Two hierarchies of $R$-recursive functions

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Abstract

In the paper some aspects of complexity of $R$-recursive functions are considered. The limit hierarchy of $R$-recursive functions is introduced by the analogy to the $\mu$-hierarchy. Then its properties and relations to the $\mu$-hierarchy are analysed.

1. Introduction

The classical theory of computation deals with the functions on enumerable (especially natural) domains. The fundamental notion in this field is the notion of a (partial) recursive function. The problem of hierarchies for these functions is also in the interest of mathematicians (for elementary, primitive recursive function, Grzegorczyk hierarchy, compare [1]).

During past years many mathematicians have been interested in creating analogous models of computation on real numbers (see for example Grzegorczyk [2], Blum, Shub, Smale [3]). An interesting approach was given by Moore. In the work [4] he defined a set of functions on the reals $\mathbb{R}$ (called $R$-recursive functions) in the analogous way to the classical recursive functions on the natural numbers $\mathbb{N}$. His model has a continuous time of computation (a continuous integration instead of a discrete recursion). The great importance in Moore's model has the zero-finding operation $\mu$, which is used to construct $\mu$-hierarchy of $R$-recursive functions.

It was shown [5] that the zero-finding operator $\mu$ can be replaced by the operation of infinite limits. This allows us to define a limit hierarchy and relate it to $\mu$-hierarchy.

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2. Preliminaries

We start with a fundamental definition of a class of real functions called $\mathcal{R}$-recursive functions [4].

**Definition 2.1** The set of $\mathcal{R}$-recursive functions is generated from the constants 0, 1 by the operations:

1) **composition:** $h(\overline{x}) = f \left( g(\overline{x}) \right)$;

2) **differential recursion:** $h(\overline{x}, 0) = f(\overline{x}),\partial_y h(\overline{x}, y) = g(\overline{x}, y, h(\overline{x}, y))$ (the equivalent formulation can be given by integrals: $h(\overline{x}, y) = f(\overline{x}) + \int_0^y g(\overline{x}, y', h(\overline{x}, y')) dy'$);

3) **$\mu$-recursion** $h(\overline{x}) = \mu_y f(\overline{x}, y) = \inf \{y : f(\overline{x}, y) = 0\}$, where infimum chooses the number $y$ with the smallest absolute value and for two $y$ with the same absolute value the negative one;

4) **vector-valued functions** can be defined by defining their components.

Several comments are needed to the above definition. A solution of a differential equation need not be unique or can diverge. Hence, we assume that if $h$ is defined by a differential recursion then $h$ is defined only where a finite and unique solution exists. This is why the set of $\mathcal{R}$-recursive functions includes also partial functions. We use (after [4]) the name of $\mathcal{R}$-recursive functions in the article, however we should remember that in reality we have partiality here (partial $\mathcal{R}$-recursive functions).

The second problem arises with the operation of infimum. Let us observe that if an infinite number of zeros accumulates just above some positive $y$ or just below some negative $y$ then the infimum operation returns that $y$ even if it itself is not a zero.

In the papers [5, 6] it was shown that if in the Moore's definition [4] $\mu$-operation is replaced by infinite limits: $h(\overline{x}) = \lim \inf_{y \to \infty} g(\overline{x}, y)$, $h(\overline{x}) = \lim \sup_{y \to \infty} g(\overline{x}, y)$ then the resulting class of functions remains the same.

This gives us also the following result (including the limit operation in the form $h(\overline{x}) = \lim_{y \to \infty} g(\overline{x}, y)$, which can be in the obvious way obtained from limsup, liminf:
Corollary 2.2 The class of $R$-recursive functions is closed under the operations of infinite limits: $h(\bar{x}) = \liminf_{y \to \infty} g(\bar{x}, y)$, $h(\bar{x}) = \limsup_{y \to \infty} g(\bar{x}, y)$, $h(\bar{x}) = \lim_{y \to \infty} g(\bar{x}, y)$.

3. Hierarchies

The operator $\mu$ is a key operator in generating the $R$-recursive functions. In a physical sense it has a property of being strongly uncomputable. This fact suggests creating a hierarchy, which is built with respect to the number of uses of $\mu$ in the definition of a given $f$.

