Superquadratically convergent methods for minimization functions

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Abstract

In the paper locally superquadratically convergent methods for minimization functions are considered. Threefold symmetric approximations to partial derivatives of the third order are constructed.

1. Introduction

Let \( f : D \subset \mathbb{R}^n \to \mathbb{R}, f \in C^3(D) \), \( D \) - open set. We want to find \( x^* \in D \) such that \( \nabla f(x^*) = 0 \). For a given \( x_0 \in D \) the Newton method defines the sequence \( \{x_k\} \) in the following way

\[
\nabla^2 f(x_k) s_k = -\nabla f(x_k), \quad x_{k+1} = x_k + s_k, \quad k = 0,1,2,\ldots
\]

If the matrix \( \nabla^2 f(x^*) \) is nonsingular then Newton method is locally quadratically convergent to \( x^* \), i.e. there exist \( c > 0 \) and \( \varepsilon > 0 \) such that, if \( \|x^* - x_0\| < \varepsilon \), then

\[
\|x_{k+1} - x^*\| \leq c \|x_k - x^*\|^2.
\]

To assure global convergence of the method one should consider a sequence

\[
x_{k+1} = x_k + t_k s_k, \quad t_k \in \mathbb{R}, \quad k = 0,1,2,\ldots
\]

and the parameter \( t_k \) should satisfy the global convergence conditions. If the matrix \( \nabla^2 f(x^*) \) is singular, then the Newton method is divergent or at most linearly convergent to \( x^* \). To assure a great speed of convergence for singular problems one applies the method of the third rate of convergence: for a given \( x_0 \in D \) the sequence \( \{x_k\} \) is defined as

\[
x_{k+1} = x_k + s_k, \quad k = 0,1,2,\ldots
\]

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where $s_k$ is the solution of the system of quadratic equations

$$
\nabla f(x_k) + \nabla^2 f(x_k)s_k + \frac{1}{2} (\nabla^3 f(x_k)s_k, s_k) = 0.
$$

(5)

When the calculation of the operator $\nabla^3 f(x_k)$ is too expensive or is not
attainable for computation then we propose a new class of the methods which
are locally superquadratically convergent to $x^*$, i.e.

$$
\lim_{k \to \infty} \left\| x_{k+1} - x^* \right\| = 0.
$$

(6)

Let $B_k = (B^1_k, B^2_k, \ldots, B^n_k)$, $B^i_k \in \mathbb{R}^{n \times n}$, $B^i_k = (B_k^i)^T$, $i = 1, 2, \ldots, n$. The
sequence $\{x_k\}$ is defined by (1.4) and

$$
\nabla f(x_k) + \nabla^2 f(x_k)s_k + \frac{1}{2} (B_k s_k, s_k) = 0.
$$

(7)

If the problem $\min_{x \in D} f(x)$ is regularly singular at $x^*$, i.e.

$$
\det(\nabla^2 f(x^*)) = 0, \text{ and } \left\| \nabla f(x) \right\| \geq c \left\| x - x^* \right\|^2, \ c > 0, \ x \in D,
$$

(8)

then the sequence $\{x_k\}$ defined by (1.4) and (1.7) is locally superlinearly
convergent to $x^*$, if the operators $B_k$ are constructed in an adequate way. In this
paper such algorithms are given.

2. The BFGS method

The DFP (Davidon [1], Fletcher and Powell [2]) method is very well known
as the method of approximation to the Hessian $\nabla^2 f(x_k)$. This formula has the
form

$$
B_{k+1} = B_k - \frac{B_k s_k s_k^T B_k}{s_k^T B_k s_k} + \frac{y_k y_k^T}{y_k^T s_k}, \ y_k = \nabla f(x_{k+1}) - \nabla f(x_k).
$$

(9)

The DFP formula, for nonsingular problems, guarantees local superlinear
convergence of the method

$$
x_{k+1} = x_k + s_k, \ B_{k+1} s_k = -\nabla f(x_k), \ k = 0, 1, 2, \ldots
$$

(10)

We may use the DFP formula to approximate the operator $\nabla^3 f(x_k)$. Namely, let

$$
B_k = (B^1_k, B^2_k, \ldots, B^n_k), \ B^i_k \in \mathbb{R}^{n \times n}, \ B^i_k = (B_k^i)^T, \ i = 1, 2, \ldots, n
$$

(11)

and let $\nabla^2_i f(x_k)$ denote $i$-th column of the matrix $\nabla^2 f(x_k)$,

$$
y'_k = \nabla^2_i f(x_{k+1}) - \nabla^2_i f(x_k). \ \text{Then}
$$

$$
B'_{k+1} = B'_k - \frac{B'_k s_k s_k^T B'_k}{s_k^T B'_k s_k} + \frac{y'_k (y'_k)^T}{(y'_k)^T s_k}, \ i = 1, 2, \ldots, n.
$$

(12)
Note that the operators $B_{k+1}$ satisfy the equation

$$B_{k+1}s_k = \nabla^2 f(x_{k+1}) - \nabla^2 f(x_k), \quad k = 0, 1, 2, \ldots \tag{13}$$

Now, from the general theory for the systems of nonlinear equations [3], [4] local superquadratic convergence of the method results (4), (7) with the update (12). On the other hand, the DFP updates do not satisfy the following properties of the partial derivatives

$$\frac{\partial^3 f(x)}{\partial x_i \partial x_j \partial x_l} = \frac{\partial^3 f(x)}{\partial x_i \partial x_j \partial x_l} = \cdots = \frac{\partial^3 f(x)}{\partial x_i \partial x_j \partial x_l}, \quad i, j, l = 1, 2, \ldots, n. \tag{14}$$

In this case, we say the operator $\nabla^3 f(x)$ is threefold symmetric (T-symmetric).

