Two hierarchies of $R$-recursive functions

Jerzy Mycka*

*E-mail address: Jerzy.Mycka@umcs.lublin.pl

Institute of Mathematics, Maria Curie-Skłodowska University, Pl. M. Curie-Skłodowskiej 1, 20-031 Lublin, Poland

Abstract

In the paper some aspects of complexity of $R$-recursive functions are considered. The limit hierarchy of $R$-recursive functions is introduced by the analogy to the $\mu$-hierarchy. Then its properties and relations to the $\mu$-hierarchy are analysed.

1. Introduction

The classical theory of computation deals with the functions on enumerable (especially natural) domains. The fundamental notion in this field is the notion of a (partial) recursive function. The problem of hierarchies for these functions is also in the interest of mathematicians (for elementary, primitive recursive function, Grzegorczyk hierarchy, compare [1].

During past years many mathematicians have been interested in creating analogous models of computation on real numbers (see for example Grzegorczyk [2], Blum, Shub, Smale [3]). An interesting approach was given by Moore. In the work [4] he defined a set of functions on the reals $R$ (called $R$-recursive functions) in the analogous way to the classical recursive functions on the natural numbers $N$. His model has a continuous time of computation (a continuous integration instead of a discrete recursion). The great importance in Moore's model has the zero-finding operation $\mu$, which is used to construct $\mu$-hierarchy of $R$-recursive functions.

It was shown [5] that the zero-finding operator $\mu$ can be replaced by the operation of infinite limits. This allows us to define a limit hierarchy and relate it to $\mu$-hierarchy.
2. Preliminaries

We start with a fundamental definition of a class of real functions called \( R \)-recursive functions [4].

**Definition 2.1** The set of \( R \)-recursive functions is generated from the constants 0,1 by the operations:

1) composition: \( h(\overline{x}) = f(\overline{g(\overline{x})}) \);

2) differential recursion: \( h(\overline{x},0) = f(\overline{x}), \partial_y h(\overline{x},y) = g(\overline{x},y,h(\overline{x},y)) \) (the equivalent formulation can be given by integrals:
   \[
   h(\overline{x},y) = f(\overline{x}) + \int_0^y g(\overline{x},y',h(\overline{x},y'))dy'.
   \]

3) \( \mu \)-recursion \( h(\overline{x}) = \mu_y f(\overline{x},y) = \inf \{ y : f(\overline{x},y) = 0 \} \), where infimum chooses the number \( y \) with the smallest absolute value and for two \( y \) with the same absolute value the negative one;

4) vector-valued functions can be defined by defining their components.

Several comments are needed to the above definition. A solution of a differential equation need not be unique or can diverge. Hence, we assume that if \( h \) is defined by a differential recursion then \( h \) is defined only where a finite and unique solution exists. This is why the set of \( R \)-recursive functions includes also partial functions. We use (after [4]) the name of \( R \)-recursive functions in the article, however we should remember that in reality we have partiality here (partial \( R \)-recursive functions).

The second problem arises with the operation of infimum. Let us observe that if an infinite number of zeros accumulates just above some positive \( y \) or just below some negative \( y \) then the infimum operation returns that \( y \) even if it itself is not a zero.

In the papers [5, 6] it was shown that if in the Moore's definition [4] \( \mu \)-operation is replaced by infinite limits: \( h(\overline{x}) = \liminf_{y \to \infty} g(\overline{x},y) \), \( h(\overline{x}) = \limsup_{y \to \infty} g(\overline{x},y) \) then the resulting class of functions remains the same.

This gives us also the following result (including the limit operation in the form \( h(\overline{x}) = \lim_{y \to \infty} g(\overline{x},y) \), which can be in the obvious way obtained from limsup, liminf:
Corollary 2.2 The class of $R$-recursive functions is closed under the operations of infinite limits: $h(\overline{x}) = \liminf_{y \to \infty} g(\overline{x}, y)$, $h(\overline{x}) = \limsup_{y \to \infty} g(\overline{x}, y)$, $h(\overline{x}) = \lim_{y \to \infty} g(\overline{x}, y)$.

