Newton-like method for singular 2-regular system of nonlinear equations

Stanisław Grzegórski*, Edyta Łukasik

Department of Computer Science, Lublin University of Technology, Nadbystrzycka 36b, 20-618 Lublin, Poland

Abstract

In this article the problem of solving a system of singular nonlinear equations will be discussed. The theory of local and Q-superlinear converegence for the nonlinear operators is developed.

1. Introduction

Let $F: D \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a nonlinear operator. The problem of solving a system of nonlinear equations consist in finding a solution $x^* \in D$ of the equation

$$F(x) = 0.$$  (1)

Definition 1

A linear operator $\Psi_2(h): \mathbb{R}^n \rightarrow \mathbb{R}^m$, $h \in \mathbb{R}^n$ is called 2-factoroperator, if

$$\Psi_2(h) = F'(x^*) + P_\perp F'(x^*)h,$$  (2)

where $P_\perp$ - denotes the orthogonal projection on $(\text{Im } F'(x))^\perp$ in $\mathbb{R}^n$ [1].

Definition 2

Operator $F$ is called 2-regular in $x^*$ on the element $h \in \mathbb{R}^n$, $h \neq 0$, if the operator $\Psi_2(h)$ has the property:

$$\text{Im } \Psi_2(h) = \mathbb{R}^m.$$  

* Corresponding author: e-mail address: grzeg@pluton.pol.lublin.pl
Definition 3

Operator $F$ is called 2-regular in $x^*$, if $F$ is 2-regular on the set $K_2(x^*)\{0\}$, where

$$K_2(x^*) = \text{Ker} F^+ (x^*) \cap \text{Ker}^2 P^+ F^- (x^*),$$

$$\text{Ker}^2 P^+ F^- (x^*) = \{ h \in R^n : P^+ F^- (x^*) h = 0 \}.$$  \hspace{1cm} (3)

We need the following assumption on $F$:

A1) completely degenerated in $x^*$:

$$\text{Im} F^- (x^*) = 0.$$ \hspace{1cm} (4)

A2) operator $F$ is 2-regular in $x^*$:

$$\text{Im} F^- (x^*) h = R^m \text{ for } h \in K_2(x^*), h \neq 0.$$ \hspace{1cm} (5)

A3)

$$\text{Ker} F^- (x^*) \neq \{0\}.$$ \hspace{1cm} (6)

If $F$ satisfies A1 in $x^*$, then

$$K_2(x^*) = \text{Ker}^2 F^- (x^*) = \{ h \in R^n : F^- (x^*) h = 0 \}.$$ \hspace{1cm} (7)

In [1] it was proved, that if $n=m$, then the sequence

$$x_{k+1} = x_k - \left\{ \hat{F}^+ (x_k) + P_k^+ F^- (x_k) h_k \right\}^{-1} \cdot \left\{ F(x_k) + P_k^+ F^- (x_k) h_k \right\},$$ \hspace{1cm} (8)

where

$P_k^+$ - denotes orthogonal projection on $\left( \text{Im} \hat{F}^+ (x_k) \right)^\perp$ in $R^n$,

$$h_k \in \text{Ker} \hat{F}^+ (x_k), \quad \| h_k \| = 1$$

converges Q-quadratically to $x^*$.

The matrices $\hat{F}^+ (x_k)$ obtained from $F^- (x_k)$ by replacing all elements, whose absolute values do not increase $\nu > 0$, by zero, where $\nu = \nu_k = \| F(x_k) \|^{(1-\alpha)/2}$, $0 < \alpha < 1$.

In the case $n = m+1$ the operator

$$\left\{ \hat{F}^+ (x_k) + P_k^+ F^- (x_k) h_k \right\}^{-1}$$

in method (8) is replaced by the operator

$$\left[ \hat{F}^+ (x_k) + P_k^+ F^- (x_k) h_k \right]^{-1}$$ \hspace{1cm} (9)

and then the method converges Q-linearly to the set of solutions [2].

Under the assumptions A1-A3, the system of equation (1) is undetermined $(n> m)$ and degenerated in $x^*$. 