It is worth remarking that the operator $\nabla^3 f(x)$ has only $P(n) = \frac{1}{6}n(n+1)(n+2)$ different elements and the DFP aproximations have $Q(n) = \frac{1}{2}n^2(n+1)$ different elements, which means that the BFGS formula is not adequate for approximation to $\nabla^3 f(x)$. In the next Section we give a new formula for the update of $B_k$ and $B_k$ will be T-symmetric.

3. New approximation to $\nabla^3 f(x)$

The approximation $B_k$ to $\nabla^3 f(x)$ satisfies secant equation (13) and operators $B_k$ should be threefold symmetric. If we take

$$B_{k+1} = B_k + E, \tag{15}$$

then

$$E s_k = \nabla^2 f(x_{k+1}) - \nabla^2 f(x_k) - B_k s_k = Y. \tag{16}$$

In that case we have to solve the problem

$$\min \|E\|^2, \quad \|E\|^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} E_{ijk}^2 \tag{17}$$

under constraints

$$Es = Y, \quad s \in R^n, \quad Y \in R^{n \times n}, \quad Y = Y^T \tag{18}$$

and

$$E_{ijk} = E_{skj} = E_{jik} = E_{jki} = E_{kij} = E_{kji}, \quad i, j, k = 1, 2, \ldots, n. \tag{19}$$

Remark. If we take another norm of the operator $E$, then we get another formula for the update $B_k$.

Let $\Lambda \in R^{n \times n}$. In our case the lagrangian has the form

$$L(E, \Lambda) = \frac{1}{2}\|E\|^2 + \sum_{i=1}^{n} \sum_{j=1}^{n} \Lambda_{ij} (\sum_{k=1}^{n} E_{ijk}s_k - Y_{ij}). \tag{20}$$
From this we have
\[ \frac{\partial L(E, \Lambda)}{\partial E_{pq}} = E_{pqr} + \Lambda_{pr}s_r = 0. \] (21)
The fact \( E_{pqr} = E_{qpr} \) implies \( \Lambda = \Lambda^T \). Since the operator \( E \) is threefold symmetric then equation (21) may be written as
\[ E_{pqr} = -\frac{1}{3}(\Lambda_{pq}s_r + \Lambda_{pr}s_q + \Lambda_{qr}s_p) \quad \text{for} \quad 1 \leq p \leq q \leq r \leq n. \] (22)
Now, the equation \( Es = Y \) has the form
\[ \sum_{i=1}^{n} (\Lambda_{ij}s_i + \Lambda_{ji}s_j + \Lambda_{ji}s_i)s_i = -3Y_{ij} \quad 1 \leq i \leq j \leq n \] (23)
or is in the matrix form
\[ \Lambda\|s\|^2 + \Lambda ss^T + ss^T \Lambda = -3Y. \] (24)
Therefore
\[ s^T \Lambda s = -\frac{1}{\|s\|^2} s^T Ys, \quad u = \Lambda s = \frac{1}{2\|s\|^2} (-3Ys + s^T Ys). \] (25)
Finally
\[ \Lambda = -\frac{1}{\|s\|^2} (3Y + us^T + su^T). \] (26)
To calculate the new threefold symmetric update \( B_{k+1} = B_k + E \) we use the formulae (22), (25) and (26).

4. Remarks on the local superquadratic convergence of the method
At first we describe the proposed algorithm:
  a) Let \( x_0 \in \mathbb{R}^n \) and \( \mathbf{B}_0 = (B_0^1, B_0^2, \ldots, B_0^n) \) - threefold operator be given. Let \( k = 0 \),
  b) Solve, using for example the Newton method, the system of quadratic equations
  \[ \nabla f(x_k) + \nabla^2 f(x_k)s_k + \frac{1}{2}(B_k s_k, s_k) = 0, \]
  c) Calculate \( x_{k+1} = x_k + s_k, \ \nabla f(x_{k+1}), \ \nabla^2 f(x_{k+1}), \)
  d) Update the operator \( B_k \) using the formulae from Section 3,
  e) If a stop criterion is not satisfied, then \( k := k + 1 \) and return to point b.
To explain a character of convergence of the method we introduce some notations. Let
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\[ H_k = \frac{1}{0} \nabla^3 f(x_k + t(x_{k+1} - x_k)) dt, \]  
\[ L_k = \{ X \in R^{n \times n \times n} : Xs_k = \nabla^2 f(x_{k+1}) - \nabla^2 f(x_k) \}, \]  
\[ Q = \{ X \in R^{n \times n \times n} : X \text{ is threefold symmetric operator} \}. \]

The set \( Q \) is linear subspace in \( R^{n \times n \times n} \) and \( H_k \in Q \). Applying Theorem 3.2.7 [5] we have

\[ H_k s_k = \nabla^2 f(x_{k+1}) - \nabla^2 f(x_k), \]

which means that \( Q \cap L_k \) is a nonempty linear set. The proposed norm is generated by inner product, so the operator \( B_{k+1} \) is defined as the orthogonal projection of the operator \( B_k \) onto the set \( Q \cap L_k \), and from Pitagoras Theorem (see [3]) we get

\[ \frac{1}{2} \sum_{k=0}^{\infty} \| B_{k+1} - B_k \|^2 < \infty. \]  

The references and the secant equation (13) assure local superquadratic convergence of the proposed algorithm [4].

References