Two hierarchies of $R$-recursive functions

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Abstract

In the paper some aspects of complexity of $R$-recursive functions are considered. The limit hierarchy of $R$-recursive functions is introduced by the analogy to the $\mu$-hierarchy. Then its properties and relations to the $\mu$-hierarchy are analysed.

1. Introduction

The classical theory of computation deals with the functions on enumerable (especially natural) domains. The fundamental notion in this field is the notion of a (partial) recursive function. The problem of hierarchies for these functions is also in the interest of mathematicians (for elementary, primitive recursive function, Grzegorczyk hierarchy, compare [1].

During past years many mathematicians have been interested in creating analogous models of computation on real numbers (see for example Grzegorczyk [2], Blum, Shub, Smale [3]). An interesting approach was given by Moore. In the work [4] he defined a set of functions on the reals $\mathbb{R}$ (called $R$-recursive functions) in the analogous way to the classical recursive functions on the natural numbers $\mathbb{N}$. His model has a continuous time of computation (a continuous integration instead of a discrete recursion). The great importance in Moore's model has the zero-finding operation $\mu$, which is used to construct $\mu$-hierarchy of $R$-recursive functions.

It was shown [5] that the zero-finding operator $\mu$ can be replaced by the operation of infinite limits. This allows us to define a limit hierarchy and relate it to $\mu$-hierarchy.

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2. Preliminaries

We start with a fundamental definition of a class of real functions called \( R \)-recursive functions [4].

**Definition 2.1** The set of \( R \)-recursive functions is generated from the constants 0,1 by the operations:

1) composition: \( h(\bar{x}) = f\left(g\left(\bar{x}\right)\right) \);

2) differential recursion: \( h(\bar{x},0) = f(\bar{x}), \partial_y h(\bar{x},y) = g(\bar{x}, y, h(\bar{x}, y)) \) (the equivalent formulation can be given by integrals: \( h(\bar{x}, y) = f(\bar{x}) + \int_0^y g(\bar{x}, y', h(\bar{x}, y')) dy' \));

3) \( \mu \)-recursion \( h(\bar{x}) = \mu_y f(\bar{x}, y) = \inf \{ y : f(\bar{x}, y) = 0 \} \), where infimum chooses the number \( y \) with the smallest absolute value and for two \( y \) with the same absolute value the negative one;

4) vector-valued functions can be defined by defining their components.

Several comments are needed to the above definition. A solution of a differential equation need not be unique or can diverge. Hence, we assume that if \( h \) is defined by a differential recursion then \( h \) is defined only where a finite and unique solution exists. This is why the set of \( R \)-recursive functions includes also partial functions. We use (after [4]) the name of \( R \)-recursive functions in the article, however we should remember that in reality we have partiality here (partial \( R \)-recursive functions).

The second problem arises with the operation of infimum. Let us observe that if an infinite number of zeros accumulates just above some positive \( y \) or just below some negative \( y \) then the infimum operation returns that \( y \) even if it itself is not a zero.

In the papers [5, 6] it was shown that if in the Moore's definition [4] \( \mu \)-operation is replaced by infinite limits: \( h(\bar{x}) = \liminf_{y \to \infty} g(\bar{x}, y) \), \( h(\bar{x}) = \limsup_{y \to \infty} g(\bar{x}, y) \) then the resulting class of functions remains the same.

This gives us also the following result (including the limit operation in the form \( h(\bar{x}) = \lim_{y \to \infty} g(\bar{x}, y) \), which can be in the obvious way obtained from limsup, liminf:
Corollary 2.2 The class of $R$-recursive functions is closed under the operations of infinite limits: $h(\overline{x}) = \liminf_{y \to \infty} g(\overline{x}, y)$, $h(\overline{x}) = \limsup_{y \to \infty} g(\overline{x}, y)$, $h(\overline{x}) = \lim_{y \to \infty} g(\overline{x}, y)$.

3. Hierarchies

The operator $\mu$ is a key operator in generating the $R$-recursive functions. In a physical sense it has a property of being strongly uncomputable. This fact suggests creating a hierarchy, which is built with respect to the number of uses of $\mu$ in the definition of a given $f$.

