z_k) \to \psi^*(b_o).
\]
Put $b_k = z_k/\psi(z_k)$ ($k \geq 1$). Then $\psi(b_k) = \psi^*(b_k) = 1$, in particular, $b_k \in \partial D$ for $k \geq 1$, and $b_k \to b_o$ as $k \to \infty$.

Fix $k$ so large that $b := b_k \in B(z_o, R)$. Put

$$G_r := \{\lambda \in \mathbb{C}; \lambda b \in B(z_o, r)\},$$
$$G_R := \{\lambda \in \mathbb{C}; \lambda b \in B(z_o, R)\}.$$

One can easily check that the sets $G_r,$ $G_R$ are open, convex, nonempty (because $\lambda_o b \in G_r$ for $\lambda_o := \psi(z_k)$, and $G_r \subset G_R$). Moreover, $G_r \subset \Delta := \{|\lambda| < 1\}$, and $1 \in G_R$.

The function $f(\lambda b)$ (resp., $g(\lambda b)$) is holomorphic in $\Delta$ (resp., in $G_R$), and $f(\lambda b) = g(\lambda b)$ for $\lambda \in G_r$. Therefore, $f(\lambda b) = g(\lambda b)$ on $\Delta \cap G_R$. It follows that $g(\lambda b)$ is an analytic continuation of $f(\lambda b)$ across $\lambda = 1$, contrary to the Fabry Theorem for $N = 1$. We have got a contradiction showing that Theorem 3 is true. □

Remark. The present proof of Theorem 3 – with no assumption on the continuity of the function $\psi^*$ – is a joint result of the author and Professor Azimbay Sadullaev.


Theorem 4. Let $f$ be a function holomorphic in a neighborhood of $0 \in \mathbb{C}^N$.

Let

$$f(z) = \sum_{|\alpha| \geq 0} Q_\alpha(z), \quad Q_\alpha(z) = \sum_{|\alpha| = j} \frac{f^{(\alpha)}(0)}{\alpha!} z^\alpha,$$

be its Taylor series development around $0$. Then there exists a sequence $\epsilon = \{\epsilon_j\}$ with $\epsilon_j \in \{-1, 1\}$ (resp., $\epsilon_j \in \{0, 1\}$) such that the function

$$f_\epsilon(z) := \sum_{j=0}^{\infty} \epsilon_j Q_j(z), \quad z \in D,$$

has no analytic continuation across any boundary point of the domain of convergence $D := \{\psi^*(z) < 1\}$ of series (3.0), where

$$\psi(z) := \limsup_{j \to \infty} \sqrt{|Q_j(z)|}.$$

For $N = 1$ this theorem (with $\epsilon_j \in \{-1, 1\}$) is due to Fatou–Hurwitz–Polya (Theorem 4.2.8 in [1]).

Now, we shall present an $N$-dimensional version of the Fatou–Hurwitz–Polya theorem for $N$-tuple power series

$$f(z) = \sum_{|\alpha| \geq 0} c_\alpha z^\alpha,$$
where $c_\alpha z^\alpha$ is a monomial of $N$ complex variables $z = (z_1, \ldots, z_N)$ of degree $|\alpha| := \alpha_1 + \cdots + \alpha_N$. The set $\mathcal{D} := \{a \in \mathbb{C}^N; \text{the series (3.1) is absolutely convergent in a neighborhood of } a\}$ is called a domain of convergence of the multiple power series (3.1).

It is known that $\mathcal{D} = \{z \in \mathbb{C}^N; h(z) < 1\}$ is a complete $N$-circular (hence, in particular, $\mathcal{D}$ is balanced) domain whose Minkowski’s functional $h \equiv h_{\mathcal{D}}$ is given by the formula $h(z) = M^*(z)$, where

$$M(z) := \limsup_{|\alpha| \to \infty} \left| \sqrt{\frac{\alpha}{|c_\alpha z^\alpha|}} \right|$$

(3.2)

Moreover, $h(z_1, \ldots, z_N) = h(|z_1|, \ldots, |z_N|)$ for all $z \in \mathbb{C}^N$, and $h$ is continuous (see [2], Lemma 1.7.1 (b)).

