Superquadratically convergent methods for minimization functions

Stanisław M. Grzegórski*, Edyta Łukasik

Department of Computer Science, Lublin University of Technology, Nadbystrzycka 36b, 20-618 Lublin, Poland

Abstract

In the paper locally superquadratically convergent methods for minimization functions are considered. Threefold symmetric approximations to partial derivatives of the third order are constructed.

1. Introduction

Let \( f : D \subset \mathbb{R}^n \rightarrow \mathbb{R}, \ f \in C^3(D), \ D \) - open set. We want to find \( x^* \in D \) such that \( \nabla f(x^*) = 0 \). For a given \( x_0 \in D \) the Newton method defines the sequence \( \{x_k\} \) in the following way

\[
\nabla^2 f(x_k) s_k = -\nabla f(x_k), \ x_{k+1} = x_k + s_k, \ k = 0,1,2, \ldots 
\]

If the matrix \( \nabla^2 f(x^*) \) is nonsingular then Newton method is locally quadratically convergent to \( x^* \), i.e. there exist \( c > 0 \) and \( \varepsilon > 0 \) such that, if \( \|x^* - x_0\| < \varepsilon \), then

\[
\|x_{k+1} - x^*\| \leq c \|x_k - x^*\|^2. \tag{2} 
\]

To assure global convergence of the method one should consider a sequence

\[
x_{k+1} = x_k + t_k s_k, \ t_k \in \mathbb{R}, \ k = 0,1,2, \ldots 
\]

and the parameter \( t_k \) should satisfy the global convergence conditions. If the matrix \( \nabla^2 f(x^*) \) is singular, then the Newton method is divergent or at most linearly convergent to \( x^* \). To assure a great speed of convergence for singular problems one applies the method of the third rate of convergence: for a given \( x_0 \in D \) the sequence \( \{x_k\} \) is defined as

\[
x_{k+1} = x_k + s_k, \ k = 0,1,2, \ldots 
\]

* Corresponding author: e-mail address: s.grzegorski@pollub.pl
where \( s_k \) is the solution of the system of quadratic equations

\[
\nabla f(x_k) + \nabla^2 f(x_k)s_k + \frac{1}{2} (\nabla^3 f(x_k)s_k,s_k) = 0. \tag{5}
\]

When the calculation of the operator \( \nabla^3 f(x_k) \) is too expensive or is not attainable for computation then we propose a new class of the methods which are locally superquadratically convergent to \( x^* \), i.e.

\[
\lim_{k \to \infty} \frac{\|x_{k+1} - x^*\|}{\|x_k - x^*\|^2} = 0. \tag{6}
\]

Let \( B_k = (B_k^1, B_k^2, \cdots, B_k^n) \), \( B_k^i \in \mathbb{R}^{n \times n} \), \( B_k^i = (B_k^i)^T \), \( i = 1, 2, \cdots, n \). The sequence \( \{x_k\} \) is defined by (1.4) and

\[
\nabla f(x_k) + \nabla^2 f(x_k)s_k + \frac{1}{2} (B_k s_k, s_k) = 0. \tag{7}
\]

If the problem \( \min_{x \in D} f(x) \) is regularly singular at \( x^* \), i.e.

\[
det(\nabla^2 f(x^*)) = 0, \quad \text{and} \quad \|\nabla f(x)\| \geq c \|x - x^*\|^2, \quad c > 0, \quad x \in D, \tag{8}
\]

then the sequence \( \{x_k\} \) defined by (1.4) and (1.7) is locally superlinearly convergent to \( x^* \), if the operators \( B_k \) are constructed in an adequate way. In this paper such algorithms are given.

