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Growth of a polynomial not vanishing in a disk

Abstract. This paper deals with the problem of finding some upper bound estimates for the maximum modulus of the derivative and higher order derivatives of a complex polynomial on a disk under the assumption that the polynomial has no zeros in another disk. The estimates obtained strengthen the well-known inequality of Ankeny and Rivlin about polynomials.

1. Introduction and statement of results. For an arbitrary entire function $f(z)$, let $M(f,r) = \max_{|z|=r} |f(z)|$. For a polynomial $P(z)$ of degree n, it is known that

$$M(P,R) \leq R^n M(P,1), \ \ R \geq 1. \tag{1.1}$$

Inequality (1.1) is a simple consequence of Maximum Modulus Principle (see [5]). It was shown by Ankeny and Rivlin [1] that if $P(z) \neq 0$ in $|z| < 1$, then (1.1) can be replaced by

$$M(P,R) \leq \frac{R^n + 1}{2} M(P,1), \ \ R \geq 1. \tag{1.2}$$

Over the last 40 years many different authors produced a large number of generalizations and refinements of inequality (1.2) by introducing coefficients of the polynomial, using higher order derivatives, by involving polar derivative of polynomials etc.

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Recently, Jain [4] obtained a generalization of (1.2) by considering polynomials with no zeros in $|z| < k$, $k \geq 1$ and has simultaneously taken into consideration the sth derivative $(0 \leq s < n)$ of the polynomial, instead of the polynomial itself. More precisely, Jain proved

Theorem A. *If $P(z)$ is a polynomial of degree n with no zeros in $|z| < k$, $k \geq 1$, then for $0 \leq s < n$,*

$$M\big(P^{(s)}, R\big) \leq \frac{1}{2}\Big\{\frac{d^s}{dR^s}(R^n + k^n)\Big\}\Big(\frac{2}{1+k}\Big)^n M(P, 1), \text{ for } R \geq k. \tag{1.3}$$

Equality holds in (1.3) (with $s = 0$ and $k = 1$) for $P(z) = z^n + 1$.

In this paper, we prove the following sharpening of Theorem A. Our theorem includes as special cases several refinements and generalizations of (1.2) as well.

Theorem 1. *Let $P(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ is a polynomial of degree n having no zeros in $|z| < k$, $k \geq 1$. If $|\zeta| = \frac{k|a_1|}{n(|a_0| - tm)}$ with $0 \leq t \leq 1$ and $m = \min_{|z|=k} |P(z)|$, then for $0 \leq s < n$ and $R \geq k$,*

$$\begin{aligned} M\big(P^{(s)}, R\big) \leq \frac{1}{2}\Big\{\frac{d^s}{dR^s}(R^n + k^n)\Big\}\Bigg[&\left(\frac{2 + 2|\zeta|}{1 + 2|\zeta|k + k^2}\right)^{\frac{n}{2}} M(P, 1) \\ &- \left\{\left(\frac{2 + 2|\zeta|}{1 + 2|\zeta|k + k^2}\right)^{\frac{n}{2}} - \frac{1}{k^n}\right\} tm\Bigg]. \end{aligned} \tag{1.4}$$

Equality holds in (1.4) (with $s = 0$ and $k = 1$) for $P(z) = z^n + 1$.

Taking $t = 0$ in Theorem 1, we get the following result.

Corollary 1. *Let $P(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ be a polynomial of degree n having no zeros in $|z| < k$, $k \geq 1$ and let $|\lambda| = \left|\frac{ka_1}{na_0}\right|$. Then for $0 \leq s < n$ and $R \geq k$,*

$$M\big(P^{(s)}, R\big) \leq \frac{1}{2}\Big\{\frac{d^s}{dR^s}(R^n + k^n)\Big\}\left(\frac{2 + 2|\lambda|}{1 + 2|\lambda|k + k^2}\right)^{\frac{n}{2}} M(P, 1). \tag{1.5}$$

Equality holds in (1.5) (with $s = 0$ and $k = 1$) for $P(z) = z^n + 1$.

