Two hierarchies of \( \mathbb{R} \)-recursive functions

Jerzy Mycka

Institute of Mathematics, Maria Curie-Skłodowska University, 
Pl. M. Curie-Skłodowskiej 1, 20-031 Lublin, Poland

Abstract

In the paper some aspects of complexity of \( \mathbb{R} \)-recursive functions are considered. The limit hierarchy of \( \mathbb{R} \)-recursive functions is introduced by the analogy to the \( \mu \)-hierarchy. Then its properties and relations to the \( \mu \)-hierarchy are analysed.

1. Introduction

The classical theory of computation deals with the functions on enumerable (especially natural) domains. The fundamental notion in this field is the notion of a (partial) recursive function. The problem of hierarchies for these functions is also in the interest of mathematicians (for elementary, primitive recursive function, Grzegorczyk hierarchy, compare [1].

During past years many mathematicians have been interested in creating analogous models of computation on real numbers (see for example Grzegorczyk [2], Blum, Shub, Smale [3]). An interesting approach was given by Moore. In the work [4] he defined a set of functions on the reals \( \mathbb{R} \) (called \( \mathbb{R} \)-recursive functions) in the analogous way to the classical recursive functions on the natural numbers \( \mathbb{N} \). His model has a continuous time of computation (a continuous integration instead of a discrete recursion). The great importance in Moore's model has the zero-finding operation \( \mu \), which is used to construct \( \mu \)-hierarchy of \( \mathbb{R} \)-recursive functions.

It was shown [5] that the zero-finding operator \( \mu \) can be replaced by the operation of infinite limits. This allows us to define a limit hierarchy and relate it to \( \mu \)-hierarchy.

\(* E-mail address: Jerzy.Mycka@umcs.lublin.pl\)
2. Preliminaries

We start with a fundamental definition of a class of real functions called $\mathbb{R}$-recursive functions [4].

**Definition 2.1** The set of $\mathbb{R}$-recursive functions is generated from the constants 0, 1 by the operations:

1) composition: $h(x) = f(g(x))$;

2) differential recursion: $h(x, 0) = f(x), h(x, y) = g(x, y, h(x, y))$ (the equivalent formulation can be given by integrals: $h(x, y) = f(x) + \int_0^y g(x, y', h(x, y')) dy'$);

3) $\mu$-recursion $h(x) = \mu_y f(x, y) =$ inf $\{y : f(x, y) = 0\}$, where infimum chooses the number $y$ with the smallest absolute value and for two $y$ with the same absolute value the negative one;

4) vector-valued functions can be defined by defining their components.

Several comments are needed to the above definition. A solution of a differential equation need not be unique or can diverge. Hence, we assume that if $h$ is defined by a differential recursion then $h$ is defined only where a finite and unique solution exists. This is why the set of $\mathbb{R}$-recursive functions includes also partial functions. We use (after [4]) the name of $\mathbb{R}$-recursive functions in the article, however we should remember that in reality we have partiality here (partial $\mathbb{R}$-recursive functions).

The second problem arises with the operation of infimum. Let us observe that if an infinite number of zeros accumulates just above some positive $y$ or just below some negative $y$ then the infimum operation returns that $y$ even if it itself is not a zero.

In the papers [5, 6] it was shown that if in the Moore's definition [4] $\mu$-operation is replaced by infinite limits: $h(x) = \liminf_{y \to \infty} g(x, y)$, $h(x) = \limsup_{y \to \infty} g(x, y)$ then the resulting class of functions remains the same.

This gives us also the following result (including the limit operation in the form $h(x) = \lim_{y \to \infty} g(x, y)$, which can be in the obvious way obtained from limsup, liminf:
Corollary 2.2 The class of $\mathbb{R}$-recursive functions is closed under the operations of infinite limits: $h(\bar{x}) = \liminf_{y \to \infty} g(\bar{x}, y)$, $h(\bar{x}) = \limsup_{y \to \infty} g(\bar{x}, y)$, $h(\bar{x}) = \lim_{y \to \infty} g(\bar{x}, y)$.

3. Hierarchies

The operator $\mu$ is a key operator in generating the $\mathbb{R}$-recursive functions. In a physical sense it has a property of being strongly uncomputable. This fact suggests creating a hierarchy, which is built with respect to the number of uses of $\mu$ in the definition of a given $f$.