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2. Extending of the system of equations

Now we construct the operator $\Phi : R^n \rightarrow R^{n-1}$ with the properties (4), (5) and such that $\Phi(x^*)=0$ [2].

Assume

A4) Let $F(x)=[f_1(x), f_2(x), ..., f_m(x)]^T$, $n>m$ is two continuously differentiable in some neighbourhood $U \subset R^n$ of the point $x^*$.

Denote:

$$H=\text{lin}\{h\} \quad \text{for} \quad h \in \text{Ker}^2 \bar{F}'(x^*), \ h \neq 0.$$  

$p = P_{H^\perp}$ denotes the orthogonal projection $R^n$ on $H^\perp$.

$$\frac{\partial f_i}{\partial x}(x) = P\left( f_i'(x) \right)^T \quad \text{for} \quad i=1,2,...,m.$$  

For each system of indices $i_1, i_2, ..., i_{n-m-1} \subset \{1, 2, ..., m\}$ and vectors $h_1, h_2, ..., h_{n-m-1} \subset R^n$ we define

$$\Phi(x) = \begin{bmatrix} F'(x) h \\ \varphi(x) \end{bmatrix}, \quad (10)$$

where

$$\varphi(x) : R^n \rightarrow R^r, \quad r=n-m-1,$$

$$\varphi(x) = PF'(x) \bar{p}, \quad \bar{p} \in [h_1, h_2, ..., h_r]^T,$$

$$\varphi(x) = M = \begin{bmatrix} \frac{\partial f_i}{\partial x}(x) h_1 \\ \frac{\partial f_i}{\partial x}(x) h_r \end{bmatrix}. \quad (11)$$

In [2] it was proved, that the sequence

$$x_{k+1} = x_k - \left[ \Phi'(x_k) \right]^\top \cdot \Phi(x_k), \quad k=0,1,2,... \quad (12)$$

quadratically converges to the solution of (1).

3. New method

We propose the Newton-like method, where the sequence $\{x_k\}$ is defined by:

$$x_{k+1} = x_k - \left( B_k \right)^\top \cdot \Phi(x_k). \quad (13)$$

The operator $\Phi'$ will be approximated by matrices $\{B_k\}$.

Let

$$s_k = x_{k+1} - x_k. \quad (14)$$

We propose matrices $B_k$ which satisfy the secant equation:

$$B_{k+1}s_k = \Phi(x_{k+1}) - \Phi(x_k) \quad \text{for} \quad k=0,1,2,... \quad (15)$$

For example, to obtain the sequence $\{B_k\}$ we can apply the Broyden method:
\[ B_{k+1} = B_k - \frac{r_k s_k^T}{s_k^T s_k} \quad \text{for } k=0,1,2,\ldots \] (16)

where
\[ r_k = \Phi(x_{k+1}) - \Phi(x_k) - B_k s_k. \] (17)

We will prove for this method:

**Q-linear convergence** to \( x^* \), i.e. there exists \( q \in (0,1) \) such that
\[ \|x_{k+1} - x^*\| \leq q \|x_k - x^*\| \quad \text{for } k = 0,1,2,\ldots \] (18)

and next **Q-superlinear convergence** to \( x^* \), i.e.:
\[ \lim_{k \to \infty} \frac{\|x_{k+1} - x^*\|}{\|x_k - x^*\|} = 0. \] (19)

We present the theorem which is an analogue of the Bounded Deterioration Theorem (Broyden, Dennis and More - [3]) for the Newton-like methods, when the operator \( F' (x^*) \) is nonsingular.

**Theorem 1** (The Bounded Deterioration Theorem)

Let \( F \) satisfies the assumptions A1-A4. If exist constants \( q_1 \geq 0 \) and \( q_2 \geq 0 \) such that matrices \( \{B_k\} \) satisfy the inequality:
\[ \left\| B_{k+1} - \Phi' (x^*) \right\| \leq (1 + q_1 r_k) \left\| B_k - \Phi' (x^*) \right\| + q_2 r_k, \] (20)

then there are constants \( \varepsilon > 0 \) i \( \delta > 0 \) such, that if
\[ \|x_0 - x^*\| \leq \varepsilon \quad \text{and} \quad \| B_0 - \Phi' (x^*) \| \leq \delta, \]

then the sequence
\[ x_{k+1} = x_k - B_k^* \Phi(x_k) \]

converges Q-linearly to \( x^* \).

