Two hierarchies of $R$-recursive functions

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Abstract

In the paper some aspects of complexity of $R$-recursive functions are considered. The limit hierarchy of $R$-recursive functions is introduced by the analogy to the $\mu$-hierarchy. Then its properties and relations to the $\mu$-hierarchy are analysed.

1. Introduction

The classical theory of computation deals with the functions on enumerable (especially natural) domains. The fundamental notion in this field is the notion of a (partial) recursive function. The problem of hierarchies for these functions is also in the interest of mathematicians (for elementary, primitive recursive function, Grzegorczyk hierarchy, compare [1].

During past years many mathematicians have been interested in creating analogous models of computation on real numbers (see for example Grzegorczyk [2], Blum, Shub, Smale [3]). An interesting approach was given by Moore. In the work [4] he defined a set of functions on the reals $R$ (called $R$-recursive functions) in the analogous way to the classical recursive functions on the natural numbers $N$. His model has a continuous time of computation (a continuous integration instead of a discrete recursion). The great importance in Moore's model has the zero-finding operation $\mu$, which is used to construct $\mu$-hierarchy of $R$-recursive functions.

It was shown [5] that the zero-finding operator $\mu$ can be replaced by the operation of infinite limits. This allows us to define a limit hierarchy and relate it to $\mu$-hierarchy.

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2. Preliminaries

We start with a fundamental definition of a class of real functions called $\mathbb{R}$-recursive functions [4].

**Definition 2.1** The set of $\mathbb{R}$-recursive functions is generated from the constants 0,1 by the operations:

1) composition: $h(x) = f(g(x))$;

2) differential recursion: $h(x,0) = f(x), \partial y h(x,y) = g(x,y,h(x,y))$ (the equivalent formulation can be given by integrals:

$$h(x,y) = f(x) + \int_0^y g(x,y',h(x,y')) dy';$$

3) $\mu$-recursion $h(x) = \mu y f(x,y) = \inf \{ y : f(x,y) = 0 \}$, where infimum chooses the number $y$ with the smallest absolute value and for two $y$ with the same absolute value the negative one;

4) vector-valued functions can be defined by defining their components.

Several comments are needed to the above definition. A solution of a differential equation need not be unique or can diverge. Hence, we assume that if $h$ is defined by a differential recursion then $h$ is defined only where a finite and unique solution exists. This is why the set of $\mathbb{R}$-recursive functions includes also partial functions. We use (after [4]) the name of $\mathbb{R}$-recursive functions in the article, however we should remember that in reality we have partiality here (partial $\mathbb{R}$-recursive functions).

The second problem arises with the operation of infimum. Let us observe that if an infinite number of zeros accumulates just above some positive $y$ or just below some negative $y$ then the infimum operation returns that $y$ even if it itself is not a zero.

In the papers [5, 6] it was shown that if in the Moore's definition [4] $\mu$-operation is replaced by infinite limits: $h(x) = \liminf_{y \to \infty} g(x,y)$, $h(x) = \limsup_{y \to \infty} g(x,y)$ then the resulting class of functions remains the same.

This gives us also the following result (including the limit operation in the form $h(x) = \lim_{y \to \infty} g(x,y)$, which can be in the obvious way obtained from limsup, liminf:
**Corollary 2.2** The class of $\mathbb{R}$-recursive functions is closed under the operations of infinite limits:  
$$h(\bar{x}) = \liminf_{y \to \infty} g(\bar{x}, y), \quad h(\bar{x}) = \limsup_{y \to \infty} g(\bar{x}, y), \quad h(\bar{x}) = \lim_{y \to \infty} g(\bar{x}, y).$$

3. Hierarchies

The operator $\mu$ is a key operator in generating the $\mathbb{R}$-recursive functions. In a physical sense it has a property of being strongly uncomputable. This fact suggests creating a hierarchy, which is built with respect to the number of uses of $\mu$ in the definition of a given $f$.