**Theorem 5.** If the domain of convergence $\mathcal{D}$ of (3.1) is not empty, then there exists a multiple sequence $\epsilon = \{\epsilon_\alpha\}$ with $\epsilon_\alpha \in \{-1, 1\}$ (resp., with $\epsilon_\alpha \in \{0, 1\}$) such that the function

$$f_\epsilon(z) := \sum_{|\alpha| \geq 0} \epsilon_\alpha c_\alpha z^\alpha, \quad z \in \mathcal{D},$$

has no analytic continuation across any boundary point of $\mathcal{D}$.

We shall see that Theorems 4 and 5 are direct consequences of the following Lemma 3.2.

Let $\mathcal{X} := \{0, 1\}^N$ (resp. $\{-1, 1\}^N$) be the space of all sequences $x = (x_1, x_2, \ldots)$ where $x_j = 0$, or $x_j = 1$ (resp. $x_j = -1$, or $x_j = 1$) for $j = 1, 2, \ldots$. Endow $\mathcal{X}$ in the topology determined by the metric

$$\rho(x, y) := \sum_{j=1}^N \frac{|x - y|_j}{2^j 1 + |x - y|_j},$$

where $|x - y|_j := \max\{|x_k - y_k|; k = 1, \ldots, j\}$.

One can easily check that $\mathcal{X}$ is a complete metric space, and therefore, it has Baire property.

Moreover, in the topology a sequence $\{x(n)\}$ of elements of $\mathcal{X}$ converges to an element $x \in \mathcal{X}$ if and only if for every $k_0 \in \mathbb{N}$ there exists $n_0 \in \mathbb{N}$ such that $x_k(n) = x_k$ for $k = 1, \ldots, k_0$, $n \geq n_0$.

**Remark 3.1.** Let $\{f_k\}$ be a sequence of holomorphic functions in an open subset $\Omega$ of $\mathbb{C}^n$. Then the following three conditions are equivalent:

(1) the series $\sum_1^\infty |f_k(z)|$ converges at each point $z \in \Omega$, and its sum $\varphi(z) := \sum_1^\infty |f_k(z)|$ is locally bounded on $\Omega$;
(2) the series $\sum_1^\infty f_k$ converges locally normally in $\Omega$, i.e. for every point $a$ of $\Omega$ there exists a neighborhood $U$ of $a$ such that the series $\sum_1^\infty \|f_k\|_U$ is convergent;

(3) the series $\sum_1^\infty |f_k|$ converges locally uniformly in $\Omega$.

**Proof.** It is clear that $(2) \Rightarrow (3) \Rightarrow (1)$.

Suppose now $(1)$ is true, and let $E(a, r) := \{z \in \mathbb{C}^n; |z_j - a_j| < r \ (j = 1, \ldots, n)\}$ be a polydisk whose closure is contained in $\Omega$. Then there is a positive constant $M$ such that $\sum_1^\infty |f_k(z)| \leq M$ for all $z \in E(a, r)$. By the Cauchy integral formula

$$|f_k(z)| \leq \left(\frac{1}{2\pi r}\right)^n \int_0^{2\pi} \cdots \int_0^{2\pi} |f_k(a_1 + re^{it_1}, \ldots, a_n + re^{it_n})| dt_1 \cdots dt_n,$$

for all $z \in E(a, \frac{r}{2})$ and $k \geq 1$.

By Lebesgue monotonous convergence theorem the series $\sum_1^\infty \mu_k$ is convergent, and so is the series $\sum_1^\infty \|f_k\|_U$ with $U := E(a, \frac{r}{2})$. □

We shall see that our extensions of the classical Fatou–Hurwitz–Polya Theorem (Theorem 4.2.8 in [1]) are a direct consequence of the following Lemma 3.2 (slight modification of Lemma 5, p. 97 in [5]).