Remark 1. Since if $P(z) = \sum_{\nu=0}^{n} a_\nu z^\nu \neq 0$ in $|z| < k$, $k \geq 1$, then from Lemma 3, we have $|\lambda| = \left|\frac{ka_1}{na_0}\right| \leq 1$. We show that in general

$$\left(\frac{2 + 2|\lambda|}{1 + 2|\lambda|k + k^2}\right)^{\frac{n}{2}} \leq \Big(\frac{2}{1+k}\Big)^n,$$

which is equivalent to showing

$$\left(\frac{2 + 2|\lambda|}{1 + 2|\lambda|k + k^2}\right) \leq \Big(\frac{2}{1+k}\Big)^2,$$

that is,

$$(1+|\lambda|)(1+k)^2 \le 2(1+2|\lambda|k+k^2),$$

which is equivalent to

$$(|\lambda|-1)(k-1)^2 \le 0,$$

which clearly holds as $|\lambda| \le 1$. This shows that (1.5) is a refinement of (1.3).

2. Lemmas. For the proof of Theorem 1, we need the following lemmas. The first lemma is due to Govil, Qazi and Rahman [3].

Lemma 1. *Let* $P(z) = \sum_{\nu=0}^{n} a_\nu z^\nu \neq 0$ *for* $|z| < k$, *where* $k \ge 1$ *and let* $\lambda = \frac{ka_1}{na_0}$. *Then for* $1 \le R \le k^2$,

$$M(P,R) \le \left(\frac{R^2+2|\lambda|Rk+k^2}{1+2|\lambda|k+k^2}\right)^{\frac{n}{2}} M(P,1).$$

The following lemma is due to Aziz and Aliya [2].

Lemma 2. *Let* $P(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j$, $1 \le \mu \le n$ *be a polynomial of degree* n *such that* $P(z) \neq 0$ *in* $|z| < k$, $k \ge 1$. *Then for* $R > 1$, $0 \le t \le 1$ *and* $m = \min_{|z|=k} |P(z)|$,

$$\left(\frac{R^\mu - 1}{R^n - 1}\right)\frac{|a_\mu|k^\mu}{|a_0| - tm} \le 1.$$

From Lemma 2, we easily get the following for $\mu = 1$.

Lemma 3. *Let* $P(z) = \sum_{\nu=0}^{n} a_\nu z^\nu \neq 0$ *in* $|z| < k$, $k \ge 1$ *and* $m = \min_{|z|=k} |P(z)|$. *Then for* $0 \le t \le 1$,

$$\frac{k|a_1|}{n(|a_0| - tm)} \le 1.$$

Lemma 4. *Let* $P(z) = \sum_{\nu=0}^{n} a_\nu z^\nu \neq 0$ *in* $|z| < k$, $k \ge 1$ *and let* $|\zeta| = \frac{k|a_1|}{n(|a_0|-tm)}$, *where* $0 \le t \le 1$ *and* $m = \min_{|z|=k} |P(z)|$. *Then for* $1 \le R \le k^2$, *we have*

$$\begin{aligned} M(P,R) \le{}& \left(\frac{R^2+2|\zeta|Rk+k^2}{1+2|\zeta|k+k^2}\right)^{\frac{n}{2}} M(P,1) \\ & - \left\{\left(\frac{R^2+2|\zeta|Rk+k^2}{1+2|\zeta|k+k^2}\right)^{\frac{n}{2}} - 1\right\} tm. \end{aligned} \tag{2.1}$$

Proof of Lemma 4. Since $P(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ has all its zeros in $|z| \ge k \ge 1$ and $m = \min_{|z|=k} |P(z)|$, therefore, $m \le |P(z)|$ for $|z| = k$. We first show that for every complex α with $|\alpha| \le 1$, the polynomial $F(z) = P(z) - \alpha m$ does not vanish in $|z| < k$, $k \ge 1$. This result is clear if $P(z)$ has a zero on

$|z| = k$ for then $m = 0$ and hence $F(z) = P(z)$. In case $P(z)$ has no zeros on $|z| = k$, that is, all the zeros of $P(z)$ lie in $|z| > k, k \geq 1$, then clearly $m > 0$ so that $\frac{m}{P(z)}$ is analytic in $|z| \leq k$ and

$$\left|\frac{m}{P(z)}\right| \leq 1, \text{ for } |z| = k. \tag{2.2}$$

Since $\frac{m}{P(z)}$ is not a constant, by the Maximum Modulus Principle, it follows that

$$m < |P(z)|, \text{ for } |z| < k. \tag{2.3}$$

Now, if $F(z) = P(z) - \alpha m$ has a zero in $|z| < k$, $k \geq 1$, say at $z = z_1$ with $|z_1| < k$, then

$$F(z_1) = P(z_1) - \alpha m = 0.$$

This gives,

$$|P(z_1)| = |\alpha| m \leq m,$$

where $|z_1| < k$, which contradicts (2.3).