**Definition 3.1** ([4]) For a given $\mathbb{R}$-recursive expression $s(\bar{x})$, let $M_{\mu}(s)$ (the $\mu$-number with respect to $x_i$) be defined as follows:

\begin{align*}
M_{\mu}(0) &= M_{\mu}(1) = M_{\mu}(-1) = 0, \\
M_{\mu}(f(g_1, g_2, \ldots)) &= \max_j \left( M_{\mu}(f) + M_{\mu}(g_j) \right), \\
M_{\mu}\left( h = f + \int_0^y g(\bar{x}, y', h) dy' \right) &= \max \left( M_{\mu}(f), M_{\mu}(g), M_h(g) \right), \\
M_{\mu}\left( h = f + \int_0^y g(\bar{x}, y', h) dy' \right) &= \max \left( M_{\mu}(g), M_h(g) \right), \\
M_{\mu}(\mu, f(\bar{x}, y)) &= \max \left( M_{\mu}(f), M_{\mu}(f) \right) + 1,
\end{align*}

where $x$ can be any $x_1, \ldots, x_n$ for $\bar{x} = (x_1, \ldots, x_n)$.

For an $\mathbb{R}$-recursive function $f$, let $M(f) = \max_{\nu}(s)$ minimized over all expressions $s$ that define $f$. Now we are ready to define $M$-hierarchy ($\mu$-hierarchy) as a family of $M_j = \{ f : M'(f) \leq j \}$.

Let us construct the analogous definition of $L$-hierarchy by replacing in the above definition $M_{\mu}$ by $L_{\mu}$ and changing line (5) to the following form (5'):

\begin{align*}
L_{\mu}\left( \liminf_{y \to \infty} g(\bar{x}, y) \right) &= L_{\mu}\left( \limsup_{y \to \infty} g(\bar{x}, y) \right) = \\
&= L_{\mu}\left( \lim g(\bar{x}, y) \right) = \max \left( L_{\mu}(f), L_{\mu}(f) \right) + 1.
\end{align*}

For an $\mathbb{R}$-recursive function $f$, let $L(f) = \max_{\nu}(s)$ minimized over all expressions $S$ that define $f$ without using the $\mu$-operation.
Definition 3.2 The \( \mathcal{L} \)-hierarchy is a family of \( L_j = \{ f : L(f) \leq j \} \).

Let us add that in Definition 3.2 we use explicitly the operator \( f(\bar{x}) = \lim_{y \to \infty} g(\bar{x}, y) \) to avoid its construction by other operators (\( \limsup \), \( \liminf \)), which would effect in a superficially higher class of a complexity of a function \( f \).

As an obvious corollary from definitions we have the following statement.

Lemma 3.3 The classes \( M_0 \) and \( M_1 \) are identical.

A function \( f \in L_0 = M_0 \) will be called (by an analogy to the case of natural recursive functions) a primitive \( \mathbb{R} \)-recursive function. After Moore [4] we can conclude that such functions as: \(-x\), \( x+y\), \( xy\), \( x/y\), \( e^x\), \( \ln x\), \( y^x\), \( \sin x\), \( \cos x\) are primitive \( \mathbb{R} \)-recursive.

We can give a few results on some levels of the limit hierarchy.

Lemma 3.4. The Kronecker \( \delta \) function, the signum function and absolute value belong to the first level \( L_1 \) of limit hierarchy.

Proof. It is sufficient to take the following definitions [5]: hence \( \delta(0) = 1 \) and for all \( x \neq 0 \) we have \( \delta(x) = 0 \) let us define \( \delta(x) = \liminf_{y \to \infty} \left( \frac{1}{1 + x^2} \right)^y \). Now from the expression \( \liminf_{y \to \infty} \arctan xy = \begin{cases} \pi/2, & \text{if } x > 0, \\ 0, & \text{if } x = 0, \\ -\pi/2, & \text{if } x < 0, \end{cases} \) we obtain

\[
\text{sgn}(x) = \frac{\liminf_{y \to \infty} \arctan xy}{2\arctan 1} \quad \text{and} \quad |x| = \text{sgn}(x)x.
\]

We should be careful with definitions of functions by cases:

Lemma 3.5 For \( h(\bar{x}) = \begin{cases} g_1(\bar{x}), & \text{if } f(\bar{x}) = 0, \\ g_2(\bar{x}), & \text{if } f(\bar{x}) = 1, \\ \ldots & \ldots \\ g_k(\bar{x}), & \text{if } f(\bar{x}) \geq k - 1 \end{cases} \) and \( g_i \in L_n \) for all \( 1 \leq i \leq k \),

