Two hierarchies of $R$-recursive functions

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Abstract

In the paper some aspects of complexity of $R$-recursive functions are considered. The limit hierarchy of $R$-recursive functions is introduced by the analogy to the $\mu$-hierarchy. Then its properties and relations to the $\mu$-hierarchy are analysed.

1. Introduction

The classical theory of computation deals with the functions on enumerable (especially natural) domains. The fundamental notion in this field is the notion of a (partial) recursive function. The problem of hierarchies for these functions is also in the interest of mathematicians (for elementary, primitive recursive function, Grzegorczyk hierarchy, compare [1].

During past years many mathematicians have been interested in creating analogous models of computation on real numbers (see for example Grzegorczyk [2], Blum, Shub, Smale [3]). An interesting approach was given by Moore. In the work [4] he defined a set of functions on the reals $R$ (called $R$-recursive functions) in the analogous way to the classical recursive functions on the natural numbers $N$. His model has a continuous time of computation (a continuous integration instead of a discrete recursion). The great importance in Moore's model has the zero-finding operation $\mu$, which is used to construct $\mu$-hierarchy of $R$-recursive functions.

It was shown [5] that the zero-finding operator $\mu$ can be replaced by the operation of infinite limits. This allows us to define a limit hierarchy and relate it to $\mu$-hierarchy.

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2. Preliminaries

We start with a fundamental definition of a class of real functions called \( \mathbb{R} \)-recursive functions [4].

**Definition 2.1** The set of \( \mathbb{R} \)-recursive functions is generated from the constants \( 0,1 \) by the operations:

1) **composition**: \( h(\vec{x}) = f(g(\vec{x})) \);

2) **differential recursion**: \( h(\vec{x},0) = f(\vec{x}), \partial_y h(\vec{x},y) = g(\vec{x},y,h(\vec{x},y)) \) (the equivalent formulation can be given by integrals: \( h(\vec{x},y) = f(\vec{x}) + \int_0^y g(\vec{x},y',h(\vec{x},y'))dy' \));

3) **\( m \)-recursion** \( h(\vec{x}) = \mu_y f(\vec{x},y) = \inf \{ y : f(\vec{x},y) = 0 \} \), where infimum chooses the number \( y \) with the smallest absolute value and for two \( y \) with the same absolute value the negative one;

4) **vector-valued functions** can be defined by defining their components.

Several comments are needed to the above definition. A solution of a differential equation need not be unique or can diverge. Hence, we assume that if \( h \) is defined by a differential recursion then \( h \) is defined only where a finite and unique solution exists. This is why the set of \( \mathbb{R} \)-recursive functions includes also partial functions. We use (after [4]) the name of \( \mathbb{R} \)-recursive functions in the article, however we should remember that in reality we have partiality here (partial \( \mathbb{R} \)-recursive functions).

The second problem arises with the operation of infimum. Let us observe that if an infinite number of zeros accumulates just above some positive \( y \) or just below some negative \( y \) then the infimum operation returns that \( y \) even if it itself is not a zero.

In the papers [5, 6] it was shown that if in the Moore's definition [4] \( \mu \) -operation is replaced by infinite limits: \( h(\vec{x}) = \liminf_{y \to \infty} g(\vec{x},y) \), \( h(\vec{x}) = \limsup_{y \to \infty} g(\vec{x},y) \) then the resulting class of functions remains the same.

This gives us also the following result (including the limit operation in the form \( h(\vec{x}) = \lim_{y \to \infty} g(\vec{x},y) \), which can be in the obvious way obtained from limsup, liminf:
**Corollary 2.2** The class of $\mathbb{R}$-recursive functions is closed under the operations of infinite limits: $h(\bar{x}) = \liminf_{y \to \infty} g(\bar{x}, y)$, $h(\bar{x}) = \limsup_{y \to \infty} g(\bar{x}, y)$, $h(\bar{x}) = \lim_{y \to \infty} g(\bar{x}, y)$.

### 3. Hierarchies

The operator $\mu$ is a key operator in generating the $\mathbb{R}$-recursive functions. In a physical sense it has a property of being strongly uncomputable. This fact suggests creating a hierarchy, which is built with respect to the number of uses of $\mu$ in the definition of a given $f$.