## 2. The BFGS method

The DFP (Davidon [1], Fletcher and Powell [2]) method is very well known as the method of approximation to the Hessian \( \nabla^2 f(x_k) \). This formula has the form

\[
B_{k+1} = B_k - \frac{B_k s_k s_k^T B_k}{s_k^T B_k s_k} + \frac{y_k y_k^T}{y_k^T s_k}, \quad y_k = \nabla f(x_{k+1}) - \nabla f(x_k). \tag{9}
\]

The DFP formula, for nonsingular problems, guarantees local superlinear convergence of the method

\[
x_{k+1} = x_k + s_k, \quad B_k s_k = -\nabla f(x_k), \quad k = 0, 1, 2, \cdots \tag{10}
\]

We may use the DFP formula to approximate the operator \( \nabla^3 f(x_k) \). Namely, let \( B_k = (B_k^1, B_k^2, \cdots, B_k^n) \), \( B_k^i \in \mathbb{R}^{n \times n} \), \( B_k^i = (B_k^i)^T \), \( i = 1, 2, \cdots, n \) (11) and let \( \nabla_i^2 f(x_k) \) denote \( i \)-th column of the matrix \( \nabla^2 f(x_k) \), \( y_k^i = \nabla_i^2 f(x_{k+1}) - \nabla_i^2 f(x_k) \). Then

\[
B_{k+1}^i = B_k^i - \frac{B_k s_k s_k^T B_k^i}{s_k^T B_k^i s_k} + \frac{y_k^i y_k^i^T}{y_k^i s_k^T s_k}, \quad i = 1, 2, \cdots, n. \tag{12}
\]
Note that the operators $B_{k+1}$ satisfy the equation
\[ B_{k+1}s_k = \nabla^2 f(x_{k+1}) - \nabla^2 f(x_k), \; k = 0, 1, 2, \ldots \]  
(13)

Now, from the general theory for the systems of nonlinear equations [3], [4] local superquadratic convergence of the method results (4), (7) with the update (12). On the other hand, the DFP updates do not satisfy the following properties of the partial derivatives
\[ \frac{\partial^3 f(x)}{\partial x_i \partial x_j \partial x_k} = \frac{\partial^3 f(x)}{\partial x_i \partial x_j \partial x_j} = \cdots = \frac{\partial^3 f(x)}{\partial x_i \partial x_j \partial x_l} , \; i, j, l = 1, 2, \ldots, n . \]
(14)

In this case, we say the operator $\nabla^3 f(x)$ is threefold symmetric (T-symmetric).

It is worth remarking that the operator $\nabla^3 f(x)$ has only $P(n) = \frac{1}{6} n(n + 1)(n + 2)$ different elements and the DFP approximations have $Q(n) = \frac{1}{2} n^2(n + 1)$ different elements, which means that the BFGS formula is not adequate for approximation to $\nabla^3 f(x)$. In the next Section we give a new formula for the update of $B_k$ and $B_k$ will be T-symmetric.

### 3. New approximation to $\nabla^3 f(x)$

The approximation $B_k$ to $\nabla^3 f(x)$ satisfies secant equation (13) and operators $B_k$ should be threefold symmetric. If we take
\[ B_{k+1} = B_k + E , \]
(15)
then
\[ ES_k = \nabla^2 f(x_{k+1}) - \nabla^2 f(x_k) - Bks_k = Y . \]
(16)

In that case we have to solve the problem
\[ \min \|E\|^2, \; \|E\|^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} E_{ijk}^2 \]
(17)
under constraints
\[ ES = Y , \; s \in R^n, \; Y \in R^{n \times n}, \; Y = Y^T \]
(18)
and
\[ E_{ijk} = E_{skj} = E_{jik} = E_{jki} = E_{kij} = E_{kji} , \; i, j, k = 1, 2, \ldots, n . \]
(19)

**Remark.** If we take another norm of the operator $E$, then we get another formula for the update $B_k$.

Let $\Lambda \in R^{n \times n}$. In our case the lagrangian has the form
\[ L(E, \Lambda) = \frac{1}{2} \|E\|^2 + \sum_{i=1}^{n} \sum_{j=1}^{n} \Lambda_{ij} (\sum_{k=1}^{n} E_{ijk}s_k - Y_{ij}) . \]
(20)
From this we have
\[ \frac{\partial L(E, \Lambda)}{\partial E_{pq}} = E_{pq} + \Lambda_{pq} s_r = 0. \] (21)

