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MOHAMED AMINE HACHANI

On the convergence of certain integrals

Abstract. Let $M(r) := \max_{|z|=r} |f(z)|$, where $f(z)$ is an entire function. Also let $\alpha > 0$ and $\beta > 1$. We discuss the behavior of the integrand $M(r)\mathrm{e}^{-\alpha(\log r)^\beta}$ as $r \to \infty$ if $\int_1^\infty M(r)\mathrm{e}^{-\alpha(\log r)^\beta}\mathrm{d}r$ is convergent.

1. Convergence of integrals vis-à-vis convergence of series. There is one fundamental property of a convergent infinite series in regard to which the analogy between infinite series and infinite integrals breaks down. If $\sum_{n=1}^\infty \theta(n)$ is convergent, then $\theta(n) \to 0$ as $n \to \infty$; but it is **not** always true, even when $\theta(r)$ is always positive, that if $\int_a^\infty \theta(r)\,\mathrm{d}r$, $a > 0$, is convergent, then $\theta(r) \to 0$ as $r \to \infty$. As a counterexample, we can consider the function given by

$$\theta_p(r) := \sum_{n=0}^{\infty} \{f_n(r,p) + g_n(r,p)\} \qquad (p > 1),$$

where the functions $f_n(r,p)$ and $g_n(r,p)$ of the real variable r are defined by

$$f_n(r,p) := \{(n+1)^p r + 1 - n(n+1)^p\}\, \mathbb{1}_{\left[n-\frac{1}{(n+1)^p};n\right[}(r)$$

and

$$g_n(r,p) := \{-(n+1)^p r + 1 + n(n+1)^p\}\, \mathbb{1}_{\left[n;n+\frac{1}{(n+1)^p}\right]}(r).$$

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Then, for every positive x,

$$\int_0^x \theta_p(r)\mathrm{d}r \leq \sum_{n=0}^{\infty} \frac{1}{(n+1)^p} < \infty,$$

while $\theta_p(r)$ does not tend to 0 as $r \to \infty$.

It is however true that if $\int_a^\infty \theta(r)\,\mathrm{d}r$ converges and $\theta(r)$ is non-negative, then

$$\liminf_{r\to\infty} r\,(\log r)(\log\log r)\cdots(\ell_k r)\,\theta(r) = 0\,,$$

where $\ell_k r$ is the k-th iterate of $\log r$. If this was not true, then there would exist positive numbers c and R_0 such that for all $R > R_0$, we would have

$$\int_R^{\mathrm{e}^R} \theta(r)\,\mathrm{d}r > \int_R^{\mathrm{e}^R} \frac{c}{r\,(\log r)(\log\log r)\cdots(\ell_k r)}\,\mathrm{d}r = c\,\left(\ell_k R - \ell_{k+1} R\right)$$

and then $\int_R^{\mathrm{e}^R} \theta(r)\,\mathrm{d}r$ could not be made arbitrarily small by taking R sufficiently large ([2, p. 376]), contradicting the convergence of the integral $\int_a^\infty \theta(r)\,\mathrm{d}r$. On the other hand, it is well known that if $\theta(r)$ is positive and non-increasing, then $\int_a^\infty \theta(r)\,\mathrm{d}r$ can converge only if $r\,\theta(r) \to 0$ as $r \to \infty$. The same conclusion can be drawn if $\theta(r)$ is the product of a monotonic function $\varphi(r)$ and a non-negative function $L(r)$ which is continuous and $L(cr) \sim L(r)$ as $r \to \infty$ (i.e. $\lim_{r\to+\infty} \frac{L(cr)}{L(r)} = 1$). This can be explained as follows. Let ε be any given positive number. Then for all sufficiently large values of u, we have

$$\begin{aligned}\varepsilon &> \left|\int_u^{2u} \varphi(r)\,L(r)\,\mathrm{d}r\right| \geq \min\left\{|\varphi(u)|\,,\,|\varphi(2u)|\right\}\int_u^{2u} L(r)\,\mathrm{d}r \\ &= |\varphi(2u)|\int_u^{2u} L(r)\,\mathrm{d}r,\end{aligned}$$