**Definition 3.1** ([4]) For a given $\mathbb{R}$-recursive expression $s(\bar{x})$, let $M_{x_i}(s)$ (the $\mu$-number with respect to $x_i$) be defined as follows:

\begin{align*}
M_{x_i}(0) &= M_{x_i}(1) = M_{x_i}(-1) = 0, \quad (1) \\
M_{x_i}(f(g_1, g_2, \ldots)) &= \max_j \left( M_{x_i}(f) + M_{x_i}(g_j) \right), \quad (2) \\
M_{x_i}(h = f + \int_0^y g(\bar{x}, y', h) dy') &= \max \left( M_{x_i}(f), M_{x_j}(g), M_h(g) \right), \quad (3) \\
M_{y_j}(h = f + \int_0^y g(\bar{x}, y', h) dy') &= \max \left( M_{y_j}(g), M_h(g) \right), \quad (4) \\
M_{x_i}((\mu, f(\bar{x}, y))) &= \max \left( M_{x_i}(f), M_{y_j}(f) \right) + 1, \quad (5)
\end{align*}

where $x$ can be any $x_1, \ldots, x_n$ for $\bar{x} = (x_1, \ldots, x_n)$.

For an $\mathbb{R}$-recursive function $f$, let $M(f) = \max_{x_i}(s)$ minimized over all expressions $s$ that define $f$. Now we are ready to define $M$-hierarchy ($\mu$-hierarchy) as a family of $M_j = \{ f : M'(f) \leq j \}$.

Let us construct the analogous definition of $L$-hierarchy by replacing in the above definition $M_{x_i}$ by $L_{x_i}$ and changing line (5) to the following form (5'):

\begin{align*}
L_{x_i}(\liminf_{y \to \infty} g(\bar{x}, y)) &= L_{x_i}(\limsup_{y \to \infty} g(\bar{x}, y)) = \\
&= L_{x_i}(\lim g(\bar{x}, y)) = \max \left( L_{x_i}(f), L_{y_j}(f) \right) + 1.
\end{align*}

For an $\mathbb{R}$-recursive function $f$, let $L(f) = \max_{x_i} L_{x_i}(s)$ minimized over all expressions $S$ that define $f$ without using the $\mu$-operation.
Definition 3.2 The $L$-hierarchy is a family of $L_j = \{ f : L(f) \leq j \}$.

Let us add that in Definition 3.2 we use explicitly the operator $f(\bar{x}) = \lim_{y \to \infty} g(\bar{x}, y)$ to avoid its construction by other operators (lim sup, lim inf), which would effect in a superficially higher class of a complexity of a function $f$.

As an obvious corollary from definitions we have the following statement.

Lemma 3.3 The classes $M_0$ and $M_1$ are identical.

A function $f \in L_0 = M_0$ will be called (by an analogy to the case of natural recursive functions) a primitive $R$-recursive function. After Moore [4] we can conclude that such functions as: $-x$, $x + y$, $xy$, $x/y$, $e^x$, $\ln x$, $y^x$, $\sin x$, $\cos x$ are primitive $R$-recursive.

We can give a few results on some levels of the limit hierarchy.

Lemma 3.4. The Kronecker $\delta$ function, the signum function and absolute value belong to the first level ($L_1$) of limit hierarchy.

Proof. It is sufficient to take the following definitions [5]: hence $\delta(0) = 1$ and for all $x \neq 0$ we have $\delta(x) = 0$ let us define $\delta(x) = \liminf_{y \to \infty} \left( \frac{1}{1 + x^2} \right)^y$. Now from the expression $\liminf_{y \to \infty} \arctan xy = \begin{cases} \pi / 2, & \text{if } x > 0, \\ 0, & \text{if } x = 0, \\ -\pi / 2, & \text{if } x < 0, \end{cases}$ we obtain

$$\text{sgn}(x) = \frac{\liminf_{y \to \infty} \arctan xy}{2\arctan 1}$$

and $|x| = \text{sgn}(x)x$.

We should be careful with definitions of functions by cases:

Lemma 3.5 For $h(\bar{x}) = \begin{cases} g_1(\bar{x}), & \text{if } f(\bar{x}) = 0, \\ g_2(\bar{x}), & \text{if } f(\bar{x}) = 1, \\ \vdots \\ g_k(\bar{x}), & \text{if } f(\bar{x}) \geq k - 1 \end{cases}$ and $g_i \in L_{n_i}$ for all $1 \leq i \leq k$.