Definition 3.1 ([4]) For a given $\mathbb{R}$-recursive expression $s(\bar{x})$, let $M_{x}(s)$ (the $\mu$-number with respect to $x_{i}$) be defined as follows:

\[ M_{x}(0) = M_{x}(1) = M_{x}(-1) = 0, \]

\[ M_{x}(f(g_{1}, g_{2}, \ldots)) = \max_{j} \left( M_{x}(f) + M_{x}(g_{j}) \right), \]

\[ M_{x}\left( h = f + \int_{0}^{y} g(\bar{x}, y', h)dy' \right) = \max \left( M_{x}(f), M_{x}(g), M_{h}(g) \right), \]

\[ M_{y}\left( h = f + \int_{0}^{y} g(\bar{x}, y', h)dy' \right) = \max \left( M_{y}(g), M_{h}(g) \right), \]

\[ M_{x}(\mu, f(\bar{x}, y)) = \max \left( M_{x}(f), M_{y}(f) \right) + 1, \]

where $x$ can be any $x_{1}, \ldots, x_{n}$ for $\bar{x} = (x_{1}, \ldots, x_{n})$.

For an $\mathbb{R}$-recursive function $f$, let $M(f) = \max_{x_{i}}(s)$ minimized over all expressions $s$ that define $f$. Now we are ready to define $M$-hierarchy ($\mu$-hierarchy) as a family of $M_{j} = \{ f : M'(f) \leq j \}$.

Let us construct the analogous definition of $L$-hierarchy by replacing in the above definition $M_{x}$ by $L_{x}$ and changing line (5) to the following form (5'):

\[ L_{x}\left( \liminf_{y \to \infty} g(\bar{x}, y) \right) = L_{x}\left( \limsup_{y \to \infty} g(\bar{x}, y) \right) = \]

\[ = L_{x}\left( \lim g(\bar{x}, y) \right) = \max \left( L_{x}(f), L_{y}(f) \right) + 1. \]

For an $\mathbb{R}$-recursive function $f$, let $L(f) = \max_{x_{i}}(s)$ minimized over all expressions $S$ that define $f$ without using the $\mu$-operation.
Definition 3.2 The $L$-hierarchy is a family of $L_j = \{ f : L(f) \leq j \}$.

Let us add that in Definition 3.2 we use explicitly the operator $f(\bar{x}) = \lim_{y \to \infty} g(\bar{x}, y)$ to avoid its construction by other operators (lim sup, lim inf), which would effect in a superficially higher class of a complexity of a function $f$.

As an obvious corollary from definitions we have the following statement.

Lemma 3.3 The classes $M_0$ and $M_1$ are identical.

A function $f \in L_0 = M_0$ will be called (by an analogy to the case of natural recursive functions) a primitive $R$-recursive function. After Moore [4] we can conclude that such functions as: $-x$, $x + y$, $xy$, $x/y$, $e^x$, $\ln x$, $y^x$, $\sin x$, $\cos x$ are primitive $R$-recursive.

We can give a few results on some levels of the limit hierarchy.

Lemma 3.4. The Kronecker $\delta$ function, the signum function and absolute value belong to the first level ($L_1$) of limit hierarchy.

Proof. It is sufficient to take the following definitions [5]: hence $\delta(0) = 1$ and for all $x \neq 0$ we have $\delta(x) = 0$ let us define $\delta(x) = \liminf_{y \to \infty} \left( \frac{1}{1 + x^2} \right)^y$. Now from the expression $\liminf_{y \to \infty} \arctan xy = \begin{cases} \pi/2, & \text{if } x > 0, \\ 0, & \text{if } x = 0, \\ -\pi/2, & \text{if } x < 0, \end{cases}$ we obtain $\sgn(x) = \frac{\liminf_{y \to \infty} \arctan xy}{2 \arctan 1}$ and $|x| = \sgn(x)x$.

We should be careful with definitions of functions by cases:

Lemma 3.5 For $h(\bar{x}) = \begin{cases} g_1(\bar{x}), & \text{if } f(\bar{x}) = 0, \\ g_2(\bar{x}), & \text{if } f(\bar{x}) = 1, \\ \cdots, & \text{if } f(\bar{x}) = k-1 \end{cases}$ and $g_i \in L_i$ for all $1 \leq i \leq k$, $f \in L_m$ the function $h$ belongs to $L_{\max(n_1 \ldots n_k, m+1)}$.
Proof. Let us see that \( eq(x, y) = \delta(x - y) \in L_1 \) and \( ge(x, y) = \frac{(\text{sgn}(x - y) + eq(x, y))}{2} + \frac{1}{2} \in L_1 \). Then of course
\[
h(\bar{x}) = \sum_{i=1}^{k-1} g_i(\bar{x}) eq\left(f(\bar{x}), i-1\right) + g_k(\bar{x}) ge\left(f(\bar{x}), k-1\right)
\]

Of course this result can be easily extended to other forms of definitions by cases.