Definition 3.1 ([4]) For a given $R$-recursive expression $s(\overline{x})$, let $M_{\overline{x}}(s)$ (the $\mu$-number with respect to $x_i$) be defined as follows:

\begin{align*}
M_{\overline{x}}(0) &= M_{\overline{x}}(1) = M_{\overline{x}}(-1) = 0, \\
M_{\overline{x}}(f(g_1, g_2, ..., g_j)) &= \max_j \left( M_{\overline{x}}(f) + M_{\overline{x}}(g_j) \right), \\
M_{\overline{x}}(h = f + \int_{0}^{x} g(\overline{x}, y', h) dy') &= \max \left( M_{\overline{x}}(f), M_{\overline{x}}(g), M_{\overline{h}}(g) \right), \\
M_{\overline{y}}(h = f + \int_{0}^{x} g(\overline{x}, y', h) dy') &= \max \left( M_{\overline{y}}(g), M_{\overline{h}}(g) \right), \\
M_{\overline{x}}(\mu, f(\overline{x}, y)) &= \max \left( M_{\overline{x}}(f), M_{\overline{y}}(f) \right) + 1,
\end{align*}

where $x$ can be any $x_i, ..., x_n$ for $\overline{x} = (x_i, ..., x_n)$.

For an $R$-recursive function $f$, let $M(f) = \max_{\overline{x}}(s)$ minimized over all expressions $s$ that define $f$. Now we are ready to define $M$-hierarchy ($\mu$-hierarchy) as a family of $M_j = \{ f : M'(f) \leq j \}$.

Let us construct the analogous definition of $L$-hierarchy by replacing in the above definition $M_{\overline{x}}$ by $L_{\overline{x}}$ and changing line (5) to the following form (5'):

\begin{align*}
L_{\overline{x}}\left( \liminf_{y \to \infty} g(\overline{x}, y) \right) &= L_{\overline{x}}\left( \limsup_{y \to \infty} g(\overline{x}, y) \right) = \\
&= L_{\overline{x}}\left( \lim_{y \to \infty} g(\overline{x}, y) \right) = \max \left( L_{\overline{x}}(f), L_{\overline{y}}(f) \right) + 1.
\end{align*}

For an $R$-recursive function $f$, let $L(f) = \max_{\overline{x}}(s)$ minimized over all expressions $S$ that define $f$ without using the $\mu$-operation.
**Definition 3.2** The L-hierarchy is a family of \( L_j = \{ f : L(f) \leq j \} \).

Let us add that in Definition 3.2 we use explicitly the operator 
\[ f(\bar{x}) = \lim_{y \to \infty} g(\bar{x}, y) \]
to avoid its construction by other operators (\( \lim \sup \), \( \lim \inf \)), which would effect in a superficially higher class of a complexity of a function \( f \).

As an obvious corollary from definitions we have the following statement.

**Lemma 3.3** The classes \( M_0 \) and \( M_1 \) are identical.

A function \( f \in L_0 = M_0 \) will be called (by an analogy to the case of natural recursive functions) a primitive \( R \)-recursive function. After Moore [4] we can conclude that such functions as: \( -x \), \( x + y \), \( xy \), \( x/y \), \( e^x \), \( \ln x \), \( y^x \), \( \sin x \), \( \cos x \) are primitive \( R \)-recursive.

We can give a few results on some levels of the limit hierarchy.

**Lemma 3.4.** The Kronecker \( \delta \) function, the signum function and absolute value belong to the first level (\( L_1 \)) of limit hierarchy.

**Proof.** It is sufficient to take the following definitions [5]: hence \( \delta (0) = 1 \) and for all \( x \neq 0 \) we have \( \delta (x) = 0 \) let us define \( \delta (x) = \lim \inf_{y \to \infty} \left( \frac{1}{1 + x^2} \right)^y \). Now from the expression \( \lim \inf_{y \to \infty} \arctan xy = \begin{cases} \pi/2, & \text{if } x > 0, \\ 0, & \text{if } x = 0, \\ -\pi/2, & \text{if } x < 0, \end{cases} \)
we obtain \( \arctan 1 = \frac{\lim \inf_{y \to \infty} \arctan xy}{2 \arctan 1} \) and \( |x| = \text{sgn}(x)x \).

