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Subclasses of typically real functions
determined by some modular inequalities

Abstract. Let $T$ be the family of all typically real functions, i.e. functions that are analytic in the unit disk $\Delta := \{ z \in \mathbb{C} : |z| < 1 \}$, normalized by $f(0) = f'(0) - 1 = 0$ and such that $\text{Im} z \text{Im} f(z) \geq 0$ for $z \in \Delta$. Moreover, let us denote:

$T(2) := \{ f \in T : f(z) = -f(-z) \text{ for } z \in \Delta \}$ and $T_{M,g} := \{ f \in T : f \prec M g \text{ in } \Delta \}$, where $M > 1$, $g \in T \cap S$ and $S$ consists of all analytic functions, normalized and univalent in $\Delta$.

We investigate classes in which the subordination is replaced with the majorization and the function $g$ is typically real but does not necessarily univalent, i.e. classes $\{ f \in T : f \ll M g \text{ in } \Delta \}$, where $M > 1$, $g \in T$, which we denote by $T_{M,g}$. Furthermore, we broaden the class $T_{M,g}$ for the case $M \in (0,1)$ in the following way: $T_{M,g} = \{ f \in T : |f(z)| \geq M |g(z)| \text{ for } z \in \Delta \}$, $g \in T$.

1. Introduction. Let $T$ be the family of all typically real functions, i.e. functions that are analytic in the unit disk $\Delta := \{ z \in \mathbb{C} : |z| < 1 \}$, normalized by $f(0) = f'(0) - 1 = 0$ and such that $\text{Im} z \text{Im} f(z) \geq 0$ for $z \in \Delta$. Let $S$ denote the class of all analytic functions, normalized as above and univalent in $\Delta$, and $\text{SR}$ – the subclass of $S$ consisting of functions with real coefficients. Moreover, let us denote: $T(2) := \{ f \in T : f(z) = -f(-z) \text{ for } z \in \Delta \}$ and $T_{M,g} := \{ f \in T : f \prec M g \text{ in } \Delta \}$, where $M > 1$, $g \in T \cap S$. The symbol $h \prec H$ denotes the subordination in $\Delta$, i.e. $h(0) = H(0)$ and $h(\Delta) \subset H(\Delta)$, where $H$ is univalent. Let us notice that for $g_1(z) = z$


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and $g_2(z) = \frac{1}{2} \log \frac{1 + z}{1 - z}$ we have $T^{M,g_1} = \{ f \in T : |f| < M \text{ in } \Delta \}$ and $T^{M,g_2} = \{ f \in T : |\text{Im} f| < M\pi/4 \text{ in } \Delta \}, M > 1$. These classes are briefly denoted by $T_M$ and $T(M)$, respectively.

The subordination in the classes $T$, $S$ and $SR$ has been investigated by several authors (for example [2], [3], [4]). The relation $T^{M,g} = \{ Mg(h/M) : h \in T_M \}$ for $g \in T \cap S$ (see [3]) provides the following formula connecting different classes of type $T^{M,g}$: $T^{M,f} = \{ Mfg^{-1}(h/M) : h \in T^{M,g} \}$, $f, g \in T \cap S$. For this reason, instead of researching a class $T^{M,f}$ one can consider a class $T^{M,g}$, for instance $T_M$ or $T(M)$. We apply this idea to obtain results in various classes $T^{M,g}$ from corresponding results in the class $T(M)$. Investigating $T(M)$ is possible because the integral formula for this class, the set of extremal points and the set of supporting points are known (see [4]).

Moreover, it is easy to prove that the class $T^{M,g} \cap T^{(2)} = \{ Mg(h/M) : h \in T_M \}$ for $g \in T^{(2)} \cap S$.

In the paper we investigate classes similar to $T^{M,g}$, in which the subordination is replaced with the majorization (the modular subordination) and the function $g$ is typically real but does not necessarily univalent, i.e. classes $T_{M,g} := \{ f \in T : f \ll Mg \text{ in } \Delta \}$, where $M > 1$, $g \in T$. The symbol $h \ll H$ denotes the majorization in $\Delta$, i.e. $|h(z)| \leq |H(z)|$ for all $z \in \Delta$.