When the system of equation is rectangular, the proof of the theorem is analogous to that for the nonsingular and quadratic system and we neglect it.

**Theorem 2** (Linear convergence)

Let \( F \) satisfies the assumptions A1-A4. Then the method
\[ x_{k+1} = x_k - \{ B_k \}^* \cdot \Phi(x_k), \]
\[ B_{k+1} = B_k - \frac{\{ \Phi(x_{k+1}) - \Phi(x_k) - B_k s_k \} s_k^T}{s_k^T s_k} \]

locally and Q-linearly converges to \( x^* \).

**Proof.**

To prove the theorem we should prove the inequality (20) from Theorem 1. Now we notice:
\[
\left\| B_{k+1} - \Phi'(x^*) \right\| = \left\| B_k - \frac{\Phi(x_{k+1}) - \Phi(x_k) - B_k s_k}{s_k^T s_k} s_k^T \right\| \leq B_k - \Phi'(x^*) \leq B_k - \Phi'(x^*) + \left\| \frac{\Phi(x_{k+1}) - \Phi(x_k) - B_k s_k}{s_k^T s_k} s_k^T \right\| \leq B_k - \Phi'(x^*) + \left\| \frac{\Phi(x_{k+1}) - \Phi(x_k) - B_k s_k}{s_k^T s_k} s_k^T \right\| + \left\| \frac{\Phi(x^*) - B_k s_k}{s_k^T s_k} \right\| \leq \Phi'(x^*) - B_k \left(1 + q_1 r_k \right) + c_1 \frac{x_{k+1}^* - x^*}{s_k^T s_k} + c_2 \frac{x_k - x^*}{s_k^T s_k} \leq \Phi'(x^*) - B_k \left(1 + q_1 r_k \right) + q_2 r_k,
\]

where \( c_1 > 0, c_2 > 0, q_1 > 0, q_2 > 0, r_k = \max \{ x_{k+1} - x^*, ||x_k - x^*|| \} \).

**Theorem 3** (Q-superlinear convergence)

Let \( F \) satisfies the assumptions A1-A4 and the sequence
\[
x_{k+1} = x_k - B_k^{-1} \Phi(x_k),
\]
\[
B_{k+1} = B_k - \frac{\Phi(x_{k+1}) - \Phi(x_k) - B_k s_k}{s_k^T s_k} s_k^T
\]
linearly converges to \( x^* \). Then the sequence \( \{x_k\} \) Q-superlinearly converges to \( x^* \).

**Proof.**

Matrices \( B_k \) satisfy secant equation (15), so
\[
B_{k+1} = P_{L_k} B_k
\]
where
\[
L_k = \left\{ X : X s_k = y_k, \text{ where } y_k = \Phi'(x_{k+1}) - \Phi'(x_k) \right\}
\]
(22)

Denote
\[
H_k = H(x_k, x_{k+1}) = \int_0^1 \Phi'(x_k + t(x_{k+1} - x_k)) dt.
\]
We have \( H_k \in L_k \) [4].
From (21) and [3] it follows:
\[ \left\| B_{k+1} - B_k \right\|^2 + \left\| B_{k+1} - H_k \right\|^2 = \left\| B_k - H_k \right\|^2, \quad \text{for } i = 0, 1, 2, \ldots \]

By lemma 2 [5] we get \( \sum_{k=1}^{\infty} \| B_{k+1} - B_k \|^2 < \infty \), thus we obtain
\[ \| B_{k+1} - B_k \| \to 0. \]

This denotes that the method (13)-(17) is Q-superlinearly convergent [6], which ends the proof. □

4. Summary

The proposed method is Q-superlinearly convergent and easier to apply than the method (12), without calculation of \( F^{(n)}(x_k) \).

References