\( f \in L_m \) the function \( h \) belongs to \( L_{\max(n_1, \ldots, n_k, m+1)} \).
Proof. Let us see that \( eq(x, y) = \delta(x - y) \in L_1 \) and \( ge(x, y) = \frac{(\text{sgn}(x - y) + eq(x, y))}{2} + \frac{1}{2} \in L_1 \). Then of course
\[
h(\bar{x}) = \sum_{i=1}^{k-1} g_i(\bar{x})eq\left(f(\bar{x}), i - 1\right) + g_k(\bar{x})ge\left(f(\bar{x}), k - 1\right)
\]

Of course this result can be easily extended to other forms of definitions by cases.

Lemma 3.6 The function \( \Theta(x) \) (equal to 1 if \( x \geq 0 \), otherwise 0), maximum \( \max(x, y) \), square-wave function \( s \) are in \( L_2 \), the function \( p(x) \) such that \( p(x) = 1 \) for \( x \in [2n, 2n+1] \) and \( p(x) = 0 \) for \( x \in [2n+1, 2n+2] \) is in \( L_2 \) and the floor function \( \lfloor x \rfloor \) is in \( L_3 \).

Proof. We give the proper definitions (from [6]) for these functions. Let
\[
\Theta(x) = \delta(x - |x|),
\]
\[
\max(x, y) = x\delta(x - y) + (1 - \delta(x - y))\left[x\Theta(x - y) + y\Theta(y - x)\right],
\]
\[
s(x) = \Theta(\sin(\pi x)).
\]

The function \( p(x) \) can be given as
\[
s(x) = \left[1 - \delta\left(\sin\left(\frac{x - 1}{2}\pi\right)\right)\right], \text{so } p \in L_2.
\]

The floor function we can define by the auxiliary function \( w(0) = 0, \partial_x w(x) = 2\Theta(-\sin(2\pi x)) \) as
\[
\lfloor x \rfloor = \begin{cases} 
2w(x/2) & \text{if } p(x) = 1, \\
2w((x - 1)/2) & \text{if } p(x) = 0.
\end{cases}
\]

From the above equation we have \( \lfloor x \rfloor \) in \( L_3 \). \( \square \)

Let us recall that if \( f : R^n \to R \) is an \( R \)-recursive function then the function \( f_{\text{iter}}(i, \bar{x}) \) is \( R \)-recursive, too.

Lemma 3.7 Let \( f : R^n \to R \) belongs to the class \( L_i \), then we have \( f_{\text{iter}} : R^{n+1} \to R \) is in \( L_{\max(2, j)} \).

Proof. The definitions, which were given by Moore [3] \( f_{\text{iter}}(i, \bar{x}) = h(2i) \), where
\[
h(0) = g(0) = \bar{x},
\]
\[ \partial_{t}g(t) = \left[ f(h(t)) - h(t) \right] s(t), \]
\[ \partial_{t}h(t) \geq \left[ \frac{g(t) - h(t)}{r(t)} \right] (1 - s(t)), \]

with \( s \) - a square wave function in \( L_{2} \) and \( r(0) = 0 \), \( \partial_{t}r(t) = 2s(t) - 1 \), \( r, s \in L_{2} \) give us the desirable statement. □

**Lemma 3.8** The \( \mathbb{R}^{1} \)-recursive functions \( \gamma_{2} : \mathbb{R}^{2} \to \mathbb{R}, \quad \gamma'_{2}, \gamma''_{2} : \mathbb{R} \to \mathbb{R} \) such that \( (\forall x, y \in \mathbb{R}) \gamma'_{2}(\gamma_{2}(x, y)) = x, \quad (\forall x, y \in \mathbb{R}) \gamma''_{2}(\gamma_{2}(x, y)) = y \), have the following properties: \( \gamma_{2}, \gamma'_{2} \) are in \( L_{10} \), \( \gamma''_{2} \) is in \( L_{14} \).