Furthermore, we broaden the class $T_{M,g}$ for the case when $M \in (0,1)$ in the following way: $T_{M,g} = \{ f \in T : |f(z)| \geq Mg(z) \text{ in } \Delta \}, g \in T$.

Moreover, we study the subclass of the class $T_{M,g}$, consisting of all odd functions, which we denote by $T^{(2)}_{M,g}$.

The class $T_{M,g}$ is not empty, because for example the function $g$ belongs to this class. Analogously, the class $T^{(2)}_{M,g}$ for $g \in T^{(2)}$ is not empty. If $M = 1$, then the class consists of only one function $g$. So we investigate the class $T_{M,g}$ for $M \in (0,1) \cup (1, \infty)$. For $g = id$ and $M \geq 1$, we have $T^{M,\text{id}} = T_{M,\text{id}}$.

In the class $T^{M,g}$ one can formulate theorems which are true for each function $g \in T \cap S$. However, in the class $T_{M,g}$ it is impossible. Indeed, theorems in the class $T_{M,g}$ in a fundamental way depends on the choice of the function $g$. It means that a theorem which is true in the class $T^{M,g_1}$ generally is not true in the class $T_{M,g_2}$, for $g_1 \neq g_2$. In each case, we connect the researching class with the class $T_M$ or $T^{(2)}_M$.

2. Some properties of the classes $T$ and $T^{(2)}$. During our investigation of the class $T_{M,g}$, we use the following relations of classes $T$ and $T^{(2)}$, which we give as lemmas. In each lemma we shall prove only one implication. The other can be proved analogously. For simplicity, instead of $h$ or $z \mapsto h(z)$ we will use $h(z)$.
Lemma 1. \( f \in T \iff \frac{1+z^2}{z} f(z^2) \in T^{(2)}. \)

Proof. Let \( f \in T. \) For \( f \in T \) we have the Robertson formula \( f(z) = \int_{-1}^{1} \frac{z}{1-2zt+z^2} d\mu(t), \) where \( \mu \) is a probability measure on \([-1, 1]\) (see \([1], [2]\)). Then
\[
\frac{(1+z^2)f(z^2)}{z} = \int_{-1}^{1} \frac{z(1+z^2)}{1-2zt+z^2} d\mu(t) = \int_{-1}^{1} \frac{z(1+z^2)}{(1+z^2)^2 - 2(1+t)z^2} d\mu(t)
\]
with \( \nu(A) = \mu(2A - 1) \) (where \( A \) is a Borel set contained in \([0, 1]\)). Clearly, \( f_{1}^{1} \frac{z(1+z^2)}{(1+z^2)^2 - 4\tau z^2} d\nu(\tau) \in T^{(2)} \) (the representation formula for functions from the class \( T^{(2)} \), see \([5]\)). Therefore, \( \frac{1+z^2}{z} f(z^2) \in T^{(2)}. \)

Lemma 2. \( f \in T^{(2)} \iff \frac{1+z^2}{1-z^4} f(iz) \in T^{(2)}. \)

Proof. Suppose that \( f \in T^{(2)}. \) From Lemma 1, the function \( h \) given by \( h(z^2) = \frac{z}{1+z^2} f(z) \) is in \( T. \) The definition of \( h \) is correct since
\[
h\left((-z)^2\right) = \frac{-z}{1+(-z)^2} f(-z) = \frac{-zf(z)}{1+z^2} = h(z^2).
\]
Hence, \( \frac{1+z^2}{1-z^2} f(iz) \in T^{(2)}. \) Because of Lemma 1 and the fact that \( h \in T \iff -h(-z) \in T, \) we receive \( -\frac{1+z^2}{z} h(-z^2) \in T^{(2)}. \) This means that \( \frac{1+z^2}{1-z^2} f(iz) - \frac{1}{i} \in T^{(2)}, \) so we have the desired result.