We should be careful with definitions of functions by cases:

**Lemma 3.5** For \( h(\bar{x}) = \begin{cases} g_1(\bar{x}), & \text{if } f(\bar{x}) = 0, \\ g_2(\bar{x}), & \text{if } f(\bar{x}) = 1, \\ \mathcal{M}, & \text{if } f(\bar{x}) \geq k, \\ g_k(\bar{x}), & \text{if } f(\bar{x}) \geq k-1 \end{cases} \)

\( f \in L_m \) the function \( h \) belongs to \( L_{\max\{m_1, \ldots, m_k, m+1\}} \) for all \( 1 \leq i \leq k \).
Proof. Let us see that \( eq(x, y) = \delta(x - y) \in L_1 \) and
\[
geq(x, y) = \frac{(\text{sgn}(x - y) + eq(x, y))}{2} + \frac{1}{2} \in L_1.
\]
Then of course
\[
h(\overline{x}) = \sum_{i=1}^{k-1} g_i(\overline{x}) eq(f(\overline{x}), i - 1) + g_k(\overline{x}) eq(f(\overline{x}), k - 1)
\]
\( \square \)

Of course this result can be easily extended to other forms of definitions by cases.

Lemma 3.6 The function \( \Theta(x) \) (equal to 1 if \( x \geq 0 \), otherwise 0), maximum \( \max(x, y) \), square-wave function \( s \) are in \( L_2 \), the function \( p(x) \) such that \( p(x) = 1 \) for \( x \in [2n, 2n+1] \) and \( p(x) = 0 \) for \( x \in [2n+1, 2n+2] \) is in \( L_2 \) and the floor function \( \lfloor x \rfloor \) is in \( L_3 \).

Proof. We give the proper definitions (from [6]) for these functions. Let
\[
\Theta(x) = \delta(x - |x|),
\]
\[
\max(x, y) = x \delta(x - y) + (1 - \delta(x - y))\left[x \Theta(x - y) + y \Theta(y - x)\right],
\]
\[
s(x) = \Theta(\sin(\pi x)).
\]

The function \( p(x) \) can be given as \( s(x)\left(1 - \delta\left(\sin\left(\frac{x - 1}{2}\pi\right)\right)\right) \), so \( p \in L_2 \).

The floor function we can define by the auxiliary function \( w(0) = 0, \partial_x w(x) = 2\Theta(-\sin(2\pi x)) \) as
\[
\lfloor x \rfloor = \begin{cases} 
2w(x/2) & \text{if } p(x) = 1, \\
2w((x-1)/2) & \text{if } p(x) = 0.
\end{cases}
\]

From the above equation we have \( \lfloor x \rfloor \) in \( L_3 \) \( \square \)

Let us recall that if \( f : \mathbb{R}^n \to \mathbb{R} \) is an \( \mathbb{R} \)-recursive function then the function \( f_{iter}(i, \overline{x}) \) is \( \mathbb{R} \)-recursive, too.

Lemma 3.7 Let \( f : \mathbb{R}^n \to \mathbb{R} \) belongs to the class \( L_i \), then we have \( f_{iter} : \mathbb{R}^{n+1} \to \mathbb{R} \) is in \( L_{\max(2, j)} \).

