Newton-like method for singular 2-regular system of nonlinear equations

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Abstract

In this article the problem of solving a system of singular nonlinear equations will be discussed. The theory of local and Q-superlinear convergence for the nonlinear operators is developed.

1. Introduction

Let $F : D \subset \mathbb{R}^n \to \mathbb{R}^m$ be a nonlinear operator. The problem of solving a system of nonlinear equations consist in finding a solution $x^* \in D$ of the equation

$$F(x) = 0.$$  \hfill (1)

Definition 1

A linear operator $\Psi_2 : \mathbb{R}^n \to \mathbb{R}^m$, $h \in \mathbb{R}^n$ is called 2-factor operator, if

$$\Psi_2(h) = F'(x^*) + P^\perp F'(x^*) h,$$  \hfill (2)

where

$P^\perp$ - denotes the orthogonal projection on $(\text{Im } F'(x))^\perp$ in $\mathbb{R}^n$ [1].

Definition 2

Operator $F$ is called 2-regular in $x^*$ on the element $h \in \mathbb{R}^n$, $h \neq 0$, if the operator $\Psi_2(h)$ has the property:

$$\text{Im } \Psi_2(h) = \mathbb{R}^m.$$  

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Definition 3
Operator F is called 2-regular in \( x^* \), if F is 2-regular on the set \( K_2(x^*)\{0\} \), where
\[
K_2(x^*) = \text{Ker} F'(x^*) \cap \text{Ker}^2 P^+ F'(x^*),
\]
where
\[
\text{Ker}^2 P^+ F'(x^*) = \left\{ h \in \mathbb{R}^n : P^+ F'(x^*)[h]^2 = 0 \right\}.
\]
We need the following assumption on F:
A1) completely degenerated in \( x^* \):
\[
\text{Im} F'(x^*) = 0.
\]
A2) operator F is 2-regular in \( x^* \):
\[
\text{Im} F'(x^*) h = \mathbb{R}^m \text{ for } h \in K_2(x^*), h \neq 0.
\]
A3)
\[
\text{Ker} F'(x^*) \neq \{0\}.
\]
If F satisfies A1 in \( x^* \), then
\[
K_2(x^*) = \text{Ker}^2 P^+ F'(x^*) = \left\{ h \in \mathbb{R}^n : F'(x^*)[h]^2 = 0 \right\}.
\]
In [1] it was proved, that if \( n=m \), then the sequence
\[
x_{k+1} = x_k - \left\{ \hat{F}'(x_k) + P_k^+ F'(x_k) h_k \right\}^{-1} \cdot \left\{ F(x_k) + P_k^+ F'(x_k) h_k \right\},
\]
where
\[
P_k^+ \text{ denotes orthogonal projection on } \left( \text{Im} \hat{F}'(x_k) \right)^\perp \text{ in } \mathbb{R}^n,
\]
h\_k \in \text{Ker} \hat{F}'(x_k), \quad \|h_k\| = 1
converges Q-quadratically to \( x^* \).
The matrices \( \hat{F}'(x_k) \) obtained from \( F'(x_k) \) by replacing all elements, whose absolute values do not increase \( v>0 \), by zero, where \( v = v_k = \|F(x_k)\|^{(1-\alpha)/2}, \quad 0<\alpha<1 \).
In the case \( n = m+1 \) the operator
\[
\left\{ \hat{F}'(x_k) + P_k^+ F'(x_k) h_k \right\}^{-1}
\]
in method (8) is replaced by the operator
\[
\left[ \hat{F}'(x_k) + P_k^+ F'(x_k) h_k \right]^+
\]
and then the method converges Q-linearly to the set of solutions [2].
Under the assumptions A1-A3, the system of equation (1) is undetermined (\( n>m \)) and degenerated in \( x^* \).
2. Extending of the system of equation

Now we construct the operator $\Phi : R^n \to R^{n-1}$ with the properties (4), (5) and such that $\Phi(x^*)=0$ [2].