**Lemma 3.2.** Let $\mathcal{X}$ denote any of the two metric spaces $\{0, 1\}^\mathbb{N}$ or $\{-1, 1\}^\mathbb{N}$. Let $\{f_k\}$ be a sequence of holomorphic functions in an open neighborhood $\Omega$ of the closure of a ball $B = B(w, r)$ such that the series $\sum_1^\infty |f_k(z)|$ converges at every point $z \in B$. Let $a$ be a boundary point of $B$.

Then, either the series $\sum_1^\infty f_k$ is normally convergent on a neighborhood of $a$, or there exists a subset $\mathcal{R}$ of $\mathcal{X}$ of the first category such that for every $x \in \mathcal{X} \setminus \mathcal{R}$ the function $f_x(z) := \sum_k x_k f_k(z)$, $z \in B$, has a singular point at $a$ (in other words, $f_x$ cannot be analytically continued to any neighborhood of $a$).

**Proof.** Given a natural number $m$, let $\mathcal{R}_m$ denote the set of all $x \in \mathcal{X}$ such that there exists a holomorphic function $f_x$ on $E_m$ (where $E_m$ is the polydisk $E_m := E(a, \frac{1}{m})$ with center $a$ and radius $\frac{1}{m}$) such that $|f_x(z)| \leq m$ on the polydisk, and $f_x(z) = f_x(z)$ for all $z \in B \cap E_m$. By definition, we put $\mathcal{R}_m = \emptyset$, if $m < 1/\text{dist}(a, \partial \Omega)$.

It is clear that the set $\mathcal{R} := \bigcup_1^\infty \mathcal{R}_m \equiv \{x \in \mathcal{X}; f_x$ has an analytic continuation across $a\}$.

The lemma will be proved if we show that the following two claims are true.

**Claim 1.** The set $\mathcal{R}_m$ is closed in the space $\mathcal{X}$.

**Claim 2.** If the interior of $\mathcal{R}_m$ is not empty, then the series $\sum_1^\infty f_k$ is normally convergent on a neighborhood of $a$. 
Indeed, if the series \( f_x := \sum_{k=1}^{\infty} x_k f_k \) converges normally on no neighborhood \( U \) of \( a \), then for every \( m \geq 1 \) the set \( \mathcal{R}_m \) is closed and has empty interior. Hence, the set \( \mathcal{R} := \bigcup_{m=1}^{\infty} \mathcal{R}_m \equiv \{ x \in \mathcal{X} : f_x \) has an analytic continuation \( \tilde{f}_x \) across \( a \} \) is of the first category, and for every \( x \in \mathcal{X} \setminus \mathcal{R} \) the function \( f_x \) has a singular point at \( a \), i.e. \( f_x \) has no analytic continuation across \( a \). We say that a function \( \tilde{f}_x \) holomorphic on a polydisk \( E \) with center \( a \) is an analytic continuation of \( f_x \) across \( a \), if \( \tilde{f}_x(z) = f_x(z) \) on \( B \cap E \).

**Proof of Claim 1.** Let \( \{ x(j) \} \) be a sequence of elements of \( \mathcal{R}_m \) convergent to \( x \in \mathcal{X} \). Let \( \{ h_j \} \equiv \{ \tilde{f}_{x(j)} \} \) be a sequence of holomorphic functions on \( E_m \) such that \( |h_j(z)| \leq m \) on \( E_m \) and \( h_j(z) = f_{x(j)}(z) \) on the intersection \( B \cap E_m \) for \( j \geq 1 \). Observe that for every \( k_0 \) there exists \( j_0 \) such that

\[
|f_{x(j)}(z) - f_x(z)| \leq \sum_{k > k_0} 2 |f_k(z)| \quad \text{for all} \quad z \in B \cap E_m \quad \text{and for all} \quad j > j_0.
\]

It follows that the sequence \( \{ h_j \} \) is convergent at each point of \( B \cap E_m \). By Vitali’s theorem the sequence \( \{ h_j \} \) is locally uniformly convergent on \( E_m \) to a holomorphic function \( h \) bounded by \( m \) and identical with \( f_x \) on \( E_m \cap B \), which shows that \( x \in \mathcal{R}_m \).