Proof. The definitions, which were given by Moore [3] \( f_{iter}(i, \overline{x}) = h(2i) \), where
\[
h(0) = g(0) = \overline{x},
\]
\[
\frac{\partial_t g(t)}{g(t)} = \left[ f(h(t)) - h(t) \right] s(t),
\]
\[
\frac{\partial_t h(t)}{h(t)} \geq \left[ \frac{g(t) - h(t)}{r(t)} \right] (1 - s(t)),
\]
with \(s\) - a square wave function in \(L_2\) and \(r(0) = 0\), \(\partial_t r(t) = 2s(t) - 1\), \(r, s \in L_2\) give us the desirable statement. □

**Lemma 3.8** The \( R^l \)-recursive functions \( \gamma_2 : R^2 \rightarrow R \), \( \gamma_2', \gamma_2'' : R \rightarrow R \) such that \((\forall x, y \in R)\gamma_2' (\gamma_2(x, y)) = x\), \((\forall x, y \in R)\gamma_2'' (\gamma_2 (x, y)) = y\), have the following properties: \( \gamma_2, \gamma_2' \) are in \( L_{10}\), \( \gamma_2'' \) is in \( L_{14}\).

**Proof.** We have the auxiliary functions \( \Gamma_2, \Gamma_2', \Gamma_2'' \), which are coding and decoding functions in the interval \((0, 1) : \Gamma_2 (x, y) = c(x) + c(y) / 10\), where
\[
c(x) = \lim_{i \to \infty} z(a(i, x)) / 10^i + b(i, x) / 10^i,
\]
and later \(z(x) = \lim_{i \to \infty} z_{\text{iter}}(i, x)\),
\[
z_{\text{iter}}(i, a_1, a_2, a_n, a_{n+1}, \ldots) = a_1 \ldots a_n 0 \ldots a_{n+1} \ldots,
\]
\[
a(i, 0, a_1, a_2, \ldots) = 0, a_i \ldots a_i
\]
\[
b(i, 0, a_1, a_2, \ldots) = 0, a_1 \ldots a_i,
\]
\[
(z'(x) = \begin{cases} 
100 \lfloor x \rfloor + 10(\lfloor x - \lfloor x \rfloor \rfloor), & \text{if } \lfloor x \rfloor \neq x, \\
\lfloor x \rfloor, & \text{if } \lfloor x \rfloor = x.
\end{cases}
\]
\(z_{\text{iter}}(i, x, y) \in L_4, a, b \in L_4\). Also \(z_{\text{iter}}\) belongs to \(L_4\), hence \(\Gamma_2 (x, y) \in L_{10}\), decoding of the first element is described in the symmetric way so \(\Gamma_2' (x)\) is in \(L_{10}\), but \(\Gamma_2'' (x) = \Gamma_2' \left( 10 - \lfloor 10x \rfloor \right) \) so \(\Gamma_2'' \in L_{14}\).

The functions \( \Gamma_2, \Gamma_2', \Gamma_2'' \) can be extended to all reals by one-to-one \( f : (0, 1) \rightarrow R \in L_0 \) without the loss of their class. □

The same method of coding and decoding by interlacing of ciphers (only the power of 10 should be changed) gives us the functions \( \gamma_n : R^n \rightarrow R \) and \( \gamma_n' : R \rightarrow R \) for \( i = 1, \ldots, n \) such that
\[
(\forall i) (\forall x_1, \ldots, x_n \in R) \gamma_n' (\gamma_n (x_1, \ldots, x_n)) = x_i
\]
in the same class: \( \gamma_n, \gamma_n' \in L_{10}\) and \( (\forall i > 1) \gamma_n' \in L_{14}\).

We finish this part with the important form of defining: a new function is given as a product of values \(f\) in some integer points.
Lemma 3.9 There exists such constant \( p \in \mathbb{N} \) that for the function

\[
\prod_{z=0}^y f(x, z) = \begin{cases} 
  f(x, 0) f(x, 1) \cdots f(x, y-1), & \text{if } y \geq 1, \\
  1, & \text{if } 0 \leq y < 1, \\
  0, & \text{if } y < 0,
\end{cases}
\]

if the function \( f \) is in the class \( L_m \) then \( \prod_{z=0}^y f(x, z) \) is in the class \( L_{m+p} \) (\( p \) is independent of \( m \)).