Assume

A4) Let $F(x)=[f_1(x), f_2(x), ..., f_m(x)]^T$, $n>m$ is two continuously differentiable in some neighbourhood $U \subset R^n$ of the point $x^*$.

Denote:

$$H=\text{lin}\{h\} \quad \text{for} \quad h \in \text{Ker} F'(x^*), \ h \neq 0.$$  

$$P = P_{H^\perp}$$  


$\frac{\partial F}{\partial x}(x) = P \left( \frac{\partial f_i}{\partial x}(x) \right)^T$  

for $i=1,2,...,m$.

For each system of indices $i_1, i_2, ..., i_{n-m-1} \subset \{1, 2, ..., m\}$ and vectors $h_1, h_2, ..., h_{n-m-1} \subset R^n$ we define

$$\Phi(x) = \begin{bmatrix} F'(x) h \\ \varphi(x) \end{bmatrix},$$

where

$$\varphi(x) : R^n \to R^r, \quad r=n-m-1,$$

$$\varphi(x) = PF'(x) h, \quad \frac{\partial \varphi}{\partial h} = [h_1, h_2, ..., h_r]^T,$$

$$\varphi(x) = \begin{bmatrix} \frac{\partial F}{\partial x}(x) h_1 \\ \vdots \\ \frac{\partial F}{\partial x}(x) h_r \end{bmatrix}.$$  

(10)

In [2] it was proved, that the sequence

$$x_{k+1} = x_k - \left[ \Phi'(x_k) \right]^{-1} \cdot \Phi(x_k), \quad k=0,1,2,...$$

quadratically converges to the solution of (1).

3. New method

We propose the Newton-like method, where the sequence $\{x_k\}$ is defined by:

$$x_{k+1} = x_k - \left( B_k \right)^+ \cdot \Phi(x_k).$$  

(13)

The operator $\Phi'$ will be approximated by matrices $\{B_k\}$.

Let

$$s_k = x_{k+1} - x_k.$$  

(14)

We propose matrices $B_k$ which satisfy the secant equation:

$$B_{k+1}s_k = \Phi(x_{k+1}) - \Phi(x_k) \quad \text{for} \quad k=0,1,2,...$$  

(15)

For example, to obtain the sequence $\{B_k\}$ we can apply the Broyden method:
\[ B_{k+1} = B_k - \frac{r_k s_k^T}{s_k^T s_k} \quad \text{for } k=0,1,2,... \quad (16) \]

where
\[ r_k = \Phi(x_{k+1}) - \Phi(x_k) - B_k s_k. \quad (17) \]

We will prove for this method:

**Q-linear convergence** to \( x^* \) i.e. there exists \( q \in (0,1) \) such that
\[
\| x_{k+1} - x^* \| \leq q \| x_k - x^* \| \quad \text{for } k = 0,1,2,... \quad (18)
\]

and next **Q-superlinear convergence** to \( x^* \), i.e.:
\[
\lim_{k \to \infty} \left( \frac{x_{k+1} - x^*}{x_k - x^*} \right) = 0. \quad (19)
\]

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We present the theorem which is an analogue of the Bounded Deterioration Theorem (Broyden, Dennis and More - [3]) for the Newton-like methods, when the operator \( \Phi' \left( x^* \right) \) is nonsingular.

**Theorem 1** (The Bounded Deterioration Theorem)

Let \( F \) satisfies the assumptions A1-A4. If exist constants \( q_1 \geq 0 \) and \( q_2 \geq 0 \) such that matrices \( \{ B_k \} \) satisfy the inequality:
\[
\left\| B_{k+1} - \Phi' \left( x^* \right) \right\| \leq \left( 1 + q_1 r_k \right) \left\| B_k - \Phi' \left( x^* \right) \right\| + q_2 r_k , \quad (20)
\]

then there are constants \( \varepsilon > 0 \) and \( \delta > 0 \) such, that if
\[
\| x_0 - x^* \| \leq \varepsilon \quad \text{and} \quad \| B_0 - \Phi' \left( x^* \right) \| \leq \delta,
\]

then the sequence
\[
x_{k+1} = x_k - B_k^* \Phi \left( x_k \right)
\]

converges Q-linearly to \( x^* \).