Definition 3.1 ([4]) For a given $R$-recursive expression $s(\bar{x})$, let $M_x(s)$ (the $\mu$-number with respect to $x_i$) be defined as follows:

1. $M_x(0) = M_x(1) = M_x(-1) = 0$,
2. $M_x(f(g_1, g_2, \ldots)) = \max_j (M_{x_j}(f) + M_x(g_j))$,
3. $M_x(h = f + \int_0^y g(\bar{x}, y', h) dy') = \max(M_x(f), M_x(g), M_h(g))$,
4. $M_y(h = f + \int_0^y g(\bar{x}, y', h) dy') = \max(M_y(g), M_h(g))$,
5. $M_x(\mu, f(\bar{x}, y)) = \max(M_x(f), M_y(f)) + 1$,

where $x$ can be any $x_1, \ldots, x_n$ for $\bar{x} = (x_1, \ldots, x_n)$.

For an $R$-recursive function $f$, let $M(f) = \max_{s} (s)$ minimized over all expressions $s$ that define $f$. Now we are ready to define $M$-hierarchy ($\mu$-hierarchy) as a family of $M_j = \{f : M'(f) \leq j\}$.

Let us construct the analogous definition of $L$-hierarchy by replacing in the above definition $M_x$ by $L_x$ and changing line (5) to the following form (5'):

$L_x\left(\liminf_{y \to \infty} g(\bar{x}, y)\right) = L_x\left(\limsup_{y \to \infty} g(\bar{x}, y)\right) = $ $L_x\left(\lim_{y \to \infty} g(\bar{x}, y)\right) = \max(L_x(f), L_y(f)) + 1.$

For an $R$-recursive function $f$, let $L(f) = \max_s (s)$ minimized over all expressions $S$ that define $f$ without using the $\mu$-operation.
Definition 3.2 The $L$-hierarchy is a family of $L_j = \{f : L(f) \leq j\}$.

Let us add that in Definition 3.2 we use explicitly the operator $f(\bar{x}) = \lim_{y \to \infty} g(\bar{x}, y)$ to avoid its construction by other operators (lim sup, lim inf), which would effect in a superficially higher class of a complexity of a function $f$.

As an obvious corollary from definitions we have the following statement.

Lemma 3.3 The classes $M_0$ and $M_1$ are identical.

A function $f \in L_0 = M_0$ will be called (by an analogy to the case of natural recursive functions) a primitive $R$-recursive function. After Moore [4] we can conclude that such functions as: $-x$, $x + y$, $xy$, $x/y$, $e^x$, $\ln x$, $y^x$, $\sin x$, $\cos x$ are primitive $R$-recursive.

We can give a few results on some levels of the limit hierarchy.

Lemma 3.4. The Kronecker $\delta$ function, the signum function and absolute value belong to the first level ($L_1$) of limit hierarchy.

Proof. It is sufficient to take the following definitions [5]: hence $\delta(0) = 1$ and for all $x \neq 0$ we have $\delta(x) = 0$ let us define $\delta(x) = \liminf_{y \to \infty} \left( \frac{1}{1 + x^2} \right)^y$. Now from the expression $\liminf_{y \to \infty} \arctan xy = \begin{cases} \pi/2, & \text{if } x > 0, \\ 0, & \text{if } x = 0, \\ -\pi/2, & \text{if } x < 0, \end{cases}$ we obtain $\sgn(x) = \frac{\liminf_{y \to \infty} \arctan xy}{2 \arctan 1}$ and $|x| = \sgn(x)x$.

We should be careful with definitions of functions by cases:

Lemma 3.5 For $h(\bar{x}) = \begin{cases} g_1(\bar{x}), & \text{if } f(\bar{x}) = 0, \\ g_2(\bar{x}), & \text{if } f(\bar{x}) = 1, \\ M, & \text{if } f(\bar{x}) \geq k - 1 \end{cases}$ and $g_i \in L_{i}$ for all $1 \leq i \leq k$, $f \in L_\infty$ the function $h$ belongs to $L_{\max(n_1, \ldots, n_k, m+1)}$. 
Proof. Let us see that \( eq(x, y) = \delta(x - y) \in L_1 \) and \( ge(x, y) = \frac{\left( \text{sgn}(x - y) + eq(x, y) \right)}{2} + \frac{1}{2} \in L_1 \). Then of course
\[
   h(\bar{x}) = \sum_{i=1}^{k-1} g_i(\bar{x}) eq(f(\bar{x}), i - 1) + g_k(\bar{x}) ge(f(\bar{x}), k - 1)
\]
\( \square \)

Of course this result can be easily extended to other forms of definitions by cases.