$f \in L_m$ the function $h$ belongs to $L_{\max(n_1, \ldots, n_k, m + 1)}$. 

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Proof. Let us see that \( eq(x, y) = \delta(x - y) \in L_1 \) and
\[
geq(x, y) = \frac{(\text{sgn}(x - y) + eq(x, y))}{2} + \frac{1}{2} \in L_1.
\]
Then of course
\[
h(\bar{x}) = \sum_{k=1}^{\infty} g_k(\bar{x}) eq(f(\bar{x}), i-1) + g_k(\bar{x}) ge(f(\bar{x}), k-1)
\]
\( \square \)

Of course this result can be easily extended to other forms of definitions by cases.

Lemma 3.6 The function \( \Theta(x) \) (equal to 1 if \( x \geq 0 \), otherwise 0), maximum \( \max(x, y) \), square-wave function \( s \) are in \( L_2 \), the function \( p(x) \) such that \( p(x) = 1 \) for \( x \in [2n, 2n+1] \) and \( p(x) = 0 \) for \( x \in [2n+1, 2n+2] \) is in \( L_2 \) and the floor function \( \lfloor x \rfloor \) is in \( L_3 \).

Proof. We give the proper definitions (from [6]) for these functions. Let
\[
\Theta(x) = \delta(x - |x|),
\]
\[
\max(x, y) = x\delta(x - y) + (1 - \delta(x - y))[x\Theta(x - y) + y\Theta(y - x)],
\]
\[
s(x) = \Theta(\sin(\pi x)).
\]

The function \( p(x) \) can be given as
\[
s(x)\left(1 - \delta\left(\sin\left(\frac{x-1}{2}\right)\pi\right)\right), \quad p \in L_2.
\]

The floor function we can define by the auxiliary function \( w(0) = 0 \), \( \partial_x w(x) = 2\Theta(-\sin(2\pi x)) \) as
\[
\lfloor x \rfloor = \begin{cases} 
2w(x/2) & \text{if } p(x) = 1, \\
2w((x-1)/2) & \text{if } p(x) = 0.
\end{cases}
\]

From the above equation we have \( \lfloor x \rfloor \) in \( L_3 \) \( \square \)

Let us recall that if \( f : R^n \to R \) is an \( R \)-recursive function then the function \( f_{iter}(i, \bar{x}) \) is \( R \)-recursive, too.

Lemma 3.7 Let \( f : R^n \to R \) belongs to the class \( L_j \), then we have \( f_{iter} : R^{n+1} \to R \) is in \( L_{\max(2, j)} \).

Proof. The definitions, which were given by Moore [3] \( f_{iter}(i, \bar{x}) = h(2i) \), where
\[
h(0) = g(0) = \bar{x},
\]
\[ \partial_t g(t) = \left[ f(h(t)) - h(t) \right] s(t), \]
\[ \partial_h r(t) = \left[ \frac{g(t) - h(t)}{r(t)} \right] (1 - s(t)), \]
with \( s \) - a square wave function in \( L_2 \) and \( r(0) = 0, \partial_r r(t) = 2s(t) - 1, r, s \in L_2 \) give us the desirable statement. □

**Lemma 3.8** The \( R^1 \)-recursive functions \( \gamma_2 : R^2 \to R, \quad \gamma_1^1, \gamma_2^2 : R \to R \) such that \((\forall x, y \in R)\gamma_2^1(\gamma_2(x, y)) = x, \quad (\forall x, y \in R)\gamma_2^2(\gamma_2(x, y)) = y, \) have the following properties: \( \gamma_2, \gamma_1^2 \) are in \( L_{10} \), \( \gamma_2^2 \) is in \( L_{14} \).