3. Hierarchies

The operator $\mu$ is a key operator in generating the $R$-recursive functions. In a physical sense it has a property of being strongly uncomputable. This fact suggests creating a hierarchy, which is built with respect to the number of uses of $\mu$ in the definition of a given $f$.

Definition 3.1 ([4]) For a given $R$-recursive expression $s(\overline{x})$, let $M_\mu(s)$ (the $\mu$-number with respect to $x_i$) be defined as follows:

$$M_\mu(0) = M_\mu(1) = M_\mu(-1) = 0,$$
$$M_\mu(f(g_1, g_2, \ldots)) = \max_j (M_{\mu_j}(f) + M_\mu(g_j)),$$
$$M_\mu(h = f + \int_0^y g(\overline{x}, y', h)dy') = \max(M_\mu(f), M_\mu(g), M_h(g)),$$
$$M_\mu(\mu, f(\overline{x}, y)) = \max(M_\mu(f), M_\mu(f)) + 1,$$

where $\overline{x}$ can be any $x_1, \ldots, x_n$ for $\overline{x} = (x_1, \ldots, x_n)$.

For an $R$-recursive function $f$, let $M(f) = \max_{\mu}(s)$ minimized over all expressions $s$ that define $f$. Now we are ready to define $M$-hierarchy ($\mu$-hierarchy) as a family of $M_j = \{f : M^\prime(f) \leq j\}$.

Let us construct the analogous definition of $L$-hierarchy by replacing in the above definition $M_\mu$ by $L_\mu$ and changing line (5) to the following form (5'):

$$L_\mu(\liminf_{y \to \infty} g(\overline{x}, y)) = L_\mu(\limsup_{y \to \infty} g(\overline{x}, y)) =$$
$$= L_\mu(\lim g(\overline{x}, y)) = \max(L_\mu(f), L_\mu(f)) + 1.$$

For an $R$-recursive function $f$, let $L(f) = \max_{\mu}, L_{\mu}(s)$ minimized over all expressions $s$ that define $f$ without using the $\mu$-operation.
Definition 3.2 The $L$-hierarchy is a family of $L_j = \{ f : L(f) \leq j \}$.

Let us add that in Definition 3.2 we use explicitly the operator $f(\bar{x}) = \lim_{y \to \infty} g(\bar{x}, y)$ to avoid its construction by other operators (lim sup, lim inf), which would effect in a superficially higher class of a complexity of a function $f$.

As an obvious corollary from definitions we have the following statement.

Lemma 3.3 The classes $M_0$ and $M_1$ are identical.

A function $f \in L_0 = M_0$ will be called (by an analogy to the case of natural recursive functions) a primitive $R$-recursive function. After Moore [4] we can conclude that such functions as: $-x$, $x + y$, $xy$, $x/y$, $e^x$, $\ln x$, $y^x$, $\sin x$, $\cos x$ are primitive $R$-recursive.

We can give a few results on some levels of the limit hierarchy.

Lemma 3.4. The Kronecker $\delta$ function, the signum function and absolute value belong to the first level ($L_1$) of limit hierarchy.

Proof. It is sufficient to take the following definitions [5]: hence $\delta(0) = 1$ and for all $x \neq 0$ we have $\delta(x) = 0$ let us define $\delta(x) = \liminf_{y \to \infty} \left( \frac{1}{1 + x^2} \right)^y$. Now from the expression $\liminf_{y \to \infty} \arctan xy = \begin{cases} \pi/2, & \text{if } x > 0, \\ 0, & \text{if } x = 0, \\ -\pi/2, & \text{if } x < 0, \end{cases}$ we obtain $\sgn(x) = \frac{\liminf_{y \to \infty} \arctan xy}{2\arctan 1}$ and $|x| = \sgn(x)x$.

We should be careful with definitions of functions by cases:

Lemma 3.5 For $h(\bar{x}) = \begin{cases} g_1(\bar{x}), & \text{if } f(\bar{x}) = 0, \\ g_2(\bar{x}), & \text{if } f(\bar{x}) = 1, \\ \text{and } g_i \in L_{n_i}, & \text{for all } 1 \leq i \leq k, \\ g_k(\bar{x}), & \text{if } f(\bar{x}) \geq k - 1 \end{cases}$ $f \in L_m$ the function $h$ belongs to $L_{\max(n_1, \ldots, n_k, m+1)}$. 