Lemma 3. \( f \in T \iff \frac{z^2}{(1-z^4) f(z)} \frac{1}{1} \in T. \)

Proof. Let \( f \in T. \) Then \( f(z) = \frac{z}{1-z^2} p(z) \) for \( p \in PR \) (the Rogosinski representation, \([2], [6]\)), where \( PR \) consists of all analytic functions \( p \) such that \( p(0) = 1, \) \( \text{Re} p(z) > 0 \) for \( z \in \Delta \) and having real coefficients. Clearly, \( \frac{1}{p} \in PR, \) so \( \frac{z^2}{(1-z^2) f(z)} \frac{1}{1} \in T. \) From this and the equality
\[
\left\{ \frac{1}{p} : p \in PR \right\} = PR,
\]
we get \( f \in T \iff \frac{z^2}{(1-z^4) f(z)} \frac{1}{1} \in T. \)

Taking \( f \in T^{(2)} \) in Lemma 3, we obtain the following relation:

Lemma 4. \( f \in T^{(2)} \iff \frac{z^2}{(1-z^2) f(z)} \frac{1}{1} \in T^{(2)}. \)

Lemma 5. \( f \in T \iff \frac{z^3}{(1-z^4)(1-z^2) f(z)} \frac{1}{1} \in T^{(2)}. \)

Proof. Let \( f \in T. \) On the basis of Lemma 1, the function \( g \) given by \( g(z) = \frac{1+z^2}{z} f(z^2) \) belongs to \( T^{(2)}. \) Hence, we have \( \frac{z^2}{(1-z^2) f(z)} \frac{1}{1} = \frac{z^3}{(1-z^2)(1-z^2) f(z^2)}. \)
From Lemma 4, we know that \( \frac{z^3}{(1-z^2)(1-z^2) f(z^2)} \frac{1}{1} \in T^{(2)} \) which is equivalent to
\[
\frac{z^3}{(1-z^2)(1-z^2) f(z^2)} \frac{1}{1} \in T^{(2)}.
\]
Lemma 6. \(f \in T^{(2)} \iff \frac{z^2}{1-z^2} f^{(iz)} \in T^{(2)}\).

Proof. Suppose that \(f \in T^{(2)}\). Let \(g(z) = \frac{1+z^2}{1-z^2} f^{(iz)}\). By Lemma 2, \(g \in T^{(2)}\). Since \(\frac{z^2}{1-z^2} g(z) = \frac{z^2}{1-z^2} f^{(iz)}\), from Lemma 4 we get \(\frac{z^2}{1-z^2} g(z) \in T^{(2)}\) i.e. \(\frac{z^2}{1-z^2} f^{(iz)} \in T^{(2)}\). \(\Box\)

3. The majorization in the class of typically real functions \(T\). At the beginning we study the case when \(M > 1\), i.e. the class \(T_{M,g} = \{f \in T : |f(z)| \leq M|g(z)| \text{ for } z \in \Delta\}, \quad g \in T\).

At first, let \(g(z) = \frac{z}{1+z}\). Clearly, \(g \in T \cap S\).

Theorem 1. If \(f \in T\) and \(|f(z)| \leq M \left|\frac{z}{1+z}\right|\) for all \(z \in \Delta\), \(M > 1\) (i.e. \(f \in T_{M,g}\) where \(g(z) = \frac{z}{1+z}\)), then \(f(z^2) \equiv \frac{z}{1+z} h(z)\) for some \(h \in T^{(2)}\).

Proof. Let \(f \in T\) and \(|f(z)| \leq M \left|\frac{z}{1+z}\right|\). Hence, \(|f(z^2)| \leq M \left|\frac{z^2}{1+z^2}\right|\). Let \(h(z) \equiv \frac{1+z^2}{z} f(z^2)\). By Lemma 1, \(h \in T^{(2)}\). Therefore, \(f(z^2) \equiv \frac{z}{1+z} h(z)\). From the above equality, we get \(\left|\frac{z}{1+z}\right| |h(z)| \leq M \left|\frac{z^2}{1+z^2}\right|\). This implies that \(|h(z)| \leq M|z| < M\), that is \(h \in T^{(2)}_M\). \(\Box\)

Now, let us consider the function \(g(z) = z + z^3\). We have \(g(z) = \frac{z}{1+z^2}(1 - z^4)\). Since \(\text{Re}(1 - z^4) > 0\) for \(z \in \Delta\), from the Rogosinski formula (see [2], [6]), we get \(g \in T\). Moreover, \(g \notin S\), because \(g'(i/\sqrt{3}) = 0\).