Lemma 3.6 The function \( \Theta(x) \) (equal to 1 if \( x \geq 0 \), otherwise 0), maximum \( \max(x, y) \), square-wave function \( s \) are in \( L_2 \), the function \( p(x) \) such that \( p(x) = 1 \) for \( x \in [2n, 2n+1] \) and \( p(x) = 0 \) for \( x \in [2n+1, 2n+2] \) is in \( L_2 \) and the floor function \( \lfloor x \rfloor \) is in \( L_3 \).

Proof. We give the proper definitions (from [6]) for these functions. Let
\[
   \Theta(x) = \delta(x - |x|), \quad \max(x, y) = x\Theta(x - y) + (1 - \delta(x - y))[x\Theta(x - y) + y\Theta(y - x)],
\]
\[
   s(x) = \Theta(\sin(\pi x)).
\]

The function \( p(x) \) can be given as
\[
   s(x) \left( 1 - \delta \left( \sin \left( \frac{(x-1)\pi}{2} \right) \right) \right), \text{ so } p \in L_2.
\]

The floor function we can define by the auxiliary function \( w(0) = 0, \partial_x w(x) = 2\Theta(-\sin(2\pi x)) \) as
\[
   \lfloor x \rfloor = \begin{cases} 
   2w(x/2) & \text{if } p(x) = 1, \\
   2w((x-1)/2) & \text{if } p(x) = 0.
   \end{cases}
\]

From the above equation we have \( \lfloor x \rfloor \) in \( L_3 \) \( \square \)

Let us recall that if \( f : R^n \to R \) is an \( R \)-recursive function then the function \( f_{iter}(i, \bar{x}) \) is \( R \)-recursive, too.

Lemma 3.7 Let \( f : R^n \to R \) belongs to the class \( L_i \), then we have \( f_{iter} : R^{n+1} \to R \) is in \( L_{\max(2, j)} \).

Proof. The definitions, which were given by Moore [3] \( f_{iter}(i, \bar{x}) = h(2i) \), where
\[
   h(0) = g(0) = \bar{x},
\]
\[
\partial_t g(t) = \left[ f(h(t)) - h(t) \right] s(t),
\]
\[
\partial_t h(t) = \left[ \frac{g(t) - h(t)}{r(t)} \right] (1 - s(t)),
\]
with \( s \) - a square wave function in \( L_2 \) and \( r(0) = 0 \), \( \partial_r(t) = 2s(t) - 1 \), \( r, s \in L_2 \) give us the desirable statement. \( \square \)

**Lemma 3.8** The \( R^l \)-recursive functions \( \gamma_2 : R^2 \to R \), \( \gamma_1^2, \gamma_2^2 : R \to R \) such that \((\forall x, y \in R) \gamma_1^2(\gamma_2(x, y)) = x\), \((\forall x, y \in R) \gamma_2^2(\gamma_1(x, y)) = y\), have the following properties: \( \gamma_2 \), \( \gamma_1^2 \) are in \( L_{10} \), \( \gamma_2^2 \) is in \( L_{14} \).

**Proof.** We have the auxiliary functions \( \Gamma_2 \), \( \Gamma_1^2 \), \( \Gamma_2^2 \), which are coding and decoding functions in the interval \((0,1) : \Gamma_2(x, y) = c(x) + c(y)/10\), where
\[
c(x) = \lim_{t \to \infty} z(a(i,x))/10^2 + b(i,x)/10^t,
\]
and later \( z(x) = \lim_{t \to \infty} z_{iter}(i,x), \)
\[
z_{iter}(i,a_1,\ldots,a_n,a_{n+1},\ldots) = a_1\ldots a_n0a_{n+1}0a_{n+2}0, \quad a(i,0,a_1,a_2,\ldots) = 0.a_1\ldots a_i, \\
b(i,0,a_1,a_2,\ldots) = 0.0a_i\ldots a_{i-1}, \]
\[(z'(x)) = \begin{cases} 100[x] + 10(x-[x]), & \text{if } x \neq \lfloor x \rfloor \\
x, & \text{if } \lfloor x \rfloor = x; \end{cases} \in L_4, a,b \in L_4. \]
Also \( z_{iter} \) belongs to \( L_4 \), hence \( \Gamma_2(x,y) \in L_{10} \), decoding of the first element is described in the symmetric way so \( \Gamma_1^2(x) \) is in \( L_{10} \), but \( \Gamma_2^2(x) = \Gamma_1^2(10-\lfloor 10x \rfloor) \) so \( \Gamma_2^2 \in L_{14} \).