**Proof.** We have the auxiliary functions \( \Gamma_{2}, \quad \Gamma'_{2}, \quad \Gamma''_{2} \), which are coding and decoding functions in the interval \( (0,1) : \Gamma_{2}(x, y) = c(x) + c(y)/10 \), where
\[
 c(x) = \lim_{t \to \infty} z(a(i,x))/10^i + b(i,x)/10^i,
\]
and later \( z(x) = \lim_{i \to \infty} z_{\text{iter}}(i,x) \),
\[
z_{\text{iter}}(i,a_{1}...a_{n}a_{n+1}...) = a_{1}...a_{n}0...a_{n+1}0...a_{n+2}...,
\]
\[
a(i,0,a_{1}...a_{n}) = 0.a_{1}...a_{n},
\]
\[
b(i,0,a_{1}...a_{n}) = 0.0...0...a_{n+1}...
\]

\[
 (z'(x)) = \begin{cases} 100[x] + 10(x-[x]), & \text{if } [x] \neq x, \\ x, & \text{if } [x] = x, \end{cases} \quad \in L_{4}, a,b \in L_{4}. \] Also \( z_{\text{iter}} \) belongs to \( L_{4} \), hence \( \Gamma_{2}(x,y) \in L_{10} \), decoding of the first element is described in the symmetric way so \( \Gamma'_{2}(x) \) is in \( L_{10} \), but \( \Gamma_{2}(x) = \Gamma'_{2}(10 - [10x]) \) so \( \Gamma''_{2} \in L_{14} \).

The functions \( \Gamma_{2}, \quad \Gamma'_{2}, \quad \Gamma''_{2} \) can be extended to all reals by one-to-one \( f : (0,1) \to \mathbb{R} \in L_{10} \) without the loss of their class. □

The same method of coding and decoding by interlacing of ciphers (only the power of 10 should be changed) gives us the functions \( \gamma_{n} : \mathbb{R}^{n} \to \mathbb{R} \) and \( \gamma'_{n} : \mathbb{R} \to \mathbb{R} \) for \( i = 1,...,n \) such that
\[
 (\forall i) (\forall x_{1},...x_{n} \in \mathbb{R}) \gamma'_{n}(\gamma_{n}(x_{1},...,x_{n})) = x_{i}
\]
in the same class: \( \gamma_{n}, \gamma'_{n} \in L_{10} \) and \( (\forall i > 1) \gamma'_{n} \in L_{14} \).

We finish this part with the important form of defining: a new function is given as a product of values \( f \) in some integer points.
Lemma 3.9 There exists such constant $p \in \mathbb{N}$ that for the function
\[
\prod_{z=0}^{y} f(\bar{x}, z) = \begin{cases} 
  f(\bar{x}, 0) f(\bar{x}, 1) \ldots f(\bar{x}, y-1), & \text{if } y \geq 1, \\
  1, & \text{if } 0 \leq y < 1, \\
  0, & \text{if } y < 0,
\end{cases}
\]
if the function $f$ is in the class $L_m$ then $\prod_{z=0}^{y} f(\bar{x}, z)$ is in the class $L_{m+p}$ ($p$ is independent of $m$).

Proof. By the definitions
\[
t(w) = \gamma_{n+2}(\gamma_{n+2}^{1,n}(w), \gamma_{n+2}^{n+1}(w) + 1, f(\gamma_{n+2}^{1,n}(w), \gamma_{n+2}^{n+1}(w)), \gamma_{n+2}^{n+2}(w))
\]
and
\[
S(\bar{x}, z) = t_{\mathcal{L}}(s(\bar{x}, 0), \ldots, t_{\mathcal{L}}(\lfloor z \rfloor, \gamma_{n+2}(\bar{x}, 0, 1)))
\]
we get the property
\[
\prod_{y=0}^{z} f(\bar{x}, y) = \gamma_{n+2}^{n+2}(S(\bar{x}, z)).
\]
From the definition of the limit hierarchy we get $\prod_{y=0}^{z} f(\bar{x}, y) \in L_{n+38}$.

In the rest of the paper we will use the constant $p$ as the number of limits used in the recursive definition of the product $\prod_{y=0}^{z} f(\bar{x}, y)$ instead of the value 38. The above constructions are tedious and can be improved with a better approximation of $p$.

4. Main results

Now we are ready to formulate two theorems which demonstrate connections between $L$-hierarchy and $M$-hierarchy.

Theorem 4.1 Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be an $\mathbb{R}$-recursive function. Then if $f \in L_i$ then $f \in M_{10i}$.