say. That $u\theta(u) \to 0$ as $u \to \infty$, now follows from the fact (see Lemma 1.1 below) that

$$\int_a^u L(r)\,\mathrm{d}r \sim uL(u)\,.$$

Lemma 1.1 (see [4, Lemma 4]). *The condition*

$$\varphi_1(t) = \int_1^t \varphi(u)\mathrm{d}u \sim t\varphi(t)$$

is equivalent to

$$\varphi(kt) \underset{t\to\infty}{\sim} \varphi(t)$$

for every fixed positive k.

2. A special kind of integrands. Let $M(r) := \max_{|z|=r} |f(z)|$, where $f(z)$ is an entire function. In his work on Carlson's theorem ([1, Chapter 9]) for entire functions of exponential type, Rahman ([7, Theorem 7]) had a situation where the integral $\int_1^\infty r^{2Q} M(r)\,\mathrm{e}^{-\pi r}\,\mathrm{d}r$ was convergent and he wanted to know the behavior of $M(r)$ for large values of r. He noted ([7, Lemma 6]) that $r^{2Q}M(r)\,\mathrm{e}^{-\pi r} \to 0$ as $r \to \infty$. In order to prove it he does not require anything more than the fact that $M(r)$ is a non-decreasing function of r. However, $M(r)$ is not just a non-decreasing function of r but also $\log M(r)$ is a downward convex function of $\log r$. Thus $r^{2Q}M(r) = \mathrm{o}\,(\mathrm{e}^{\pi r})$ was not expected to be all that the convergence of $\int_1^\infty r^{2Q} M(r)\,\mathrm{e}^{-\pi r}\,\mathrm{d}r$ would imply. Recently, Qazi [5] has proved the following stronger result, which is "essentially" best possible.

Theorem 2.1. *Let $M(r) := \max_{|z|=r} |f(z)|$, where f is an entire function and suppose that $\int_0^\infty r^\alpha\, M(r)\,\mathrm{e}^{-\beta r}\,\mathrm{d}r < \infty$ for some $\alpha > 0$ and some $\beta > 0$. Then $\sqrt{r}\cdot r^\alpha M(r)\,\mathrm{e}^{-\beta r} = \mathrm{O}(1)$ as $r \to \infty$.*

3. The main result. An entire function f is a polynomial if and only if there exists a positive number k such that $M(r) := \max_{|z|=r} |f(z)| = \mathrm{O}(r^k)$ as $r \to \infty$. The degree n of f is the infimum of all such numbers k. In this case, we have

$$\lim_{r\to\infty} \frac{\log M(r)}{\log r} = n.$$

If f is a transcendental entire function, then (see a remark following Theorem 3.1)

$$\frac{\log M(r)}{\log r} \longrightarrow \infty \quad \text{as} \quad r \to \infty\,;$$

however, $M(r)\,\mathrm{e}^{-\alpha(\log r)^\beta}$ may tend to zero as $r \to \infty$ for some $\alpha > 0$ and some $\beta > 1$. This can happen if f is an entire function of order 0, that is, if

$$\limsup_{r\to\infty} \frac{\log\log M(r)}{\log r} = 0\,.$$

In connection with Theorem 2.1, one may then ask the following question: *What can we say about the behavior of $M(r)$ as $r \to \infty$ if f is an entire function such that $\int_1^\infty M(r)\,\mathrm{e}^{-\alpha(\log r)^\beta}\,\mathrm{d}r$ converges for some $\alpha > 0$ and some $\beta > 1$?*

We give an answer to this question. The proof of Theorem 2.1 as given by Qazi [5] is based on the use of the well-known Stirling's formula for Euler's Gamma function. This was somehow natural because of the integrand in $\int_0^\infty r^\alpha\, M(r)\,\mathrm{e}^{-\beta r}\,\mathrm{d}r$ having $\mathrm{e}^{-\beta r}$ as a factor. Since the integrand does not anymore have such a factor, the use of Stirling's formula is more or less out of the question. So, we have to use some other ideas. In addition to Stirling's formula, Qazi's proof of Theorem 2.1 uses Hadamard's three-circles theorem. That remains available to us and we have tried to use it as efficiently as we could.