When the system of equation is rectangular, the proof of the theorem is analogous to that for the nonsingular and quadratic system and we neglect it.

**Theorem 2** (Linear convergence)

Let \( F \) satisfies the assumptions A1-A4. Then the method
\[
x_{k+1} = x_k - \left\{ B_k \right\}^* \cdot \Phi \left( x_k \right),
\]
\[
B_{k+1} = B_k - \frac{\left\{ \Phi \left( x_{k+1} \right) - \Phi \left( x_k \right) - B_k s_k \right\} s_k^T}{s_k^T s_k}
\]

locally and Q-linearly converges to \( x^* \).

**Proof.**

To prove the Theorem we should prove the inequality (20) from Theorem 1.

Now we notice:
\[
\|B_{k+1} - \Phi'(x^*)\| 
\leq \|B_k - \Phi'(x^*)\| + \left\| \frac{\Phi(x_{k+1}) - \Phi(x_k) - B_k s_k}{s_k^T s_k} \right\| \|s_k\| 
\leq \|B_k - \Phi'(x^*)\| + \left\| \frac{\Phi(x_{k+1}) - \Phi(x_k) - B_k s_k}{s_k^T s_k} \right\| \|s_k\|
\]

\[
\leq \|B_k - \Phi'(x^*)\| + \|s_k\| \|s_k\| \left\| \frac{\Phi(x_{k+1}) - \Phi(x_k) - B_k s_k}{s_k^T s_k} \right\| + \left\| \frac{\Phi'(x^*) - B_k}{s_k^T s_k} \right\| \|s_k\| 
+ c_2 \left\| \frac{x_k - x^*}{s_k^T s_k} \right\| \|s_k\| \|s_k\| \left\| \frac{\Phi'(x^*) - B_k}{s_k^T s_k} \right\| + c_1 \left\| \frac{x_k - x^*}{s_k^T s_k} \right\| \|s_k\| \|s_k\| \left\| \frac{\Phi'(x^*) - B_k}{s_k^T s_k} \right\| + q_1 r_k + q_2 r_k,
\]

where \(c_1 > 0, c_2 > 0, q_1 > 0, q_2 > 0, r_k = \max\{\|x_{k+1} - x^*\|, \|x_k - x^*\|\}\). \qed

**Theorem 3** (Q-superlinear convergence)

Let \(F\) satisfies the assumtions A1-A4 and the sequence

\[
x_{k+1} = x_k - B_k^{-1} \cdot \Phi(x_k),
\]

\[
B_{k+1} = B_k - \frac{\Phi(x_{k+1}) - \Phi(x_k) - B_k s_k}{s_k^T s_k}
\]

linearly converges to \(x^*\). Then the sequence \(\{x_k\}\) Q-superlinearly converges to \(x^*\).

**Proof.**

Matrices \(B_k\) satisfy secant equation (15), so

\[
B_{k+1} = P_{L_k} B_k
\]

where

\[
L_k = \left\{ X : X s_k = y_k, \text{ where } y_k = \Phi'(x_{k+1}) - \Phi'(x_k) \right\}
\]

Denote

\[
H_k = H(x_k, x_{k+1}) = \int_0^1 \Phi'(x_k + t(x_{k+1} - x_k)) dt.
\]

We have \(H_k \in L_k\) [4].
From (21) and [3] it follows:

$$\|B_{k+1} - B_k\|^2 + \|B_{k+1} - H^2_k\|^2 = \|B_k - H_k\|^2,$$ for $i = 0, 1, 2, \ldots$.

By lemma 2 [5] we get $\sum_{k=1}^{\infty} \|B_{k+1} - B_k\|^2 < \infty$, thus we obtain

$$\|B_{k+1} - B_k\| \to 0.$$ This denotes that the method (13)-(17) is Q-superlinearly convergent [6], which ends the proof. □

4. Summary

The proposed method is Q-superlinearly convergent and easier to apply than the method (12), without calculation of $F''(x_k)$.

References