**Proof of Claim 2.** If \( \mathcal{R}_m \) has a nonempty interior, then there exist \( x(0) = (x_1(0), x_2(0), \ldots) \in \mathcal{R}_m \) and a natural number \( k_0 \) such that

\[
(*) \quad x \in \mathcal{X}, \; x_j = x_j(0) \quad (j = 1, \ldots, k_0) \quad \Rightarrow \quad x \in \mathcal{R}_m.
\]

Put

\[
M := \sup \left\{ \sum_{k=1}^{k_0} |f_k(z)| ; \; z \in E_m \right\}, \quad u_k := \Re f_k, \quad v_k := \Im f_k.
\]

By implication (2) \( \Rightarrow \) (3) of Remark 3.1 it is sufficient to show that

\[
(**) \quad \sum_{k=1}^{\infty} |f_k(z)| \leq M + 4m, \quad z \in E_m.
\]

Let \( A \) be a finite subset of \( \mathbb{N} \setminus [1, k_0] \). Given a fixed point \( z \) of \( E_m \), put

\[
A_1 := \{ k \in A ; \; u_k(z) \geq 0 \}, \quad A_2 := \{ k \in A ; \; u_k(z) < 0 \}.
\]

It is clear that \( A = A_1 \cup A_2, \; A_1 \cap A_2 = \emptyset \). Consider two cases.

**Case 1:** \( \mathcal{X} = \{0, 1\}^{\mathbb{N}} \). Let \( x(j) = (x_1(j), x_2(j), \ldots) \) \( (j = 1, 2) \) be two points of the interior of \( \mathcal{R}_m \) defined by the formulas:

\[
\begin{align*}
x_k(j) &= x_k(0), \quad k = 1, \ldots, k_0, \quad j = 1; \\
x_k(j) &= x_k(0), \quad k > k_0, \quad k \notin A, \quad j = 1; \\
x_k(1) &= 1, \quad k \in A_1; \\
x_k(1) &= 0, \quad k \in A_2.
\end{align*}
\]

Indeed, if the series \( f_x := \sum_{k=1}^{\infty} x_k f_k \) converges normally on no neighborhood \( U \) of \( a \), then for every \( m \geq 1 \) the set \( \mathcal{R}_m \) is closed and has empty interior. Hence, the set \( \mathcal{R} := \bigcup_{m=1}^{\infty} \mathcal{R}_m \equiv \{ x \in \mathcal{X} : f_x \) has an analytic continuation \( \tilde{f}_x \) across \( a \} \) is of the first category, and for every \( x \in \mathcal{X} \setminus \mathcal{R} \) the function \( f_x \) has a singular point at \( a \), i.e. \( f_x \) has no analytic continuation across \( a \). We say that a function \( \tilde{f}_x \) holomorphic on a polydisk \( E \) with center \( a \) is an analytic continuation of \( f_x \) across \( a \), if \( \tilde{f}_x(z) = f_x(z) \) on \( B \cap E \).
Then
\[ \sum_{k \in A} |u_k(z)| \leq \left| \sum_{k \in A} (x_k(1) - x_k(2)) f_k(z) \right| = |\tilde{f}(x(1)) - \tilde{f}(x(2))| \leq 2m. \]

By the arbitrary property of \( A \) and \( z \) one gets
\[ \sum_{k=k_0+1}^{\infty} |u_k(z)| \leq 2m, \quad z \in E_m. \]

