Newton-like method for singular 2-regular system of nonlinear equations

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Abstract

In this article the problem of solving a system of singular nonlinear equations will be discussed. The theory of local and Q-superlinear convergence for the nonlinear operators is developed.

1. Introduction

Let \( F : D \subset \mathbb{R}^n \to \mathbb{R}^m \) be a nonlinear operator. The problem of solving a system of nonlinear equations consist in finding a solution \( x^* \in D \) of the equation

\[
F(x) = 0.
\]  

Definition 1

A linear operator \( \Psi_2(h) : \mathbb{R}^n \to \mathbb{R}^m \), \( h \in \mathbb{R}^n \) is called 2-factor operator, if

\[
\Psi_2(h) = F'(x^*) + P^\perp F^\perp(x^*) h,
\]

where

\( P^\perp \) - denotes the orthogonal projection on \((\text{Im} F'(x))^\perp\) in \( \mathbb{R}^n \) [1].

Definition 2

Operator \( F \) is called 2-regular in \( x^* \) on the element \( h \in \mathbb{R}^n \), \( h \neq 0 \), if the operator \( \Psi_2(h) \) has the property:

\[
\text{Im} \Psi_2(h) = \mathbb{R}^m.
\]

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Definition 3
Operator $F$ is called 2-regular in $x^*$, if $F$ is 2-regular on the set $K_2(x^*)\{0\}$, where

$$K_2(x^*) = \text{Ker} F^* x^* \cap \text{Ker}^2 P^\perp F^* x^* ,$$

$$\text{Ker}^2 P^\perp F^* x^* = \{ h \in R^n : P^\perp F^* (x^*) [h]^2 = 0 \}.$$  

We need the following assumption on $F$:

A1) completely degenerated in $x^*$:

$$\text{Im} F^* (x^*) = 0.$$  

A2) operator $F$ is 2-regular in $x^*$:

$$\text{Im} F^* (x^*) h = R^m \text{ for } h \in K_2(x^*), h \neq 0.$$  

A3)

$$\text{Ker} F^* (x^*) \neq \{0\} .$$

If $F$ satisfies A1 in $x^*$, then

$$K_2(x^*) = \text{Ker}^2 P^\perp F^* (x^*) = \{ h \in R^n : F^* (x^*) [h]^2 = 0 \} .$$

In [1] it was proved, that if $n=m$, then the sequence

$$x_{k+1} = x_k - \left\{ \hat{F}^* (x_k) + P^\perp h_k \right\}^{-1} \cdot \left\{ F(x_k) + P^\perp F^* (x_k) h_k \right\} ,$$

where

$$P^\perp h_k \text{ denotes orthogonal projection on } \left( \text{Im} \hat{F}^* (x_k) \right)^\perp \text{ in } R^n ,$$

$$h_k \in \text{Ker} \hat{F}^* (x_k) , \| h_k \| = 1$$

converges Q-quadratically to $x^*$.

The matrices $\hat{F}^* (x_k)$ obtained from $F^* (x_k)$ by replacing all elements, whose absolute values do not increase $\nu > 0$, by zero, where $\nu = \nu_k = \| F(x_k) \|^{(1-\alpha)/2} , 0 < \alpha < 1$.

In the case $n = m+1$ the operator

$$\left\{ \hat{F}^* (x_k) + P^\perp h_k \right\}^{-1}$$

in method (8) is replaced by the operator

$$\left[ \hat{F}^* (x_k) + P^\perp F^* (x_k) h_k \right]^{-}$$

and then the method converges Q-linearly to the set of solutions [2].

Under the assumptions A1-A3, the system of equation (1) is undetermined ($n>m$) and degenerated in $x^*$.
2. Extending of the system of equation

Now we construct the operator $\Phi : R^n \rightarrow R^{n-1}$ with the properties (4), (5) and such that $\Phi(x^*) = 0$ [2].

Assume

A4) Let $F(x) = [f_1(x), f_2(x), \ldots, f_m(x)]^T$, $n > m$ is two continuously differentiable in some neighbourhood $U \subset R^n$ of the point $x^*$.

Denote:

$H = \text{lin}\{h\}$ for $h \in \text{Ker} F^2(x^*)$, $h \neq 0$.