**Proof.** We have the auxiliary functions \( \Gamma_2, \Gamma_1^1, \Gamma_2^2 \), which are coding and decoding functions in the interval \((0, 1) : \Gamma_2(x, y) = c(x) + c(y)/10\), where
\[
c(x) = \lim_{i \to \infty} z(i, x) = 10^{i^0} + b(i, x)/10^i,
\]
and later \( z(x) = \lim_{i \to \infty} z_{iter}(i, x) \),
\[
z_{iter}(i, a_1...a_n, a_{n+1}... = a_1...a_n 0.a_{n+1}... \quad a(i, 0.a_1a_2...a_{n-1}) = 0.a_1...a_i \quad b(i, 0.a_1a_2...a_{n-1}) = 0.0.a_1...a_i, \]

\[
(z'(x)) = \begin{cases} 
100[x] + 10(x - [x]), & \text{if } \lfloor x \rfloor \neq x, \\
x, & \text{if } \lfloor x \rfloor = x, 
\end{cases}, \quad \text{if } x \notin L_4, a, b \in L_4. \text{ Also } z_{iter} \text{ belongs to } L_4, \text{ hence } \Gamma_2(x, y) \in L_{10}, \text{ decoding of the first element is described in the symmetric way so } \Gamma_1^1(x) \text{ is in } L_{10}, \text{ but } \Gamma_2^2(x) = \Gamma_2^2(10 - [10x]) \text{ so } \Gamma_2^2 \in L_{14}. \]

The functions \( \Gamma_2, \Gamma_1^1, \Gamma_2^2 \) can be extended to all reals by one-to-one \( f : (0, 1) \to R \in L_0 \) without the loss of their class. □

The same method of coding and decoding by interlacing of ciphers (only the power of 10 should be changed) gives us the functions \( \gamma_n : R^n \to R \) and \( \gamma_i^i : R \to R \) for \( i = 1,...,n \) such that
\[
(\forall i)(\forall x_1, ..., x_n \in R)\gamma_i^i(\gamma_n(x_1, ..., x_n)) = x_i
\]
in the same class: \( \gamma_n, \gamma_i^i \in L_{10} \) and \( (\forall i > 1)\gamma_i^i \in L_{14} \).

We finish this part with the important form of defining: a new function is given as a product of values \( f \) in some integer points.
Lemma 3.9 There exists such constant \( p \in \mathbb{N} \) that for the function

\[
\prod_{z=0}^{y} f(\bar{x}, z) = \begin{cases} 
  f(\bar{x}, 0) f(\bar{x}, 1) \ldots f(\bar{x}, y-1), & \text{if } y \geq 1, \\
  1, & \text{if } 0 \leq y < 1, \\
  0, & \text{if } y < 0,
\end{cases}
\]

if the function \( f \) is in the class \( L_m \) then \( \prod_{z=0}^{y} f(\bar{x}, z) \) is in the class \( L_{m+p} \) (\( p \) is independent of \( m \)).

Proof. By the definitions

\[
t(w) = \gamma_{n+2}^{1,n}(w), \gamma_{n+2}^{n+1}(w) + 1, f(\gamma_{n+2}^{1,n}(w), \gamma_{n+2}^{n+1}(w)), \gamma_{n+2}^{n+2}(w) \]

and

\[
S(\bar{x}, z) = t(\ldots s(\bar{x}, 0)) = t_{10} \left[ \left\lfloor \frac{z}{n} \right\rfloor, \gamma_{n+2}^{1,n}(\bar{x}, 0, 1) \right]
\]

we get the property

\[
\prod_{y=0}^{z} f(\bar{x}, y) = \gamma_{n+2}^{n+2}(S(\bar{x}, z)).
\]

From the definition of the limit hierarchy we get \( \prod_{y=0}^{z} f(\bar{x}, y) \in L_{m+38} \). 

In the rest of the paper we will use the constant \( p \) as the number of limits used in the recursive definition of the product \( \prod_{y=0}^{z} f(\bar{x}, y) \) instead of the value 38. The above constructions are tedious and can be improved with a better approximation of \( p \).

4. Main results

Now we are ready to formulate two theorems which demonstrate connections between \( L \)-hierarchy and \( M \)-hierarchy.

Theorem 4.1 Let \( f : \mathbb{R}^n \rightarrow \mathbb{R} \) be an \( \mathbb{R} \)-recursive function. Then if \( f \in L_i \) then \( f \in M_{10i} \).