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Proof. Let us see that \(eq(x, y) = \delta(x - y) \in L_1\) and \(ge(x, y) = \frac{(\sgn(x - y) + eq(x, y))}{2} + \frac{1}{2} \in L_1\). Then of course

\[h(\bar{x}) = \sum_{i=1}^{k-1} g_i(\bar{x})eq(f(\bar{x}), i-1) + g_k(\bar{x})ge(f(\bar{x}), k-1)\]

Of course this result can be easily extended to other forms of definitions by cases.

**Lemma 3.6** The function \(\Theta(x)\) (equal to 1 if \(x \geq 0\), otherwise 0), maximum \(\max(x, y)\), square-wave function \(s\) are in \(L_2\), the function \(p(x)\) such that \(p(x) = 1 \text{ for } x \in [2n, 2n+1]\) and \(p(x) = 0 \text{ for } x \in [2n+1, 2n+2]\) is in \(L_2\) and the floor function \(\lfloor x \rfloor\) is in \(L_3\).

**Proof.** We give the proper definitions (from [6]) for these functions. Let

\[\Theta(x) = \delta(x - \lfloor x \rfloor),\]

\[\max(x, y) = x\delta(x - y) + (1 - \delta(x - y))[x\Theta(x - y) + y\Theta(y - x)],\]

\[s(x) = \Theta(\sin(x \pi)).\]

The function \(p(x)\) can be given as \(s(x)^2 - \delta(\sin(x \pi))\), so \(p \in L_2\).

The floor function we can define by the auxiliary function \(w(0) = 0, \partial_x w(x) = 2\Theta(-\sin(2\pi x))\) as

\[
\lfloor x \rfloor = \begin{cases} 
2w(x/2) & \text{if } p(x) = 1, \\
2w((x - 1)/2) & \text{if } p(x) = 0.
\end{cases}
\]

From the above equation we have \(\lfloor x \rfloor\) in \(L_3\).

Let us recall that if \(f : R^n \to R\) is an \(R\)-recursive function then the function \(f_{\text{iter}}(i, \bar{x})\) is \(R\)-recursive, too.

**Lemma 3.7** Let \(f : R^n \to R\) belongs to the class \(L_i\), then we have \(f_{\text{iter}} : R^{n+1} \to R\) is in \(L_{\text{max}(2, j)}\).

**Proof.** The definitions, which were given by Moore [3] \(f_{\text{iter}}(i, \bar{x}) = h(2i)\), where

\[h(0) = g(0) = \bar{x},\]
\[
\partial_t g(t) = \left[ f(h(t)) - h(t) \right] s(t),
\]
\[
\partial_t h(t) = \geq \left[ \frac{g(t) - h(t)}{r(t)} \right] (1 - s(t)),
\]
with \( s \) - a square wave function in \( L_2 \) and \( r(0) = 0 \), \( \partial_t r(t) = 2s(t) - 1 \), \( r, s \in L_2 \) give us the desirable statement. □

**Lemma 3.8** The \( R^l \)-recursive functions \( \gamma_2 : R^2 \to R \), \( \gamma_1^1, \gamma_2^2 : R \to R \) such that \((\forall x, y \in R)\gamma_2^1(\gamma_2(x, y)) = x \), \((\forall x, y \in R)\gamma_2^2(\gamma_2(x, y)) = y \), have the following properties: \( \gamma_2, \gamma_1^2 \) are in \( L_{10} \), \( \gamma_2^2 \) is in \( L_{14} \).