$P = P_{H^\perp}$ denotes the orthogonal projection $R^n$ on $H^\perp$.

$\frac{\partial^2 q(x)}{\partial x_i} = P\left(\frac{\partial f_i}{\partial x}(x)\right)^T$ for $i = 1, 2, \ldots, m$.

For each system of indices $i_1, i_2, \ldots, i_{n-m-1} \subset \{1, 2, \ldots, m\}$ and vectors $h_1, h_2, \ldots, h_{n-m-1} \subset R^n$ we define

$$\Phi(x) = \begin{bmatrix} F_i(x) h \\ \varphi(x) \end{bmatrix}, \quad \text{(10)}$$

where

$$\varphi(x) : R^n \rightarrow R^r, \quad r = n - m - 1,$$

$$\varphi(x) = PF_i(x)^{\varphi^2}, \quad \frac{\partial^2 \varphi}{\partial h_i} = \begin{bmatrix} f_i^2(x) h_1 \\ \vdots \\ f_i^2(x) h_r \end{bmatrix}^T,$$

$$\varphi(x) = \begin{bmatrix} f_i^2(x) h_1 \\ \vdots \\ f_i^2(x) h_r \end{bmatrix}. \quad \text{(11)}$$

In [2] it was proved, that the sequence

$$x_{k+1} = x_k - \left[ \Phi\left( x_k \right) \right]^+ \cdot \Phi\left( x_k \right), \quad k = 0, 1, 2, \ldots \quad \text{(12)}$$

quadratically converges to the solution of (1).

3. New method

We propose the Newton-like method, where the sequence $\{x_k\}$ is defined by:

$$x_{k+1} = x_k - \left( B_k \right)^+ \cdot \Phi(x_k). \quad \text{(13)}$$

The operator $\Phi^2$ will be approximated by matrices $\{B_k\}$.

Let

$$s_k = x_{k+1} - x_k. \quad \text{(14)}$$

We propose matrices $B_k$ which satisfy the secant equation:

$$B_{k+1}s_k = \Phi\left( x_{k+1} \right) - \Phi\left( x_k \right) \quad \text{for } k = 0, 1, 2, \ldots \quad \text{(15)}$$

For example, to obtain the sequence $\{B_k\}$ we can apply the Broyden method:
\[ B_{k+1} = B_k - \frac{r_k s_k^T}{s_k^T s_k} \quad \text{for } k=0,1,2,... \]  
(16)

where
\[ r_k = \Phi(x_{k+1}) - \Phi(x_k) - B_k s_k. \]  
(17)

We will prove for this method:

**Q-linear convergence** to \( x^* \) i.e. there exists \( q \in (0,1) \) such that
\[ \| x_{k+1} - x^* \| \leq q^k \| x_k - x^* \| \quad \text{for } k = 0,1,2,... \]  
(18)

and next Q-superlinear convergence to \( x^* \), i.e.:
\[ \lim_{k \to \infty} \frac{\| x_{k+1} - x^* \|}{\| x_k - x^* \|} = 0. \]  
(19)

We present the theorem which is an analogue of the Bounded Deterioration Theorem (Broyden, Dennis and More - [3]) for the Newton-like methods, when the operator \( \Phi' (x^*) \) is nonsingular.

**Theorem 1** (The Bounded Deterioration Theorem)

Let \( F \) satisfies the assumptions A1-A4. If exist constants \( q_1 \geq 0 \) and \( q_2 \geq 0 \) such that matrices \( \{ B_k \} \) satisfy the inequality:
\[ \| B_{k+1} - \Phi' (x^*) \| \leq (1 + q_1 r_k) \| B_k - \Phi' (x^*) \| + q_2 r_k, \]  
(20)

then there are constants \( \epsilon > 0 \) and \( \delta > 0 \) such, that if
\[ \| x_0 - x^* \| \leq \epsilon \quad \text{and} \quad \| B_0 - \Phi' (x^*) \| \leq \delta, \]

then the sequence \( x_{k+1} = x_k - B_k^* \Phi (x_k) \) converges Q-linearly to \( x^* \).

When the system of equation is rectangular, the proof of the theorem is analogous to that for the nonsingular and quadratic system and we neglect it.