Theorem 2. If \(f \in T^{(2)}\) and \(|f(z)| \leq M|z + z^3|\) for all \(z \in \Delta\), \(M > 1\) (i.e. \(f \in T^{(2)}_{M,g}\) where \(g(z) = z + z^3\)), then \(f(z) \equiv \frac{1+z^2}{z} h(z^2)\) for some \(h \in T_M\).

Proof. Suppose that \(f \in T^{(2)}\) and \(|f(z)| \leq M|z + z^3|\). By Lemma 1, the function \(h\) given by \(h(z^2) \equiv \frac{z}{1+z^2} f(z)\) is in \(T\). Therefore, \(f(z) \equiv \frac{1+z^2}{z} h(z^2)\). From the second assumption, we have \(\left|\frac{1+z^2}{z}\right| |h(z^2)| \leq M|z + z^3|\). Then \(|h(z^2)| \leq M|z^2| < M\), i.e. \(h \in T_M\). \(\Box\)

Let us study the next function \(g(z) = z + z^3(1+z^2)\). We have \(g(z) = \frac{z}{1-z^2}(1 + z^2)\). Since \(\text{Re}(1 + z^2) > 0\) for \(z \in \Delta\), from the Rogosinski formula, \(g \in T\). Furthermore, \(g \notin S\), because \(g'(\sqrt{\sqrt{3} - 2} i) = 0\).

Theorem 3. If \(f \in T^{(2)}\) and \(|f(z)| \leq M \left|\frac{z+z^3}{1-z^2}\right|\) for all \(z \in \Delta\), \(M > 1\) (i.e. \(f \in T^{(2)}_{M,g}\) where \(g(z) = \frac{z+z^3}{1-z^2}\)), then \(f(z) \equiv \frac{1+z^2}{1-z^2} h^{(iz)}\) for some \(h \in T^{(2)}_M\).

Proof. Assume that \(f \in T^{(2)}\) and \(|f(z)| \leq M \left|\frac{z+z^3}{1-z^2}\right|\). Let \(h^{(iz)} \equiv \frac{1+z^2}{1-z^2} f(z)\). By Lemma 2, \(h \in T^{(2)}\). Hence, \(f(z) \equiv \frac{1+z^2}{1-z^2} h^{(iz)}\). From the above equality,
we get \( \frac{1+z^2}{1-z^2} |h(iz)| \leq M \left| \frac{z+z^3}{1-z^2} \right| \). Therefore, \( |h(iz)| \leq M|z| < M \), that is \( h \in T_M^{(2)} \).

In the further investigation we consider the case when \( M \in (0,1) \), i.e. the class

\[
T_{M,g} = \{ f \in T : |f(z)| \geq M|g(z)| \text{ for } z \in \Delta \}, \quad g \in T.
\]

Suppose that \( g(z) = \frac{z}{1-z^2} \). Since \( g(z) = \frac{1}{1-z^2} \) and \( \text{Re} \left( \frac{1}{1-z^2} \right) > 0 \) for \( z \in \Delta \), hence \( g \in T \). We have also \( g'(i/\sqrt{3}) = 0 \), and it follows that \( g \notin S \).

**Theorem 4.** If \( f \in T \) and \( |f(z)| \geq M \left| \frac{z}{1-z^2} \right| \) for all \( z \in \Delta \), \( M \in (0,1) \) (i.e. \( f \in T_{M,g} \) where \( g(z) = \frac{z}{1-z^2} \)), then \( f(z) = \frac{z^2}{(1-z^2)^2} \frac{1}{h(z)} \) for some \( h \in T_{1/M}^{(2)} \).