The fact \( E_{pqr} = E_{qpr} \) implies \( \Lambda = \Lambda^T \). Since the operator \( E \) is threefold symmetric then equation (21) may be written as
\[ E_{pqr} = -\frac{1}{3}(\Lambda_{pq}s_r + \Lambda_{pr}s_q + \Lambda_{qr}s_p) \quad \text{for} \quad 1 \leq p \leq q \leq r \leq n. \] (22)

Now, the equation \( E s = Y \) has the form
\[ \sum_{i=1}^{n}(\Lambda_{ij}s_i + \Lambda_{ji}s_j + \Lambda_{jj}s_j)s_i = -3Y_i \quad 1 \leq i \leq j \leq n \] (23)
or is in the matrix form
\[ \Lambda\|s\|^2 + \Lambda s s^T + s s^T \Lambda = -3Y. \] (24)

Therefore
\[ s^T \Lambda s = \frac{1}{\|s\|^2} s^T Y s, \ u = \Lambda s = \frac{1}{2\|s\|^2}(-3Y + s^T Y s). \] (25)

Finally
\[ \Lambda = \frac{1}{\|s\|^2} (3Y + u u^T). \] (26)

To calculate the new threefold symmetric update \( B_{k+1} = B_k + E \) we use the formulae (22), (25) and (26).

4. Remarks on the local superquadratic convergence of the method

At first we describe the proposed algorithm:

a) Let \( x_0 \in R^n \) and \( B_0 = (B_0^1, B_0^2, \ldots, B_0^n) \) - threefold operator be given. Let \( k = 0 \),
b) Solve, using for example the Newton method, the system of quadratic equations
\[ \nabla f(x_k) + \nabla^2 f(x_k)s_k + \frac{1}{2}(B_k s_k, s_k) = 0, \]
c) Calculate \( x_{k+1} = x_k + s_k, \ \nabla f(x_{k+1}), \ \nabla^2 f(x_{k+1}), \)
d) Update the operator \( B_k \) using the formulae from Section 3,
e) If a stop criterion is not satisfied, then \( k := k + 1 \) and return to point b.
To explain a character of convergence of the method we introduce some notations. Let
Superquadratically convergent methods for minimization functions

\[ H_k = \int_{0}^{1} \nabla^3 f(x_k + t(x_{k+1} - x_k)) dt, \]  
(27)

\[ L_k = \{ X \in \mathbb{R}^{n \times n \times n} : Xs_k = \nabla^2 f(x_{k+1}) - \nabla^2 f(x_k) \}, \]  
(28)

\[ Q = \{ X \in \mathbb{R}^{n \times n \times n} : X \text{ is threefold symmetric operator} \}. \]  
(29)

The set \( Q \) is linear subspace in \( \mathbb{R}^{n \times n \times n} \) and \( H_k \in Q \). Applying Theorem 3.2.7 [5] we have

\[ H_k s_k = \nabla^2 f(x_{k+1}) - \nabla^2 f(x_k), \]  
(30)

which means that \( Q \cap L_k \) is a nonempty linear set. The proposed norm is generated by inner product, so the operator \( B_{k+1} \) is defined as the orthogonal projection of the operator \( B_k \) onto the set \( Q \cap L_k \), and from Pitagoras Theorem (see [3]) we get

\[ \|B_{k+1} - H_k\|^2 + \|B_k - B_{k+1}\|^2 = \|B_k - H_k\|^2 \quad k = 0,1,2,\ldots \]  
(31)

The inequality \( \|B_{k+1} - H_k\| \leq \|B_k - H_k\| \) implies local linear convergence of the sequence \( \{x_k\} \). From equations (31) it results additionally

\[ \sum_{k=0}^{\infty} \|B_{k+1} - B_k\|^2 < \infty. \]  
(32)

The last inequality and the secant equation (13) assure local superquadratic convergence of the proposed algorithm [4].

References