**Lemma 3.6** The function \( \Theta(x) \) (equal to 1 if \( x \geq 0 \), otherwise 0), maximum \( \max(x, y) \), square-wave function \( s \) are in \( L_2 \), the function \( p(x) \) such that \( p(x) = 1 \) for \( x \in [2n, 2n+1] \) and \( p(x) = 0 \) for \( x \in [2n+1, 2n+2] \) is in \( L_2 \) and the floor function \( \lfloor x \rfloor \) is in \( L_3 \).

**Proof.** We give the proper definitions (from [6]) for these functions. Let
\[
\Theta(x) = \delta\left(x - \lfloor x \rfloor\right),
\]
\[
\max(x, y) = x \delta(x - y) + (1 - \delta(x - y))\left[x \Theta(x - y) + y \Theta(y - x)\right],
\]
\[
s(x) = \Theta(\sin(\pi x))
\]
The function \( p(x) \) can be given as \( s(x)\left(1 - \delta\left(\frac{\sin\left(\frac{x-1}{2}\pi\right)}{2}\right)\right) \), so \( p \in L_2 \).

The floor function we can define by the auxiliary function \( w(0) = 0 \), \( \partial_x w(x) = 2\Theta(-\sin(2\pi x)) \) as
\[
\lfloor x \rfloor = \begin{cases} 
2w(x/2) & \text{if } p(x) = 1, \\
2w((x-1)/2) & \text{if } p(x) = 0.
\end{cases}
\]

From the above equation we have \( \lfloor x \rfloor \) in \( L_3 \).

Let us recall that if \( f : R^n \to R \) is an \( R \)-recursive function then the function \( f_{iter}(i, \bar{x}) \) is \( R \)-recursive, too.

**Lemma 3.7** Let \( f : R^n \to R \) belongs to the class \( L_i \), then we have \( f_{iter} : R^{n+1} \to R \) is in \( L_{\max(2, j)} \).

**Proof.** The definitions, which were given by Moore [3] \( f_{iter}(i, \bar{x}) = h(2i) \), where
\[
h(0) = g(0) = \bar{x},
\]
\[ \partial_{t}g(t) = \left[ f(h(t)) - h(t) \right] s(t), \]
\[ \partial_{t}h(t) = \geq \left[ \frac{g(t) - h(t)}{r(t)} \right] (1 - s(t)), \]

with \( s \) - a square wave function in \( L_2 \), \( r(0) = 0 \), \( \partial_{t}r(t) = 2s(t) - 1 \), \( r, s \in L_2 \) give us the desirable statement. □

**Lemma 3.8** The \( R^l \)-recursive functions \( \gamma_2 : R^2 \to R \), \( \gamma_1^1, \gamma_1^2 : R \to R \) such that \((\forall x, y \in R) \gamma_2^1(\gamma_2(x, y)) = x, \ (\forall x, y \in R) \gamma_2^2(\gamma_2(x, y)) = y\), have the following properties: \( \gamma_2, \gamma_1^2 \) are in \( L_{10} \), \( \gamma_2^2 \) is in \( L_{14} \).

**Proof.** We have the auxiliary functions \( \Gamma_2, \Gamma_1^1, \Gamma_1^2 \), which are coding and decoding functions in the interval \((0,1) : \Gamma_2(x, y) = c(x) + c(y)/10 \), where

\[ c(x) = \lim_{i \to \infty} z(i, x), \]

and later \( z(x) = \lim_{i \to \infty} z_{\text{iter}}(i, x) \),

\[ z_{\text{iter}}(i, a_1, a_2, a_3, \ldots) = a_1 \ldots a_n 0 \ldots a_{n+1} \ldots, \]

\[ a(i, 0, a_2, a_3, \ldots) = 0 a_1 \ldots a_i \]

\[ b(i, 0, a_2, a_3, \ldots) = 0.0 \ldots a_{i+1} \ldots, \]

\[ z'(x) = \begin{cases} 100 \lfloor x \rfloor + 10(x - \lfloor x \rfloor), & \text{if } \lfloor x \rfloor \neq x, \\ x, & \text{if } \lfloor x \rfloor = x. \end{cases} \]

with \( s \) - a square wave function in \( L_2 \) and \( r(0) = 0 \), \( \partial_{t}r(t) = 2s(t) - 1 \), \( r, s \in L_2 \) give us the desirable statement. □

The functions \( \Gamma_2, \Gamma_1^1, \Gamma_1^2 \) can be extended to all reals by one-to-one \( f : (0,1) \to R \in L_0 \) without the loss of their class. □

The same method of coding and decoding by interlacing of ciphers (only the power of 10 should be changed) gives us the functions \( \gamma_n : R^n \to R \) and \( \gamma_n^i : R \to R \) for \( i = 1, \ldots, n \) such that

\[ (\forall i)(\forall x_1, \ldots, x_n \in R) \gamma_n^i(\gamma_n(x_1, \ldots, x_n)) = x_i \]

in the same class: \( \gamma_n, \gamma_n^1 \in L_{10} \) and \( (\forall i > 1) \gamma_n^i \in L_{14} \).