**Proof.** Let \( f \in T \) and \( |f(z)| \geq M \left| \frac{z}{1-z^2} \right| \). By Lemma 3, the function \( h \) given by \( h(z) = \frac{z^2}{(1-z^2)^2} \frac{1}{f(z)} \) belongs to \( T \). So \( f(z) = \frac{z^2}{(1-z^2)^2} \frac{1}{h(z)} \). From the second assumption, we have \( \left| \frac{z^2}{(1-z^2)^2} \frac{1}{h(z)} \right| \geq M \left| \frac{z}{1-z^2} \right| \), i.e. \( |h(z)| \leq |z|/M < 1/M \). Hence, \( h \in T_{1/M}^{(2)} \) and the proof is complete. \( \square \)

Analogously, using Lemma 4, we prove the following theorem:

**Theorem 5.** If \( f \in T^{(2)} \) and \( |f(z)| \geq M \left| \frac{z}{1-z^2} \right| \) for all \( z \in \Delta \), \( M \in (0,1) \) (i.e. \( f \in T_{M,g} \) where \( g(z) = \frac{z}{1-z^2} \)), then \( f(z) = \frac{z^2}{(1-z^2)^2} \frac{1}{h(z)} \) for some \( h \in T_{1/M}^{(2)} \).

Now, let us consider the function \( g(z) = \frac{z}{1-z^2} \). Clearly, \( g(z) = \frac{z}{1-z^2} \frac{1}{1-z} \) and \( \text{Re} \left( \frac{1}{1-z} \right) > 0 \) for \( z \in \Delta \), so \( g \in T \). We have also

\[
g' \left( (i/\sqrt{7}) - 1/4 \right) = 0,
\]

which means that \( g \notin S \).

**Theorem 6.** If \( f \in T \) and \( |f(z)| \geq M \left| \frac{z}{1-z^2} \right| \) for all \( z \in \Delta \), \( M \in (0,1) \) (i.e. \( f \in T_{M,g} \) where \( g(z) = \frac{z}{1-z^2} \)), then \( f(z^2) = \frac{z^3}{(1-z^2)^2} \frac{1}{h(z)} \) for some \( h \in T_{1/M}^{(2)} \).

**Proof.** Suppose that \( f \in T \) and \( |f(z)| \geq M \left| \frac{z}{1-z^2} \right| \). By Lemma 5, the function \( h(z) = \frac{z^3}{(1-z^2)^2} \frac{1}{f(z^2)} \) is in \( T^{(2)} \). Hence, \( f(z^2) = \frac{z^3}{(1-z^2)^2} \frac{1}{h(z)} \). From the second assumption, we get \( \left| \frac{z^3}{(1-z^2)^2} \frac{1}{h(z)} \right| \geq M \left| \frac{z^2}{1-z^2} \right| \).
so $|h(z)| \leq |z|/M < 1/M$. This means that $h \in T^{(2)}_{1/M}$, so we have the desired result. □

Now let us study the function $g(z) = \frac{z}{1-z^2}$. Because $g(z) = \frac{z}{1-z^2} \frac{1}{1+z^2}$ and $\text{Re} \left( \frac{1}{1+z^2} \right) > 0$ for $z \in \Delta$, so $g \in T$. Moreover, $g \in T^{(2)}$ and $g \notin S$, because $g'(i + 1)/\sqrt{12} = 0$.

**Theorem 7.** If $f \in T^{(2)}$ and $|f(z)| \geq M \left| \frac{z}{1-z^2} \right|$ for all $z \in \Delta$, $M \in (0,1)$ (i.e. $f \in T^{(2)}_{M,g}$ where $g(z) = \frac{z}{1-z^2}$), then $f(iz) \equiv \frac{z^2}{1-z^2} h(z)$ for some $h \in T^{(2)}_{1/M}$.

**Proof.** Let $f \in T^{(2)}$ and $|f(z)| \geq M \left| \frac{z}{1-z^2} \right|$. By Lemma 6, the function $h(z) \equiv \frac{z^2}{1-z^2} f'(iz)$ belongs to $T^{(2)}$. So $f(iz) \equiv \frac{z^2}{1-z^2} h(z)$. From the second assumption, we have $\left| \frac{z^2}{1-z^2} \frac{1}{|h(z)|} \right| \geq M \left| \frac{iz}{1-z^2} \right|$ i.e. $|h(z)| \leq |z|/M < 1/M$. Therefore, $h \in T^{(2)}_{1/M}$ and the proof is complete. □

The converses to Theorems 1–7 are also true.

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