Proof. By the definitions

\[
t(w) = \gamma_{n+2} \left( \gamma_{n+2}(w), \gamma_{n+2}^{n+1}(w) + 1, f \left( \gamma_{n+2}(w), \gamma_{n+2}^{n+1}(w) \right) \right)
\]

and

\[
S(x, z) = \left( \frac{x}{z} \right) \left( s(x, 0) \right) \cdots = \left( \frac{z}{y} \right) \left( \gamma_{n+2}(x, 0, 1) \right)
\]

we get the property

\[
\prod_{y=0}^z f(x, y) = \gamma_{n+2}^{n+2} \left( S(x, z) \right).
\]

From the definition of the limit hierarchy we get \( \prod_{y=0}^z f(x, y) \in L_{m+38} \)

In the rest of the paper we will use the constant \( p \) as the number of limits used in the recursive definition of the product \( \prod_{y=0}^z f(x, y) \) instead of the value 38. The above constructions are tedious and can be improved with a better approximation of \( p \).

4. Main results

Now we are ready to formulate two theorems which demonstrate connections between \( L \)-hierarchy and \( M \)-hierarchy.

Theorem 4.1 Let \( f : \mathbb{R}^n \rightarrow \mathbb{R} \) be an \( \mathbb{R} \)-recursive function. Then if \( f \in L_i \) then \( f \in M_{10i} \).

Proof. We use a simple induction here. The case \( i = 0 \) is given in Lemma 3.3. Now let us suppose that the thesis is true for \( i = n \). Let \( f \in L_{n+1} \) be defined as

\[
f(x) = \lim_{y \to \infty} g(x, y) \quad \text{for} \quad g \in L_n.
\]

Then we can recall Theorem 4.2 from [6] which gives us the following result: to define \( f \) from \( g \) it is necessary to use at
most 10 µ-operation. Hence for \( g \in M_{10n} \) the function \( f \) satisfies \( f \in M_{10n+10} \). Similar inferences hold for \( \operatorname{lim\ inf}, \operatorname{lim\ sup} \).

Now we can give the result about the 'limit complexity' of the infimum operator \( \mu \).

**Lemma 4.2** If \( f(\bar{x},y): R^{n+1} \to R \) is in the class \( L_n \) then the function \( g: R^n \to R, \quad g(\bar{x}) = \mu_y f(\bar{x},y) \) is in the class \( L_{n+3p+9} \) is from Lemma 3.9.

**Proof.** Here we must employ the results from [6]. There we defined the function \( g: R^n \to R, \quad g(\bar{x}) = \mu_y f(\bar{x},y) \) for \( f(\bar{x},y): R^{n+1} \to R \) (f - R-recursive) replacing the \( \mu \)-operator by limit operation. First we introduced the function

\[
Z'(\bar{x},z) = \begin{cases} 
\inf_y \left\{ f : K'(\bar{x},y) = 0 \right\}, & \text{if } z = 0 \text{ and } \exists y K'(\bar{x},y) = 0, \\
\text{undefined} & \text{if } z = 0 \text{ and } \forall y K'(\bar{x},y) \neq 0, \\
1 & \text{if } z \neq 0,
\end{cases}
\]

given in the following way:

\[
Z'(\bar{x},z) = \begin{cases} 
\text{undefined} & \text{if } (z = 0) \land \left( S'(\bar{x}) < \frac{1}{12} \right), \\
\sqrt{S'(\bar{x}) - \frac{1}{12}} & \text{if } (z = 0) \land \left( S'(\bar{x}) \geq \frac{1}{12} \right) \\
\land f \left( \bar{x}, \sqrt{S'(\bar{x}) - \frac{1}{12}} \right) = 0, \\
-\sqrt{S'(\bar{x}) - \frac{1}{12}} & \text{if } (z = 0) \land \left( S'(\bar{x}) \geq \frac{1}{12} \right) \\
\land f \left( \bar{x}, -\sqrt{S'(\bar{x}) - \frac{1}{12}} \right) = 0, \\
1, & \text{if } z \neq 0,
\end{cases}
\]

where \( S'(\bar{x}) = \lim_{t \to \infty} S_1'(\bar{x},t) + \lim_{t \to \infty} S_2'(\bar{x},t) \). Both functions \( S_1', S_2' \) are defined by an integration

\[
S_i'(\bar{x},t) = \int y^2 \left( 1 - h_i' \left( \bar{x}, (-1)^{i+1} y - 1/2, (-1)^{i+1} y + 1/2 \right) \right) dy, \quad i = 1, 2
\]

from \( h_i' (\bar{x},a,b) = \liminf_{t \to \infty} \prod_{w=0}^{n} K' \left( \bar{x}, a + w \frac{b-a}{z} \right) \) where \( K' \) is the characteristic function of \( f \).