Proof. We use a simple induction here. The case $i = 0$ is given in Lemma 3.3. Now let us suppose that the thesis is true for $i = n$. Let $f \in L_{n+1}$ be defined as $f(\bar{x}) = \lim_{y \rightarrow \infty} g(\bar{x}, y)$ for $g \in L_n$. Then we can recall Theorem 4.2 from [6] which gives us the following result: to define $f$ from $g$ it is necessary to use at
most 10 μ-operation. Hence for \( g \in M_{10n} \) the function \( f \) satisfies \( f \in M_{10n+10} \).

Similar inferences hold for \( \lim \inf, \lim \sup \).

Now we can give the result about the 'limit complexity' of the infimum operator \( \mu \).

**Lemma 4.2** If \( f(\bar{x},y) : R^{n+1} \to R \) is in the class \( L_m \) then the function \( g : R^n \to R \), \( g(\bar{x}) = \mu_y f(\bar{x},y) \) is in the class \( L_{m+3p+9} \) from Lemma 3.9.

**Proof.** Here we must employ the results from [6]. There we defined the function \( g : R^n \to R \), \( g(\bar{x}) = \mu_y f(\bar{x},y) \) for \( f(\bar{x},y) : R^{n+1} \to R \) (\( f \) - \( R \)-recursive) replacing the \( \mu \)-operator by limit operation. First we introduced the function

\[
Z_f'(\bar{x},z) = \begin{cases} 
\text{undefined} & \text{if } z = 0 \text{ and } \exists yK'f(\bar{x},y) = 0, \\
\inf_y \left\{ f : K'(\bar{x},y) = 0 \right\} & \text{if } z = 0 \text{ and } \forall yK'(\bar{x},y) \neq 0, \\
1 & \text{if } z \neq 0,
\end{cases}
\]

given in the following way:

\[
Z_f'(\bar{x},z) = \begin{cases} 
\text{undefined} & \text{if } (z = 0) \land \left( S_f'(\bar{x}) < \frac{1}{12} \right), \\
\sqrt{S_f'(\bar{x})} - \frac{1}{12}, & \text{if } (z = 0) \land \left( S_f'(\bar{x}) \geq \frac{1}{12} \right) \\
0, & \land f(\bar{x},\sqrt{S_f'(\bar{x})} - \frac{1}{12}) = 0, \\
-\sqrt{S_f'(\bar{x})} - \frac{1}{12}, & \text{if } (z = 0) \land \left( S_f'(\bar{x}) \geq \frac{1}{12} \right) \\
0, & \land f(\bar{x},-\sqrt{S_f'(\bar{x})} - \frac{1}{12}) = 0, \\
1, & \text{if } z \neq 0,
\end{cases}
\]

where \( S_f'(\bar{x}) = \lim_{t \to \infty} S_1'(\bar{x},t) + \lim_{t \to \infty} S_2'(\bar{x},t) \). Both functions \( S_1', S_2' \) are defined by an integration

\[
S_i'(\bar{x},t) = \int y^2 \left( 1 - h_i'(\bar{x},(-1)^{i+1} y - 1/2,(-1)^{i+1} y + 1/2) \right) dy, \quad i = 1,2
\]

from \( h_i'(\bar{x},a,b) = \lim_{\gamma \to -\infty} \prod_{w=0}^{\infty} K'(\bar{x},a + w \frac{b-a}{z}) \) where \( K' \) is the characteristic function of \( f \).

Hence we can conclude that if \( K' \) is in the \( L_s \) then \( Z_f' \) is in the class \( L_{s+p+3} \).

Let us finish with the definition of the characteristic function of the infimum of zeros of \( f \) (see Theorem 4.2 from [5]
where \( G^f (\bar{x}, z, a, b, y) \) divides the interval \([a, b]\) into \(2^{|z|}\) equal subintervals and gives the value 1 for \(y\) from the subintervals, which contains the least zero of \(f\) in \([a, b]\) and value 0 otherwise. Precisely for \(y\) from \(\left[a, a + \frac{b-a}{2^{|z|}}\right]\)

\[
G^f (\bar{x}, z, a, b, y) = \begin{cases} 
1, & \text{if } h^f \left( \bar{x}, a + \frac{b-a}{2^{|z|}}, a + \frac{k(b-a)}{2^{|z|}} \right) = 0, \\
0, & \text{otherwise}
\end{cases}
\]