We finish this part with the important form of defining: a new function is given as a product of values \( f \) in some integer points.
Lemma 3.9 There exists such constant \( p \in \mathbb{N} \) that for the function
\[
\prod_{z=0}^{y} f(\bar{x}, z) = \begin{cases} 
  f(\bar{x}, 0)f(\bar{x}, 1)\ldots f(\bar{x}, y-1), & \text{if } y \geq 1, \\
  1, & \text{if } 0 \leq y < 1, \\
  0, & \text{if } y < 0,
\end{cases}
\]
if the function \( f \) is in the class \( L_m \) then \( \prod_{z=0}^{y} f(\bar{x}, z) \) is in the class \( L_{m+p} \) (\( p \) is independent of \( m \)).

**Proof.** By the definitions
\[
t(\bar{w}) = \gamma_{n+2}(\gamma_{n+2}^1(\bar{w}), \gamma_{n+2}^{n+1}(\bar{w}) + 1, f(\gamma_{n+2}^1(\bar{w}), \gamma_{n+2}^{n+1}(\bar{w})), \gamma_{n+2}^{n+2}(\bar{w}))
\]
and
\[
S(\bar{x}, \bar{z}) = t(\bar{x}, t(s(\bar{x}, 0))\ldots) = t_{ier}(\lceil \bar{z} \rceil, \gamma_{n+2}(\bar{x}, 0, 1))
\]
we get the property
\[
\prod_{y=0}^{z} f(\bar{x}, y) = \gamma_{n+2}(S(\bar{x}, \bar{z})).
\]
From the definition of the limit hierarchy we get \( \prod_{y=0}^{z} f(\bar{x}, y) \in L_{m+38} \).

In the rest of the paper we will use the constant \( p \) as the number of limits used in the recursive definition of the product \( \prod_{y=0}^{z} f(\bar{x}, y) \) instead of the value 38. The above constructions are tedious and can be improved with a better approximation of \( p \).

4. Main results

Now we are ready to formulate two theorems which demonstrate connections between \( L \)-hierarchy and \( M \)-hierarchy.

**Theorem 4.1** Let \( f : \mathbb{R}^n \rightarrow \mathbb{R} \) be an \( \mathbb{R} \)-recursive function. Then if \( f \in L_i \) then \( f \in M_{10i} \).

**Proof.** We use a simple induction here. The case \( i = 0 \) is given in Lemma 3.3. Now let us suppose that the thesis is true for \( i = n \). Let \( f \in L_{n+1} \) be defined as \( f(\bar{x}) = \lim_{y \rightarrow \infty} g(\bar{x}, y) \) for \( g \in L_n \). Then we can recall Theorem 4.2 from [6] which gives us the following result: to define \( f \) from \( g \) it is necessary to use at
most 10 μ-operation. Hence for \( g \in M_{10n} \) the function \( f \) satisfies \( f \in M_{10n+10} \). Similar inferences hold for \( \lim \inf, \lim \sup \).

Now we can give the result about the 'limit complexity' of the infimum operator \( \mu \).

**Lemma 4.2** If \( f(\bar{x},y):R^{n+1} \to R \) is in the class \( L_m \) then the function \( g:R^n \to R, \ g(\bar{x})=\mu_y f(\bar{x},y) \) is in the class \( L_{m+3+p+9} \) is from Lemma 3.9.