The same argument gives
\[ \sum_{k=k_0+1}^{\infty} |v_k(z)| \leq 2m, \quad z \in E_m. \]

Hence
\[ \sum_{k=1}^{\infty} |f_k(z)| = \left( \sum_{k=1}^{k_0} + \sum_{k=k_0+1}^{\infty} \right) |f_k(z)| \leq M + 4m, \quad z \in E_m. \]

**Case 2:** \( \mathcal{X} = \{-1, 1\}^N \). Now we define two elements \( x(1), x(2) \) of the interior of \( \mathcal{R}_m \) by the formulas:
\[ x_k(j) = x_k(0), \quad k = 1, \ldots, k_0, \quad j = 1, 2; \]
\[ x_k(j) = x_k(0), \quad k > k_0, \quad k \notin A, \quad j = 1, 2; \]
\[ x_k(1) = 1, \quad x_k(2) = -1, \quad k \in A_1; \]
\[ x_k(1) = -1, \quad x_k(2) = 1, \quad k \in A_2. \]

Then
\[ 2 \sum_{k \in A} |u_k(z)| \leq \left| \sum_{k \in A} (x_k(1) - x_k(2)) f_k(z) \right| = |\tilde{f}(x(1)) - \tilde{f}(x(2))| \leq 2m. \]

Hence, by the analogous argument as in the proof of the case 1, we get
\[ \sum_{k=1}^{\infty} |f_k(z)| \leq M + 4m, \quad z \in E_m, \]
which ends the proof of the case 2. \( \square \)

**Corollary 3.3.** Let \( \{f_k\} \) be a sequence of holomorphic functions on an open set \( \Omega \subset \mathbb{C}^N \). Let \( D \) denote the set of all points \( a \) in \( \Omega \) such that the series \( \sum_1^{\infty} f_k \) is absolutely convergent at every point of a neighborhood of \( a \). Assume that the sum \( \varphi(z) := \sum_1^{\infty} |f_k(z)| \) is locally bounded in \( D \), and \( \bar{D} \subset \Omega \). Let \( \mathcal{X} \) be any of the two metric spaces \( \{0, 1\}^N \) or \( \{-1, 1\}^N \).

Then there exists a subset \( \mathcal{R} \) of \( \mathcal{X} \) of the first category such that for every point \( x \in \mathcal{X} \setminus \mathcal{R} \) the holomorphic function \( f_x(z) := \sum_{k=1}^{\infty} x_k f_k(z), \quad z \in D \), cannot be continued analytically across any boundary point of \( D \).
Proof. Let \( \{w_j\} \) be the sequence of all rational points of \( D \) (or any countable dense subset of \( D \)). Let \( a_j \) be a point of \( \partial D \) such that \( ||w_j - a_j|| = \text{dist}(w_j, \partial D) \). By Lemma 3.2 for every \( j \) there exists a subset \( R_j \) of \( \mathcal{X} \) of the first category such that for every \( x \in \mathcal{X} \setminus R_j \) the function \( f_x \) has a singular point at \( a_j \). The set \( R := \bigcup R_j \) is again of the first category such that for every \( x \in \mathcal{X} \setminus R \) the function \( f_x \) has analytic extension across no boundary point of \( D \). □

Proof of Theorems 4 and 5. It is sufficient to apply Lemma 3.2 with \( \Omega = \mathbb{C}^N \), with \( f_k = Q_k \) and \( f_k = c_{\alpha(k)} z^{\alpha(k)} \) \( (k \in \mathbb{Z}_+) \), respectively, where \( \alpha : \mathbb{Z}_+ \ni k \mapsto \alpha(k) \in \mathbb{Z}_N^+ \) is a one-to-one mapping, and with \( D \) replaced by the domain of convergence \( D \) of the corresponding power series. □

Remark 3.4. The author would like to draw reader’s attention to the fact that, unfortunately, the proofs of Theorems 4 and 5 published in [6] contain flaws.

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Received May 16, 2011