for \(y \in \left(a + \frac{(k-1)(b-a)}{2^{|z|}}, a + \frac{k(b-a)}{2^{|z|}}\right)\) (where \(k = 2, 3, \ldots, 2^n\)) we have:

\[
G^f (\bar{x}, z, a, b, y) = \begin{cases} 
1, & \text{if } \prod_{i=1}^{k-1} h^f \left( \bar{x}, a + \frac{(i-1)(b-a)}{2^{|z|}}, a + \frac{i(b-a)}{2^{|z|}} \right) \neq 0, \\
\wedge h^f \left( \bar{x}, a + \frac{b-a}{2^{|z|}}, a + \frac{k(b-a)}{2^{|z|}} \right) = 0, \\
0, & \text{otherwise}
\end{cases}
\]

and for \(Y \not\in [A, B]\) the function \(g^f_x\) is equal to 2.

The definition of \(G^f\) is given by the cases with respect to the value of the expression given by \(\prod h^f\), since for \(f \in L_m\), the function \(h^f \in L_{m+p+2}\) and \(G^f \in L_{m+2p+3}\). Then we have \(K^f_x \in L_{m+2p+6}\). Now we must use the function \(K^f_x\) in the same way as \(K^f\) which gives us \(Z^f\) in the class \(L_{m+3p+9}\). The final definition of \(g (\bar{x}) = \mu, f (\bar{x}, y)\) ([5] Theorem 4.3) given below
\[
g(\bar{x}) = \begin{cases} 
Z^{+}_i(\bar{x},0) - Z^{-}_i(\bar{x},0), & \text{if } S^{+}_i(\bar{x}) < \frac{1}{12} \wedge S^{-}_i(\bar{x}) < \frac{1}{12}, \\
Z^{+}_i(\bar{x},0), & \text{if } \left( S^{+}_i(\bar{x}) \geq \frac{1}{12} \wedge S^{-}_i(\bar{x}) < \frac{1}{12} \right) \\
Z^{-}_i(\bar{x},0), & \text{or } \left( S^{+}_i(\bar{x}) < \frac{1}{12} \wedge S^{-}_i(\bar{x}) < \frac{1}{12} \wedge Z^{+}_i(\bar{x},0) < Z^{-}_i(\bar{x},0) \right), \\
-Z^{-}_i(\bar{x},0), & \text{if } \left( S^{+}_i(\bar{x}) < \frac{1}{12} \wedge S^{-}_i(\bar{x}) \geq \frac{1}{12} \right) \\
-Z^{+}_i(\bar{x},0), & \text{or } \left( S^{+}_i(\bar{x}) < \frac{1}{12} \wedge S^{-}_i(\bar{x}) < \frac{1}{12} \wedge Z^{+}_i(\bar{x},0) \geq Z^{-}_i(\bar{x},0) \right), 
\end{cases}
\]

where \( f^+(\bar{x},y) = \begin{cases} f(\bar{x},y), & y \geq 0, \\
1, & y < 0; \end{cases} \)
\( f^-(\bar{x},y) = \begin{cases} f(\bar{x},-y), & y > 0, \\
1, & y \leq 0; \end{cases} \)
remains the class of \( g \) identical to the class of \( Z^i \), i.e. \( g \in L_{m+3,p+9} \).

**Theorem 4.3** Let \( f : R^n \rightarrow R \) be an \( R \)-recursive function. Then for all \( i \geq 0 \) if \( f \in M_i \) then \( f \in L_{(3,p+9)i} \).

The above statement is a simple consequence of the fact \( M_0 = L_0 \) and Lemma 4.2.

### 5. Conclusions

In the paper we give the first rough approximation of 'a complexity' of limit operations in the terms of the \( \mu \)-operator and conversely. The results, interpreted in the intuitional way, can suggest what kind of connection exists between infinite limits and a \( \mu \)-operator.

We also establish the proper relation between the levels of the limit hierarchy and \( \mu \)-hierarchy. Let us point out that in consequence we may investigate analogies which exist for the limit hierarchy (also \( \mu \)-hierarchy) and Baire classes.
[7]. Also the kind of a connection between the $\sum_\infty^0$–measurable functions and R-recursive functions is an open problem.

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References