Proof. We use a simple induction here. The case \( i = 0 \) is given in Lemma 3.3. Now let us suppose that the thesis is true for \( i = n \). Let \( f \in L_{n+1} \) be defined as \( f(\bar{x}) = \lim_{y \to \infty} g(\bar{x}, y) \) for \( g \in L_n \). Then we can recall Theorem 4.2 from [6] which gives us the following result: to define \( f \) from \( g \) it is necessary to use at
most $10\mu$-operation. Hence for $g \in M_{10\mu}$ the function $f$ satisfies $f \in M_{10\mu+10}$. Similar inferences hold for $\lim\inf, \lim\sup$. □

Now we can give the result about the 'limit complexity' of the infimum operator $\mu$.

**Lemma 4.2** If $f(\bar{x},y): R^{n+1} \to R$ is in the class $L_{\mu}$ then the function $g: R^n \to R$, $g(\bar{x}) = \mu_y, f(\bar{x},y)$ is in the class $L_{\mu+3\mu+9}$ is from Lemma 3.9.

**Proof.** Here we must employ the results from [6]. There we defined the function $g: R^n \to R$, $g(\bar{x}) = \mu_y, f(\bar{x},y)$ for $f(\bar{x},y): R^{n+1} \to R$ ($f$ - $R$-recursive) replacing the $\mu$-operator by limit operation. First we introduced the function $Z(\bar{x},z) = \begin{cases} \inf_y \{f : K'(\bar{x},y) = 0\}, & \text{if } z = 0 \text{ and } \exists y K'(\bar{x},y) = 0, \\ \text{undefined}, & \text{if } z = 0 \text{ and } \forall y K'(\bar{x},y) \neq 0, \\ 1, & \text{if } z \neq 0, \end{cases}$

given in the following way:

$$Z'(\bar{x},z) = \begin{cases} \text{undefined,} & \text{if } (z = 0) \land (S'(\bar{x}) < \frac{1}{12}), \\ \sqrt{S'(\bar{x}) - \frac{1}{12}}, & \text{if } (z = 0) \land (S'(\bar{x}) \geq \frac{1}{12}) \land f(\bar{x},\sqrt{S'(\bar{x}) - \frac{1}{12}}) = 0, \\ -\sqrt{S'(\bar{x}) - \frac{1}{12}}, & \text{if } (z = 0) \land (S'(\bar{x}) \geq \frac{1}{12}) \land f(\bar{x},-\sqrt{S'(\bar{x}) - \frac{1}{12}}) = 0, \\ 1, & \text{if } z \neq 0. \end{cases}$$

where $S'(\bar{x}) = \lim_{t \to \infty} S'_1(\bar{x},t) + \lim_{t \to \infty} S'_2(\bar{x},t)$. Both functions $S'_1, S'_2$ are defined by an integration

$$S'_i(\bar{x},t) = \int y^2 \left(1 - h'(\bar{x},(-1)^{i+1}y - 1/2,(-1)^{i+1}y + 1/2)\right)dy, \quad i = 1,2$$

from $h'(\bar{x},a,b) = \liminf_{y \to a} \prod_{w=0}^{\infty} K'(\bar{x},a + w \frac{b-a}{z})$ where $K'$ is the characteristic function of $f$.

Hence we can conclude that if $K'$ is in the $L_\mu$ then $Z'$ is in the class $L_{\mu+3\mu+9}$. Let us finish with the definition of the characteristic function of the infimum of zeros of $f$ (see Theorem 4.2 from [5]
\[ K^f_\mu (y) = \lim_{a \to \infty} \lim_{b \to \infty} \lim_{z \to \infty} G^f (\bar{x}, z, a, b, y), \]

where \( G^f (\bar{x}, z, a, b, y) \) divides the interval \([a, b]\) into \(2^{z+j}\) equal subintervals and gives the value 1 for \(y\) from the subintervals, which contains the least zero of \(f\) in \([a, b]\) and value 0 otherwise. Precisely for \(y\) from \(a, a + \frac{b-a}{2^{z+j}}\):

\[
G^f (\bar{x}, z, a, b, y) = \begin{cases} 
1, & \text{if } h^f \left( \bar{x}, a + \frac{b-a}{2^{z+j}}, a + \frac{k(b-a)}{2^{z+j}} \right) = 0, \\
0, & \text{otherwise}
\end{cases}
\]