**Proof.** We have the auxiliary functions \( \Gamma_2, \Gamma_1^1, \Gamma_2^2 \), which are coding and decoding functions in the interval \((0,1) : \Gamma_2^1(x, y) = c(x) + c(y)/10 \), where
\[
c(x) = \lim_{i \to \infty} z(a(i, x)) / 10^i + b(i, x) / 10^i,
\]
and later
\[
z_{iter}(i, a_1 \ldots a_n a_{n+1} \ldots) = a_1 \ldots a_n 0 \ldots a_{n+1} \ldots,
\]
\[
a(i, 0, a_1 a_2 \ldots) = 0 a_1 \ldots a_i
\]
\[
b(i, 0, a_1 a_2 \ldots) = 0.0 \xi_1 \ldots,
\]
\[
(z')(x) = \begin{cases} 100 \lfloor x \rfloor + 10 (x - \lfloor x \rfloor), & \text{if } \lfloor x \rfloor \neq x, \\ x, & \text{if } \lfloor x \rfloor = x; \end{cases}
\]
and later \( z_{iter} \) belongs to \( L_4 \), hence \( \Gamma_2(x, y) \in L_{10} \), decoding of the first element is described in the symmetric way so \( \Gamma_2^1(x) \) is in \( L_{10} \), but \( \Gamma_2^2(x) = \Gamma_2^1(10 - \lfloor 10x \rfloor) \) so \( \Gamma_2^2 \in L_{14} \).

The functions \( \Gamma_2, \Gamma_1^1, \Gamma_2^2 \) can be extended to all reals by one-to-one \( f : (0,1) \to R \in L_{10} \) without the loss of their class. □

The same method of coding and decoding by interlacing of ciphers (only the power of 10 should be changed) gives us the functions \( \gamma_n : R^n \to R \) and \( \gamma_i^i : R \to R \) for \( i = 1, \ldots, n \) such that
\[
(\forall x_i, x_n \in R)\gamma_i^i(\gamma_n(x_1, \ldots, x_n)) = x_i
\]
in the same class: \( \gamma_n, \gamma_i^i \in L_{10} \) and \( (\forall i > 1)\gamma_i^i \in L_{14} \).

We finish this part with the important form of defining: a new function is given as a product of values \( f \) in some integer points.
Lemma 3.9 There exists such constant \( p \in \mathbb{N} \) that for the function
\[
\prod_{z=0}^{y} f(\overline{x}, z) = \begin{cases} 
\frac{f(\overline{x}, 0) f(\overline{x}, 1) \ldots f(\overline{x}, y-1)}{y!}, & \text{if } y \geq 1, \\
1, & \text{if } 0 \leq y < 1, \\
0, & \text{if } y < 0,
\end{cases}
\]
if the function \( f \) is in the class \( L_m \) then \( \prod_{z=0}^{y} f(\overline{x}, z) \) is in the class \( L_{m+p} \) (\( p \) is independent of \( m \)).

Proof. By the definitions
\[
t(w) = \gamma_{n+2}^{1,n}(w) \cdot \gamma_{n+2}^{n+1}(w) + 1, \quad f = \gamma_{n+2}^{1,n}(w) \cdot \gamma_{n+2}^{n+1}(w) \cdot \gamma_{n+2}^{n+2}(w)
\]
and
\[
S(\overline{x}, z) = t(\overline{x}, z, \lim_{y \to \infty} \gamma_{n+2}^{n+2}(S(\overline{x}, z)))
\]
we get the property
\[
\prod_{y=0}^{z} f(\overline{x}, y) = \gamma_{n+2}^{n+2}(S(\overline{x}, z)).
\]
From the definition of the limit hierarchy we get \( \prod_{y=0}^{z} f(\overline{x}, y) \in L_{m+38} \)

In the rest of the paper we will use the constant \( p \) as the number of limits used in the recursive definition of the product \( \prod_{y=0}^{z} f(\overline{x}, y) \) instead of the value 38. The above constructions are tedious and can be improved with a better approximation of \( p \).

4. Main results

Now we are ready to formulate two theorems which demonstrate connections between \( L \)-hierarchy and \( M \)-hierarchy.

**Theorem 4.1** Let \( f : \mathbb{R}^n \to \mathbb{R} \) be an \( \mathbb{R} \)-recursive function. Then if \( f \in L_i \) then \( f \in M_{10i} \).