Hadamard's three-circles theorem [8, p. 172] can be stated as follows:

Theorem 3.1. *Let $f(z)$ be an analytic function, regular for $r_1 \le |z| \le r_3$. Furthermore, let $r_1 < r_2 < r_3$, and let M_1, M_2, M_3 be the maxima of $|f(z)|$ on the three circles $|z| = r_1, r_2, r_3$, respectively. Then*

$$M_2^{\log(r_3/r_1)} \le M_1^{\log(r_3/r_2)}\, M_3^{\log(r_2/r_1)}\,. \tag{3.1}$$

Since we may write (3.1) in the form

$$\log M(r_2) \le \frac{\log r_3 - \log r_2}{\log r_3 - \log r_1} \log M(r_1) + \frac{\log r_2 - \log r_1}{\log r_3 - \log r_1} \log M(r_3), \tag{3.2}$$

Hadamard's three-circles theorem may be interpreted by saying that $\log M(r)$ is a convex function of $\log r$. If f is a transcendental entire function, then inequality (3.2) leads to the existence of a positive number r_0 such that $r \mapsto \frac{\log M(r)}{\log r}$ is a strictly increasing and unbounded function of r, for $r \ge r_0$.

Now we can state our theorem:

Theorem 3.2. *Let $M(r) := \max_{|z|=r} |f(z)|$, where f is an entire function and suppose that $\int_1^\infty M(r)\, \mathrm{e}^{-\alpha\,(\log r)^\beta}\, \mathrm{d}r < \infty$ for some $\alpha > 0$ and some $\beta > 1$. Then, for any $\varepsilon > 0$,*

$$\lim_{r\to\infty} r\,(\log r)^{-\gamma-\varepsilon} \cdot M(r)\, \mathrm{e}^{-\alpha\,(\log r)^\beta} = 0\,,$$

where $\gamma := \max\{0\,,\,(\beta-2)/2\}$.

4. Proof of Theorem 3.2. We present the proof in several steps.
Step I. First we prove that

$$\frac{R}{(\log R)^{\beta-1}}\, M(R)\, \mathrm{e}^{-\alpha(\log R)^\beta} \to 0 \ \text{ as } \ R \to \infty\,. \tag{4.1}$$

Take any $\varepsilon > 0$ and note that

$$\frac{M(r)}{(\log r)^{\beta-1}(1/r)}$$

is an increasing function of r for all large r. Hence, if R is large enough, then

$$\frac{R\,M(R)}{\alpha\beta(\log R)^{\beta-1}} \int_R^{R^2} \alpha\,\beta\,(\log r)^{\beta-1} \left(\frac{1}{r}\right)\, \mathrm{e}^{-\alpha\,(\log r)^\beta}\, \mathrm{d}r$$
$$\le \int_R^{R^2} M(r)\, \mathrm{e}^{-\alpha\,(\log r)^\beta}\, \mathrm{d}r < \varepsilon\,,$$

that is,

$$\frac{R}{(\log R)^{\beta-1}} M(R) \left(\mathrm{e}^{-\alpha\,(\log R)^\beta} - \mathrm{e}^{-\alpha\,(2\log R)^\beta} \right) < \alpha\beta\varepsilon\,,$$

which implies (4.1).