for \(y \in \left( a + \frac{(k-1)(b-a)}{2^{z+j}}, a + \frac{k(b-a)}{2^{z+j}} \right)\) (where \(k = 2, 3, ..., 2^n\)) we have:

\[
G^f (\bar{x}, z, a, b, y) = \begin{cases} 
1, & \text{if } \prod_{i=1}^{k-1} h^f \left( \bar{x}, a + \frac{(i-1)(b-a)}{2^{z+j}}, a + \frac{i(b-a)}{2^{z+j}} \right) \neq 0, \\
\cap h^f \left( \bar{x}, a + \frac{(k-1)(b-a)}{2^{z+j}}, a + \frac{k(b-a)}{2^{z+j}} \right) = 0, \\
0, & \text{otherwise}
\end{cases}
\]

and for \(Y \notin [A, B]\) the function \(g^f_x\) is equal to 2.

The definition of \(G_f\) is given by the cases with respect to the value of the expression given by \(\prod_i h^f\), since for \(f \in L_m\), the function \(h_f \in L_{m+p+2}\) and \(G^f \in L_{m+2+p+3}\). Then we have \(K^f_\mu \in L_{m+2+p+6}\). Now we must use the function \(K^f_\mu\) in the same way as \(K^f\) which gives us \(Z_f\) in the class \(L_{m+3+p+9}\). The final definition of \(g (\bar{x}) = \mu_y f (\bar{x}, y)\) ([5] Theorem 4.3) given below...
\[
g(\bar{x}) = \begin{cases} 
Z^{f^+}(\bar{x},0) - Z^{f^-}(\bar{x},0), & \text{if } S^{f^+}(\bar{x}) < \frac{1}{12} \land S^{f^-}(\bar{x}) < \frac{1}{12}, \\
Z^{f^+}(\bar{x},0), & \text{if } \left( S^{f^+}(\bar{x}) \geq \frac{1}{12} \land S^{f^-}(\bar{x}) < \frac{1}{12} \right) \\
\text{or} & \left( S^{f^+}(\bar{x}) < \frac{1}{12} \land S^{f^-}(\bar{x}) < \frac{1}{12} \land Z^{f^+}(\bar{x},0) < Z^{f^-}(\bar{x},0) \right), \\
-Z^{f^+}(\bar{x},0), & \text{if } \left( S^{f^+}(\bar{x}) < \frac{1}{12} \land S^{f^-}(\bar{x}) \geq \frac{1}{12} \right) \\
\text{or} & \left( S^{f^+}(\bar{x}) < \frac{1}{12} \land S^{f^-}(\bar{x}) < \frac{1}{12} \land Z^{f^+}(\bar{x},0) \geq Z^{f^-}(\bar{x},0) \right),
\end{cases}
\]

where \( f^+ (\bar{x}, y) = \begin{cases} 
f(\bar{x}, y), & y \geq 0, \\
1, & y < 0;
\end{cases} \)

\( f^- (\bar{x}, y) = \begin{cases} 
f(\bar{x}, -y), & y > 0, \\
1, & y \leq 0;
\end{cases} \)

remains the class of \( g \) identical to the class of \( Z^f \), i.e. \( g \in L_{m+3,p+y} \). 

**Theorem 4.3** Let \( f : \mathbb{R}^n \to \mathbb{R} \) be an \( \mathbb{R} \)-recursive function. Then for all \( i \geq 0 \) if \( f \in M_i \) then \( f \in L_{(i,p+9)i} \).

The above statement is a simple consequence of the fact \( M_0 = L_0 \) and Lemma 4.2.

5. Conclusions

In the paper we give the first rough approximation of 'a complexity' of limit operations in the terms of the \( \mu \)-operator and conversely. The results, interpreted in the intuitionist way, can suggest what kind of connection exists between infinite limits and a \( \mu \)-operator.

We also establish the proper relation between the levels of the limit hierarchy and \( \mu \)-hierarchy. Let us point out that in consequence we may investigate analogies which exist for the limit hierarchy (also \( \mu \)-hierarchy) and Baire classes...
[7]. Also the kind of a connection between the $\sum_0^\infty$-measurable functions and $\mathbb{R}$-recursive functions is an open problem.

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References