The functions \( \Gamma_2 \), \( \Gamma_1^2 \), \( \Gamma_2^2 \) can be extended to all reals by one-to-one \( f : (0,1) \to R \in L_0 \) without the loss of their class. \( \square \)

The same method of coding and decoding by interlacing of ciphers (only the power of 10 should be changed) gives us the functions \( \gamma_n : R^n \to R \) and \( \gamma_n^i : R \to R \) for \( i = 1, \ldots, n \) such that
\[(\forall i)(\forall x_1,\ldots,x_n \in R) \gamma_n^i(\gamma_n(x_1,\ldots,x_n)) = x_i, \]
in the same class: \( \gamma_n, \gamma_n^1 \in L_{10} \) and \( \forall i > 1 \gamma_n^i \in L_{14} \).

We finish this part with the important form of defining: a new function is given as a product of values \( f \) in some integer points.
Lemma 3.9 There exists such constant \( p \in N \) that for the function

\[
\prod_{z=0}^{y} f(\bar{x}, z) = \begin{cases} 
  f(\bar{x}, 0) f(\bar{x}, 1) \ldots f(\bar{x}, y-1), & \text{if } y \geq 1, \\
  1, & \text{if } 0 \leq y < 1, \\
  0, & \text{if } y < 0,
\end{cases}
\]

if the function \( f \) is in the class \( L_m \) then \( \prod_{z=0}^{y} f(\bar{x}, z) \) is in the class \( L_{m+p} \) (\( p \) is independent of \( m \)).

Proof. By the definitions

\[
t(w) = \gamma_{n+2}(\gamma_{n+2}^{1,1}(w), \gamma_{n+2}^{1,1}(w)) + 1, \quad f(\bar{x}, 0) f(\bar{x}, 1) \ldots f(\bar{x}, y-1)
\]

and

\[
S(\bar{x}, z) = t(s(s(\bar{x}, 0)) \ldots) = t_{\text{iter}}(s(s(\bar{x}, 0)), \gamma_{n+2}(\bar{x}, 0, 1))
\]

we get the property

\[
\prod_{y=0}^{z} f(\bar{x}, y) = \gamma_{n+2}^{1,1}(S(\bar{x}, z)).
\]

From the definition of the limit hierarchy we get \( \prod_{y=0}^{z} f(\bar{x}, y) \in L_{m+38} \).

In the rest of the paper we will use the constant \( p \) as the number of limits used in the recursive definition of the product \( \prod_{y=0}^{z} f(\bar{x}, y) \) instead of the value 38. The above constructions are tedious and can be improved with a better approximation of \( p \).

4. Main results

Now we are ready to formulate two theorems which demonstrate connections between \( L \)-hierarchy and \( M \)-hierarchy.

Theorem 4.1 Let \( f : \mathbb{R}^n \rightarrow \mathbb{R} \) be an \( R \)-recursive function. Then if \( f \in L_i \) then \( f \in M_{10i} \).

Proof. We use a simple induction here. The case \( i = 0 \) is given in Lemma 3.3. Now let us suppose that the thesis is true for \( i = n \). Let \( f \in L_{n+1} \) be defined as \( f(\bar{x}) = \lim_{y \to \infty} g(\bar{x}, y) \) for \( g \in L_n \). Then we can recall Theorem 4.2 from [6] which gives us the following result: to define \( f \) from \( g \) it is necessary to use at
most 10 μ-operation. Hence for \( g \in M_{10n} \) the function \( f \) satisfies \( f \in M_{10n+10} \).
Similar inferences hold for \( \lim \inf \), \( \lim \sup \).