Step II. Next, we prove that for all large r,

$$M(S)\,\mathrm{e}^{-\alpha\,(\log S)^\beta} < \frac{(\log S)^\gamma}{S} \quad \text{for some } \; S = S(r) \in \left(r\,,\, r + \frac{r}{(\log r)^\gamma} \right). \tag{4.2}$$

If this was not true, then for all $t \in (r\,,\, r + r/(\log r)^\gamma)$, which in the case where $1 < \beta \le 2$ means "for all $t \in (r\,,\, 2r)$", we would have

$$M(t)\,\mathrm{e}^{-\alpha\,(\log t)^\beta} \ge \frac{(\log t)^\gamma}{t}\,.$$

This would imply that

$$\int_r^{r+r/(\log r)^\gamma} M(t)\,\mathrm{e}^{-\alpha\,(\log t)^\beta}\,\mathrm{d}t \ge \int_r^{r+r/(\log r)^\gamma} \frac{(\log t)^\gamma}{t}\,\mathrm{d}t$$
$$= \frac{1}{\gamma+1} \left\{ \left(\log\left(r + \frac{r}{(\log r)^\gamma} \right) \right)^{\gamma+1} - (\log r)^{\gamma+1} \right\}.$$

It is easily checked that the last expression is equal to $\log 2$ if γ is zero and is $1 + \mathrm{o}\,(1)$ if γ is positive. Thus the integral $\int_1^\infty M(t)\,\mathrm{e}^{-\alpha\,(\log t)^\beta}\,\mathrm{d}t$ would not be convergent, contradicting our hypothesis. Hence (4.2) holds. This means that for all large r,

$$M(\lambda r) < \frac{(\log \lambda r)^\gamma}{\lambda r}\,\mathrm{e}^{\alpha\,(\log \lambda r)^\beta} \quad \text{for some } \; \lambda \in \left(1\,,\, 1 + \frac{1}{(\log r)^\gamma} \right). \tag{4.3}$$

Step III. Since $\log M(r)$ is a convex function of $\log r$, we have

$$(M(r))^2 \le M\left(\frac{r}{\lambda}\right) M(\lambda r) \qquad (\lambda > 0)\,. \tag{4.4}$$

This is our main tool. We use (4.1) and (4.3) in (4.4) to conclude that

$$\lim_{r\to\infty} r\,(\log r)^{-(\gamma+\beta-1)/2}\, M(r)\,\mathrm{e}^{-\alpha\,(\log r)^\beta} = 0\,. \tag{4.5}$$

If r is sufficiently large and

$$\lambda \in \left(1\,,\, 1 + \frac{1}{(\log r)^\gamma} \right) \tag{4.6}$$

is chosen such that (possible by (4.3))

$$M(\lambda r) < \frac{(\log \lambda r)^\gamma}{\lambda r}\,\mathrm{e}^{\alpha\,(\log \lambda r)^\beta},$$

then using this and (4.1) in (4.4), we obtain

$$(M(r))^2 \le c_1(r)\,\frac{(\log(r/\lambda))^{\beta-1}}{r/\lambda}\,\mathrm{e}^{\alpha\,(\log(r/\lambda))^\beta} \cdot \frac{(\log(\lambda r))^\gamma}{\lambda r}\,\mathrm{e}^{\alpha\,(\log(\lambda r))^\beta},$$

where $c_1(r) = \mathrm{o}(1)$ as $r \to \infty$.

Now, note that

$$\begin{aligned}(\log(r/\lambda))^{\beta-1}\,(\log(\lambda r))^{\gamma} \\ &= (\log r)^{\gamma+\beta-1}\left\{\left(1-\frac{\log\lambda}{\log r}\right)^{\beta-1}\left(1+\frac{\log\lambda}{\log r}\right)^{\gamma}\right\} \\ &\le (\log r)^{\gamma+\beta-1}\left\{\left(1-\frac{\log\lambda}{\log r}\right)^{\beta-1}\left(1+\frac{\log\lambda}{\log r}\right)^{\beta-1}\right\} \\ &< (\log r)^{\gamma+\beta-1}\end{aligned}$$

because $\gamma < \beta - 1$. Hence

$$\begin{aligned}&(M(r))^2 \\ &\le c_1(r)\frac{(\log r)^{\gamma+\beta-1}}{r^2}\,\exp\left\{\alpha\,(\log r)^{\beta}\left(\left(1-\frac{\log\lambda}{\log r}\right)^{\beta}+\left(1+\frac{\log\lambda}{\log r}\right)^{\beta}\right)\right\} \\ &= c_1(r)\frac{(\log r)^{\gamma+\beta-1}}{r^2}\,\exp\left\{\alpha\,(\log r)^{\beta}\left(2+(\beta\,(\beta-1)+c_2(r))\left(\frac{\log\lambda}{\log r}\right)^{2}\right)\right\} \\ &= c_1(r)\frac{(\log r)^{\gamma+\beta-1}}{r^2}\,\exp\left\{\alpha\,(\log r)^{\beta}\left(2+(\beta\,(\beta-1)+c_2(r))\,(\log r)^{-2\gamma-2}\right)\right\},\end{aligned}$$