**Theorem 2** (Linear convergence)

Let \( F \) satisfies the assmuptions A1-A4. Then the method
\[ x_{k+1} = x_k - B_k^* \Phi (x_k), \]
\[ B_{k+1} = B_k - \frac{\Phi(x_{k+1}) - \Phi(x_k) - B_k s_k}{s_k^T s_k} s_k^T \]  
locally and Q-linearly converges to \( x^* \).

**Proof.**

To prove the Theorem we should prove the inequality (20) from Theorem 1. Now we notice:
where $c_1 > 0$, $c_2 > 0$, $q_1 > 0$, $q_2 > 0$, $r_k = \max \{ \| x_{k+1} - x^* \|, \| x_k - x^* \| \}$.

\begin{align*}
\| B_{k+1} - \Phi' (x^*) \| &= \left\| B_k - \frac{\{ \Phi(x_{k+1}) - \Phi(x_k) - B_k s_k \} s_k^T}{s_k^T s_k} - \Phi' (x^*) \right\| \\
&\leq \left\| B_k - \Phi' (x^*) \right\| + \left\| \frac{\{ \Phi(x_{k+1}) - \Phi(x_k) - B_k s_k \} s_k^T}{s_k^T s_k} \right\| \\
&\leq \left\| B_k - \Phi' (x^*) \right\| + \left\| \frac{\{ \Phi(x_{k+1}) - \Phi(x_k) - \Phi' (x^*) (x_{k+1} - x^*) \} s_k^T}{s_k^T s_k} \right\| \\
&+ \left\| \frac{\{ \Phi(x_k) - \Phi' (x^*) (x_k - x^*) \} s_k^T}{s_k^T s_k} \right\| \\
&= \left\| \Phi' (x^*) - B_k \right\| (1 + q_1 r_k) + c_1 \left\| x_{k+1} - x^* \right\|^2 \left\| s_k \right\| \\
&+ c_2 \left\| x_k - x^* \right\|^2 \left\| s_k \right\| \\
&\leq \left\| \Phi' (x^*) - B_k \right\| (1 + q_1 r_k) + q_2 r_k,
\end{align*}

where $c_1 > 0$, $c_2 > 0$, $q_1 > 0$, $q_2 > 0$, $r_k = \max \{ \| x_{k+1} - x^* \|, \| x_k - x^* \| \}$.

**Theorem 3** (Q-superlinear convergence)

Let $F$ satisfies the assumptions A1-A4 and the sequence

\begin{align*}
x_{k+1} &= x_k - \{ B_k \}^{-1} \cdot \Phi (x_k), \\
B_{k+1} &= B_k - \frac{\{ \Phi(x_{k+1}) - \Phi(x_k) - B_k s_k \} s_k^T}{s_k^T s_k}
\end{align*}

linearly converges to $x^*$. Then the sequence $\{ x_k \}$ Q-superlinearly converges to $x^*$.

**Proof.**

Matrices $B_k$ satisfy secant equation (15), so

\begin{equation}
B_{k+1} = P_{L_k}^{+} B_k
\end{equation}

where

\begin{equation}
L_k = \left\{ X : X s_k = y_k, \ \text{where} \ y_k = \Phi' (x_{k+1}) - \Phi' (x_k) \right\}
\end{equation}

Denote

\begin{equation}
H_k = H(x_k, x_{k+1}) = \int_0^1 \Phi' (x_k + t (x_{k+1} - x_k)) dt.
\end{equation}

We have $H_k \in L_k$ [4].
From (21) and [3] it follows:
\[ \|B_{k+1} - B_k\|^2 + \|B_{k+1} - H_k\|^2 = \|B_k - H_k\|^2, \quad \text{for } i = 0, 1, 2, \ldots. \]
By lemma 2 [5] we get \( \sum_{k=1}^{\infty} \|B_{k+1} - B_k\|^2 < \infty \), thus we obtain
\[ \|B_{k+1} - B_k\| \to 0. \]
This denotes that the method (13)-(17) is Q-superlinearly convergent [6], which ends the proof. □

4. Summary

The proposed method is Q-superlinearly convergent and easier to apply than the method (12), without calculation of \( F^{\nu}(x_k) \).

References