Hence we can conclude that if \( K' \) is in the \( L_s \) then \( Z_f \) is in the class \( L_{s+p+3} \).

Let us finish with the definition of the characteristic function of the infimum of zeros of \( f \) (see Theorem 4.2 from [5].
\[
K^f_{\mu}(y) = 1 - \lim_{a \to -\infty} \lim_{b \to +\infty} \lim_{z \to +\infty} G^f(x, z, a, b, y),
\]

where \( G^f(x, z, a, b, y) \) divides the interval \([a, b]\) into \(2^z\) equal subintervals and gives the value 1 for \(y\) from the subintervals, which contains the least zero of \(f\) in \([a, b]\) and value 0 otherwise. Precisely for \(y\) from \([a, a + \frac{b-a}{2^z}]\)

\[
G^f(x, z, a, b, y) = \begin{cases} 
1, & \text{if } h^f\left(\frac{x, a + \frac{b-a}{2^z}}{a + \frac{b-a}{2^z}}\right) = 0, \\
0, & \text{otherwise}
\end{cases}
\]

for \(y \in \left\{a + \frac{(k-1)(b-a)}{2^z}, a + \frac{k(b-a)}{2^z}\right\}\) (where \(k = 2, 3, ..., 2^n\)) we have:

\[
G^f(x, z, a, b, y) = \begin{cases} 
1, & \text{if } \prod_{i=1}^{k-1} h^f\left(\frac{x, a + \frac{(i-1)(b-a)}{2^z}, a + \frac{i(b-a)}{2^z}}{a + \frac{b-a}{2^z}}\right) \neq 0, \\
\wedge h^f\left(\frac{x, a + \frac{(k-1)(b-a)}{2^z}, a + \frac{k(b-a)}{2^z}}{a + \frac{b-a}{2^z}}\right) = 0, \\
0, & \text{otherwise}
\end{cases}
\]

and for \(Y \not\in [A, B]\) the function \(g^f\) is equal to 2.

The definition of \(G_f\) is given by the cases with respect to the value of the expression given by \(\prod h^f\), since for \(f \in L_m\), the function \(h_f \in L_{m+p+2}\) and \(G^f \in L_{m+2,p+3}\). Then we have \(K^f_{\mu} \in L_{m+2,p+6}\). Now we must use the function \(K^f_{\mu}\) in the same way as \(K^f\) which gives us \(Z_f\) in the class \(L_{m+3,p+9}\). The final definition of \(g(x) = \mu, f(x, y)\) ([5] Theorem 4.3) given below...
Theorem 4.3 Let $f : \mathbb{R} \to \mathbb{R}$ be an $\mathbb{R}$-recursive function. Then for all $i \geq 0$ if $f \in M_i$, then $f \in L^{(3,p+9)}_{i}$. The above statement is a simple consequence of the fact $M_0 = L_0$ and Lemma 4.2.

5. Conclusions

In the paper we give the first rough approximation of 'a complexity' of limit operations in the terms of the $\mu$-operator and conversely. The results, interpreted in the intuitional way, can suggest what kind of connection exists between infinite limits and a $\mu$-operator.

We also establish the proper relation between the levels of the limit hierarchy and $\mu$-hierarchy. Let us point out that in consequence we may investigate analogies which exist for the limit hierarchy (also $\mu$-hierarchy) and Baire classes.
[7]. Also the kind of a connection between the $\sum_n^0$-measurable functions and $\mathbb{R}$-recursive functions is an open problem.

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**References**