Now we can give the result about the 'limit complexity' of the infimum operator \( \mu \).

**Lemma 4.2** If \( f(x,y):R^{n+1} \rightarrow R \) is in the class \( L_m \) then the function 
\[ g:R^n \rightarrow R, \quad g(x) = \mu_y f(x,y) \]
is in the class \( L_{m+3p+9} \) is from Lemma 3.9.

**Proof.** Here we must employ the results from [6]. There we defined the function 
\[ g(x) = \mu_y f(x,y) \]
for \( f(x,y):R^{n+1} \rightarrow R \) \( (f \; \text{ - } \; R\text{-recursive}) \)
replacing the μ-operator by limit operation. First we introduced the function 
\[ Z'(x,z) = \begin{cases} 
\inf_y \{ f : K'(x,y) = 0 \}, & \text{if } z = 0 \text{ and } \exists y K'(x,y) = 0, \\
\text{undefined} & \text{if } z = 0 \text{ and } \forall y K'(x,y) \neq 0, \\
1 & \text{if } z \neq 0,
\end{cases} \]
given in the following way:
\[ Z'(x,z) = \begin{cases} 
\text{undefined} & \text{if } (z=0) \land (S'(x) < \frac{1}{12}), \\
\sqrt{S'(x)} - \frac{1}{12}, & \text{if } (z=0) \land (S'(x) \geq \frac{1}{12}) \land f(x,\sqrt{S'(x)} - \frac{1}{12}) = 0, \\
-\sqrt{S'(x)} - \frac{1}{12}, & \text{if } (z=0) \land (S'(x) \geq \frac{1}{12}) \land f(x,-\sqrt{S'(x)} - \frac{1}{12}) = 0, \\
1, & \text{if } z \neq 0,
\end{cases} \]
where \( S'(x) = \lim_{t \rightarrow -\infty} S'_1(x,t) + \lim_{t \rightarrow +\infty} S'_2(x,t) \). Both functions \( S'_1 \), \( S'_2 \) are defined by an integration 
\[ S'_i(x,t) = \int y^2 \left(1-h'(x,-1)^{i+1}y-1/2,(-1)^{i+1}y+1/2) \right) dy, \quad i = 1,2 \]
from \( h'(x,a,b) = \liminf_{j \rightarrow \infty} \prod_{w=0}^{i+1} K'(x,a+b \frac{b-a}{z}) \) where \( K' \) is the characteristic function of \( f \).

Hence we can conclude that if \( K' \) is in the \( L_s \) then \( Z'_f \) is in the class \( L_{s+p+3} \).

Let us finish with the definition of the characteristic function of the infimum of zeros of \( f \) (see Theorem 4.2 from [5]

\[ K_\mu^f(y) = 1 - \lim_{a \to -\infty} \lim_{b \to +\infty} \lim_{z \to +\infty} G^f(\bar{x}, z, a, b, y), \]

where \( G^f(\bar{x}, z, a, b, y) \) divides the interval \([a, b]\) into \(2^{z} \) equal subintervals and gives the value 1 for \( y \) from the subintervals, which contains the least zero of \( f \) in \([a, b]\) and value 0 otherwise. Precisely for \( y \) from \([a, a + \frac{b-a}{2^z}]\)

\[
G^f(\bar{x}, z, a, b, y) = \begin{cases} 
1, & \text{if } h^f(\bar{x}, a, a + \frac{b-a}{2^z}) = 0, \\
0, & \text{otherwise}
\end{cases}
\]

for \( y \in \left(a + \frac{(k-1)(b-a)}{2^z}, a + \frac{k(b-a)}{2^z}\right) \) (where \( k = 2, 3, ..., 2^z \)) we have:

\[
G^f(\bar{x}, z, a, b, y) = \begin{cases} 
1, & \text{if } \prod_{i=1}^{k-1} h^f(\bar{x}, a + \frac{(i-1)(b-a)}{2^z}, a + \frac{i(b-a)}{2^z}) \neq 0, \\
0, & \text{otherwise}
\end{cases}
\]

and for \( Y \notin [A, B] \) the function \( g^f_x \) is equal to 2.