where $c_2(r) = \mathrm{o}(1)$ as $r \to \infty$, and where we have used (4.6) in the last line. Note that $\beta - 2\gamma - 2$ is negative if $1 < \beta < 2$ and zero if $\beta \ge 2$. Hence $(\log r)^{\beta-2\gamma-2} = \mathrm{O}(1)$ as $r \to \infty$. This allows us to conclude that

$$M(r) \le c_3(r)\,\frac{(\log r)^{(\gamma+\beta-1)/2}}{r}\,\mathrm{e}^{\alpha\,(\log r)^{\beta}} = c_3(r)\,\frac{(\log r)^{\gamma+(\beta-1-\gamma)/2}}{r}\,\mathrm{e}^{\alpha\,(\log r)^{\beta}},$$

where $c_3(r) = \mathrm{o}(1)$ as $r \to \infty$, which is equivalent to (4.5).

Inequality (4.5) is considerably stronger than (4.1) and provides a better estimate for $M(r/\lambda)$ in (4.4). Using (4.5) and (4.3) in (4.4) the way (4.1) and (4.3) were used above in (4.4), we obtain

$$\lim_{r\to\infty} r\,(\log r)^{-\gamma-(\beta-1-\gamma)/2^2}\,M(r)\,\mathrm{e}^{-\alpha\,(\log r)^{\beta}} = 0\,, \tag{4.7}$$

which may in turn be used to conclude that

$$\lim_{r\to\infty} r\,(\log r)^{-\gamma-(\beta-1-\gamma)/2^3}\,M(r)\,\mathrm{e}^{-\alpha\,(\log r)^{\beta}} = 0\,. \tag{4.8}$$

Clearly, (4.8) is stronger than (4.7). Since this process can go on indefinitely, we see that for any positive integer k, we have

$$\lim_{r\to\infty} r\,(\log r)^{-\gamma-(\beta-1-\gamma)/2^k}\,M(r)\,\mathrm{e}^{-\alpha\,(\log r)^{\beta}} = 0\,,$$

from which the desired result follows. □

Remark. The property of the function $M(r)$ by which $\log M(r)$ is a convex function of $\log r$ is shared by some other functions associated with an entire function f, which makes the proof of Theorem 3.2 applicable to these associated functions. In fact, let $f(z) := \sum_{\nu=0}^{\infty} a_\nu z^\nu$ be an entire function; for any $r > 0$, we define the function $\mathcal{M}_p(r) = \mathcal{M}_p(f; r)$ by

$$\mathcal{M}_p(f; r) := \left(\frac{1}{2\pi} \int_0^{2\pi} \left| f(r\,\mathrm{e}^{\mathrm{i}\theta}) \right|^p \mathrm{d}\theta \right)^{1/p}, \quad p > 0,$$

and the maximum term of f, denoted by $\mu(r)$, is given by the maximum of $|a_\nu| r^\nu$ for $\nu \in \{0, 1, 2, \dots\}$. Then, $\log \mu(r)$ and $\log \mathcal{M}_p(r)$ (for any $p > 0$) are two convex functions of $\log r$. The proof of this statement was given by G. Valiron ([9, pp. 30–31]) for the function $\log \mu(r)$ and by G. H. Hardy [3] for $\log \mathcal{M}_p(r)$. The reader might find [6] to be of some interest in this connection.

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Mohamed Amine Hachani
Département de Mathématiques et de Statistique
Université de Montréal
Montréal, Québec H3C 3J7
Canada
e-mail: hachani@dms.umontreal.ca

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