**Proof.** Here we must employ the results from [6]. There we defined the function \( g:R^n \to R, \ g(\bar{x})=\mu_y f(\bar{x},y) \) for \( f(\bar{x},y):R^{n+1} \to R \) (\( f \) - \( R \)-recursive) replacing the \( \mu \)-operator by limit operation. First we introduced the function

\[
Z'(\bar{x},z) = \begin{cases} 
\text{undefined} & \text{if } z=0 \text{ and } \exists y K'(\bar{x},y)=0, \\
1 & \text{if } z \neq 0,
\end{cases}
\]

given in the following way:

\[
Z'(\bar{x},z) = \begin{cases} 
\text{undefined} & \text{if } (z=0) \land \left( S'(\bar{x}) < \frac{1}{12} \right), \\
\sqrt{S'(\bar{x})-\frac{1}{12}} & \text{if } (z=0) \land \left( S'(\bar{x}) \geq \frac{1}{12} \right) \\
-\sqrt{S'(\bar{x})-\frac{1}{12}} & \text{if } (z=0) \land \left( S'(\bar{x}) \geq \frac{1}{12} \right) \\
1 & \text{if } z \neq 0.
\end{cases}
\]

where \( S'(\bar{x}) = \lim_{t \to \infty} S'_{1}(\bar{x},t) + \lim_{t \to \infty} S'_{2}(\bar{x},t) \). Both functions \( S'_{1}, S'_{2} \) are defined by an integration

\[
S'_{i}(\bar{x},t) = \int y^2 \left( 1-h'_{i}(\bar{x},(-1)^{i+1}y-1/2,(-1)^{i+1}y+1/2) \right) dy, \ i=1,2
\]

from \( h'_{i}(\bar{x},a,b) = \liminf_{t \to \infty} \prod_{w=0}^{\infty} K'(\bar{x},a+w\frac{b-a}{z}) \) where \( K' \) is the characteristic function of \( f \).

Hence we can conclude that if \( K' \) is in the \( L_s \) then \( Z' \) is in the class \( L_{s+p+3} \).

Let us finish with the definition of the characteristic function of the infimum of zeros of \( f \) (see Theorem 4.2 from [5]
The definition of $G_f$ is given by the cases with respect to the value of the expression given by $\prod h^i$, since for $f \in L_m$, the function $h_f \in L_{m+p+2}$ and $G_f \in L_{m+2+p+3}$. Then we have $K^f \in L_{m+2+p+6}$. Now we must use the function $K^f$ in the same way as $K_f$ which gives us $Z_f$ in the class $L_{m+3,p+9}$. The final definition of $g(\bar{x}) = \mu_f(\bar{x}, y)$ ([5] Theorem 4.3) given below
\[
g(\overline{x}) = \begin{cases} Z^f(\overline{x},0) - Z^f(\overline{x},0), & \text{if } S^f(\overline{x}) < \frac{1}{12} \land S^f(\overline{x}) < \frac{1}{12}, \\ Z^f(\overline{x},0), & \text{if } \left( S^f(\overline{x}) \geq \frac{1}{12} \land S^f(\overline{x}) < \frac{1}{12} \right) \\
& \lor \\
& \left( S^f(\overline{x}) < \frac{1}{12} \land S^f(\overline{x}) < \frac{1}{12} \right) \\
& \land Z^f(\overline{x},0) < Z^f(\overline{x},0), \\
& \lor \\
- Z^f(\overline{x},0), & \text{if } \left( S^f(\overline{x}) < \frac{1}{12} \land S^f(\overline{x}) \geq \frac{1}{12} \right) \\
& \lor \\
& \left( S^f(\overline{x}) < \frac{1}{12} \land S^f(\overline{x}) < \frac{1}{12} \right) \\
& \land Z^f(\overline{x},0) \geq Z^f(\overline{x},0), \\
\end{cases}
\]

where \( f^+(\overline{x},y) = \begin{cases} f(\overline{x},y), & y \geq 0, \\
1, & y < 0; \\
\end{cases} \)
and \( f^-(\overline{x},y) = \begin{cases} f(\overline{x},-y), & y > 0, \\
1, & y \leq 0; \\
\end{cases} \)
remains the class of \( g \) identical to the class of \( Z^f \), i.e. \( g \in L_{m+3,p+q} \).

**Theorem 4.3** Let \( f : R^n \to R \) be an \( R \)-recursive function. Then for all \( i \geq 0 \) if \( f \in M_i \) then \( f \in L_{(i+p+9)i} \).

The above statement is a simple consequence of the fact \( M_0 = L_0 \) and Lemma 4.2.

**5. Conclusions**

In the paper we give the first rough approximation of 'a complexity' of limit operations in the terms of the \( \mu \)-operator and conversely. The results, interpreted in the intuitional way, can suggest what kind of connection exists between infinite limits and a \( \mu \)-operator.

We also establish the proper relation between the levels of the limit hierarchy and \( \mu \)-hierarchy. Let us point out that in consequence we may investigate analogies which exist for the limit hierarchy (also \( \mu \)-hierarchy) and Baire classes.
[7]. Also the kind of a connection between the $\sum_0^\infty$ measurable functions and $R$-recursive functions is an open problem.

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References