The definition of \( G^f \) is given by the cases with respect to the value of the expression given by \( \prod h^f \), since for \( f \in L_m \), the function \( h^f \in L_{m+p+2} \) and \( G^f \in L_{m+2,p+3} \). Then we have \( K_\mu^f \in L_{m+2,p+6} \). Now we must use the function \( K_\mu^f \) in the same way as \( K^f \) which gives us \( Z_f \) in the class \( L_{m+3,p+9} \). The final definition of \( g(\bar{x}) = \mu, f(\bar{x}, y) \) ([5] Theorem 4.3) given below
\[
\begin{align*}
Z^f (\bar{x},0) - Z^f (\bar{x},0), & \quad \text{if } S^f (\bar{x}) < \frac{1}{12} \wedge S^f (\bar{x}) < \frac{1}{12}, \\
Z^f (\bar{x},0), & \quad \text{if } \left( S^f (\bar{x}) \geq \frac{1}{12} \wedge S^f (\bar{x}) < \frac{1}{12} \right) \\
& \quad \text{or} \\
& \quad \left( S^f (\bar{x}) < \frac{1}{12} \wedge S^f (\bar{x}) < \frac{1}{12} \wedge Z^f (\bar{x},0) < Z^f (\bar{x},0) \right), \\
-Z^f (\bar{x},0), & \quad \text{if } \left( S^f (\bar{x}) < \frac{1}{12} \wedge S^f (\bar{x}) \geq \frac{1}{12} \right) \\
& \quad \text{or} \\
& \quad \left( S^f (\bar{x}) < \frac{1}{12} \wedge S^f (\bar{x}) < \frac{1}{12} \wedge Z^f (\bar{x},0) \geq Z^f (\bar{x},0) \right),
\end{align*}
\]

g (\bar{x}) = \begin{cases} 
Z^f (\bar{x},0) - Z^f (\bar{x},0), & \text{if } S^f (\bar{x}) < \frac{1}{12} \wedge S^f (\bar{x}) < \frac{1}{12}, \\
Z^f (\bar{x},0), & \text{if } \left( S^f (\bar{x}) \geq \frac{1}{12} \wedge S^f (\bar{x}) < \frac{1}{12} \right) \\
& \text{or} \\
& \left( S^f (\bar{x}) < \frac{1}{12} \wedge S^f (\bar{x}) < \frac{1}{12} \wedge Z^f (\bar{x},0) < Z^f (\bar{x},0) \right), \\
-Z^f (\bar{x},0), & \text{if } \left( S^f (\bar{x}) < \frac{1}{12} \wedge S^f (\bar{x}) \geq \frac{1}{12} \right) \\
& \text{or} \\
& \left( S^f (\bar{x}) < \frac{1}{12} \wedge S^f (\bar{x}) < \frac{1}{12} \wedge Z^f (\bar{x},0) \geq Z^f (\bar{x},0) \right),
\end{cases}
\]

where \( f^+ (\bar{x}, y) = \begin{cases} f (\bar{x}, y), & y \geq 0, \\
1, & y < 0; \end{cases} \)
and \( f^- (\bar{x}, y) = \begin{cases} f (\bar{x}, -y), & y > 0, \\
1, & y \leq 0; \end{cases} \) remains the class of \( g \) identical to the class of \( Z^f \), i.e. \( g \in L_{m+3,p+9} \).

**Theorem 4.3** Let \( f : \mathbb{R}^n \to \mathbb{R} \) be an \( \mathbb{R} \)-recursive function. Then for all \( i \geq 0 \) if \( f \in M_i \), then \( f \in L_{(i)+p+9,i} \).

The above statement is a simple consequence of the fact \( M_0 = L_0 \) and Lemma 4.2.

5. Conclusions

In the paper we give the first rough approximation of 'a complexity' of limit operations in the terms of the \( \mu \)-operator and conversely. The results, interpreted in the intuitional way, can suggest what kind of connection exists between infinite limits and a \( \mu \)-operator.

We also establish the proper relation between the levels of the limit hierarchy and \( \mu \)-hierarchy. Let us point out that in consequence we may investigate analogies which exist for the limit hierarchy (also \( \mu \)-hierarchy) and Baire classes.
[7]. Also the kind of a connection between the $\sum_1^\infty$–measurable functions and $\mathbb{R}$-recursive functions is an open problem.

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References