Proof. We use a simple induction here. The case \( i = 0 \) is given in Lemma 3.3. Now let us suppose that the thesis is true for \( i = n \). Let \( f \in L_{n+1} \) be defined as \( f(\overline{x}) = \lim_{y \to \infty} g(\overline{x}, y) \) for \( g \in L_n \). Then we can recall Theorem 4.2 from [6] which gives us the following result: to define \( f \) from \( g \) it is necessary to use at
most 10 μ-operation. Hence for $g \in M_{10n}$ the function $f$ satisfies $f \in M_{10n+10}$. Similar inferences hold for $\lim \inf, \lim \sup$. □

Now we can give the result about the 'limit complexity' of the infimum operator $\mu$.

**Lemma 4.2** If $f(x,y): R^{n+1} \to R$ is in the class $L_m$ then the function $g: R^n \to R$, $g(x) = \mu_y f(x,y)$ is in the class $L_{m+3p+9}$ is from Lemma 3.9.

**Proof.** Here we must employ the results from [6]. There we defined the function $g(x) = \mu_y f(x,y)$ for $f(x,y): R^{n+1} \to R$ ($f$ - $R$-recursive) replacing the $\mu$-operator by limit operation. First we introduced the function

$$Z^f(x,z) = \begin{cases} \inf_y \{f : K^f(x,y) = 0\}, & \text{if } z = 0 \text{ and } \exists y K^f(x,y) = 0, \\ \text{undefined} & \text{if } z = 0 \text{ and } \forall y K^f(x,y) \neq 0, \\ 1 & \text{if } z \neq 0, \end{cases}$$

given in the following way:

$$Z^f(x,z) = \begin{cases} \text{undefined} & \text{if } (z=0) \land (S^f(x) < \frac{1}{12}), \\ \sqrt{S^f(x)} - \frac{1}{12}, & \text{if } (z=0) \land (S^f(x) \geq \frac{1}{12}) \land f(x,\sqrt{S^f(x)} - \frac{1}{12}) = 0, \\ -\sqrt{S^f(x)} - \frac{1}{12}, & \text{if } (z=0) \land (S^f(x) \geq \frac{1}{12}) \land f(x,-\sqrt{S^f(x)} - \frac{1}{12}) = 0, \\ 1, & \text{if } z \neq 0, \end{cases}$$

where $S^f(x) = \lim_{t \to 0} S^f_1(x,t) + \lim_{t \to 0} S^f_2(x,t)$. Both functions $S^f_1, S^f_2$ are defined by an integration

$$S^f_i(x,t) = \int y^2 \left(1 - h^f(x,(-1)^{i+1} y - 1/2,(-1)^{i+1} y + 1/2)\right) dy, \quad i = 1, 2$$

from $h^f(x,a,b) = \lim \inf_{t \to 0} \prod_{w=0}^{i+1} K^f \left(x, a + \frac{b-a}{z}\right)$ where $K^f$ is the characteristic function of $f$.

Hence we can conclude that if $K^f$ is in the $L_s$ then $Z^f$ is in the class $L_{s+p+3}$. Let us finish with the definition of the characteristic function of the infimum of zeros of $f$ (see Theorem 4.2 from [5].
\[ K^f_\mu(y) = 1 - \lim_{a \to -\infty \atop b \to +\infty} \lim_{z \to \infty} G^f(\bar{x}, z, a, b, y), \]

where \( G^f(\bar{x}, z, a, b, y) \) divides the interval \([a,b]\) into \(2^{\lfloor z \rfloor}\) equal subintervals and gives the value 1 for \(y\) from the subintervals, which contains the least zero of \(f\) in \([a,b]\) and value 0 otherwise. Precisely for \(y\) from \([a, a + \frac{b-a}{2^{\lfloor z \rfloor}}]\):

\[
G^f(\bar{x}, z, a, b, y) = \begin{cases} 
1, & \text{if } h^f(\bar{x}, a, a + \frac{b-a}{2^{\lfloor z \rfloor}}) = 0, \\
0, & \text{otherwise}
\end{cases}
\]

for \(y \in \left[a + \frac{(k-1)(b-a)}{2^{\lfloor z \rfloor}}, a + \frac{k(b-a)}{2^{\lfloor z \rfloor}}\right]\) (where \(k = 2,3,\ldots,2^n\)) we have:

\[
G^f(\bar{x}, z, a, b, y) = \begin{cases} 
1, & \text{if } \prod_{i=1}^{k-1} h^f(\bar{x}, a + \frac{(i-1)(b-a)}{2^{\lfloor z \rfloor}}, a + \frac{i(b-a)}{2^{\lfloor z \rfloor}}) \neq 0, \\
\land h^f(\bar{x}, a + \frac{(k-1)(b-a)}{2^{\lfloor z \rfloor}}, a + \frac{k(b-a)}{2^{\lfloor z \rfloor}}) = 0, & \text{otherwise}
\end{cases}
\]

and for \(Y \notin [A,B]\) the function \(g^f_x\) is equal to 2.

The definition of \(G_f\) is given by the cases with respect to the value of the expression given by \(\prod h^f\), since for \(f \in L_m\), the function \(h_f \in L_{m+p+2}\) and \(G^f \in L_{m+2+p+3}\). Then we have \(K^f_\mu \in L_{m+2+p+6}\). Now we must use the function \(K^f_\mu\) in the same way as \(K^f\) which gives us \(Z_f\) in the class \(L_{m+3,p+9}\). The final definition of \(g(\bar{x}) = \mu_f(\bar{x}, y)\) ([5] Theorem 4.3) given below.
\[
g(\vec{x}) = \begin{cases} 
Z_{f^-}(\vec{x},0) - Z_{f^+}(\vec{x},0), & \text{if } S_{f^-}(\vec{x}) < \frac{1}{12} \land S_{f^+}(\vec{x}) < \frac{1}{12}, \\
Z_{f^-}(\vec{x},0), & \text{if } \left( S_{f^-}(\vec{x}) \geq \frac{1}{12} \land S_{f^+}(\vec{x}) < \frac{1}{12} \right) \lor \\
-Z_{f^+}(\vec{x},0), & \text{if } \left( S_{f^-}(\vec{x}) < \frac{1}{12} \land S_{f^+}(\vec{x}) \geq \frac{1}{12} \right) \lor \\
& \left( S_{f^-}(\vec{x}) < \frac{1}{12} \land S_{f^+}(\vec{x}) < \frac{1}{12} \right) \land Z_{f^-}(\vec{x},0) < Z_{f^+}(\vec{x},0), \\
& \left( S_{f^-}(\vec{x}) < \frac{1}{12} \land S_{f^+}(\vec{x}) \geq \frac{1}{12} \right) \land Z_{f^-}(\vec{x},0) \geq Z_{f^+}(\vec{x},0),
\end{cases}
\]

where \( f^+ (\vec{x}, y) = \begin{cases} 
f(\vec{x}, y), & y \geq 0, \\
1, & y < 0;
\end{cases} \quad f^- (\vec{x}, y) = \begin{cases} 
f(\vec{x}, -y), & y > 0, \\
1, & y \leq 0;
\end{cases} \) remains the class of \( g \) identical to the class of \( Z_f \), i.e. \( g \in L_{m+3,p+9} \).

**Theorem 4.3** Let \( f : \mathbb{R}^n \to \mathbb{R} \) be an \( \mathbb{R} \)-recursive function. Then for all \( i \geq 0 \) if \( f \in M_i \) then \( f \in L_{(i,p+9);i} \).

The above statement is a simple consequence of the fact \( M_0 = L_0 \) and Lemma 4.2.

### 5. Conclusions

In the paper we give the first rough approximation of 'a complexity' of limit operations in the terms of the \( \mu \)-operator and conversely. The results, interpreted in the intuitional way, can suggest what kind of connection exists between infinite limits and a \( \mu \)-operator.

We also establish the proper relation between the levels of the limit hierarchy and \( \mu \)-hierarchy. Let us point out that in consequence we may investigate analogies which exist for the limit hierarchy (also \( \mu \)-hierarchy) and Baire classes.
[7]. Also the kind of a connection between the $\sum^0_n$-measurable functions and $R$-recursive functions is an open problem.

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References