Hence, we conclude that the polynomial $F(z)$ does not vanish in $|z| < k$, $k \geq 1$. Applying Lemma 1 to the polynomial

$$F(z) = P(z) - \alpha m = (a_0 - \alpha m) + \sum_{j=1}^{n} a_j z^j,$$

we get for every complex α with $|\alpha| \leq 1$ and $1 \leq R \leq k^2$,

$$\max_{|z|=R} \left|P(z) - \alpha m\right| \leq \left(\frac{R^2 + 2|\eta| Rk + k^2}{1 + 2|\eta| k + k^2}\right)^{\frac{n}{2}} \max_{|z|=1} \left|P(z) - \alpha m\right|, \tag{2.4}$$

where $\eta = \frac{ka_1}{n(a_0 - \alpha m)}$.

Since for every α with $|\alpha| \leq 1$, we have

$$|a_0 - \alpha m| \geq |a_0| - |\alpha| m,$$

and by (2.3)

$$|\alpha| m \leq m < |P(0)| = |a_0|,$$

we get

$$|\eta| = \left|\frac{ka_1}{n(a_0 - \alpha m)}\right| \leq \frac{k|a_1|}{n(|a_0| - |\alpha| m)} = |\zeta|.$$

Also, by Lemma 3, we have $\frac{k|a_1|}{n(|a_0| - |\alpha| m)} \leq 1$ and since $\left(\frac{R^2 + 2Rkx + k^2}{1 + 2kx + k^2}\right)$ is an increasing function of x in [0, 1], it follows from (2.4) that for every α with

$|\alpha| \leq 1$ and $1 \leq R \leq k^2$,

$$\max_{|z|=R} \big|P(z) - \alpha m\big| \leq \left(\frac{R^2 + 2|\zeta|Rk + k^2}{1 + 2|\zeta|k + k^2}\right)^{\frac{n}{2}} \max_{|z|=1} \big|P(z) - \alpha m\big|. \tag{2.5}$$

Let z_0 on $|z| = 1$, be such that

$$\max_{|z|=1} \big|P(z) - \alpha m\big| = \big|P(z_0) - \alpha m\big|, \tag{2.6}$$

and let z_1 on $|z| = R$, be such that $|P(z_1)| = \max_{|z|=R}|P(z)|$. Then the inequalities (2.5) and (2.6) together imply

$$\max_{|z|=R} \big|P(z) - \alpha m\big| \leq \left(\frac{R^2 + 2|\zeta|Rk + k^2}{1 + 2|\eta|k + k^2}\right)^{\frac{n}{2}} \big|P(z_0) - \alpha m\big|, \tag{2.7}$$

for every α with $|\alpha| \leq 1$ and $1 \leq R \leq k^2$. Inequality (2.7) in particular gives

$$\big|P(z_1) - \alpha m\big| \leq \left(\frac{R^2 + 2|\zeta|Rk + k^2}{1 + 2|\zeta|k + k^2}\right)^{\frac{n}{2}} \big|P(z_0) - \alpha m\big|, \tag{2.8}$$

for every α with $|\alpha| \leq 1$ and $1 \leq R \leq k^2$. If we choose the argument of α, so that

$$|P(z_0) - \alpha m| = |P(z_0)| - |\alpha|m,$$

we get from (2.8) that for $1 \leq R \leq k^2$,

$$|P(z_1)| - |\alpha|m \leq \left(\frac{R^2 + 2|\zeta|Rk + k^2}{1 + 2|\zeta|k + k^2}\right)^{\frac{n}{2}} \Big(|P(z_0)| - |\alpha|m\Big). \tag{2.9}$$

The fact that the quantity $\big(|P(z_0)| - |\alpha|m\big)$ in the right hand side of (2.9) is positive follows from (2.2) and (2.3). This gives

$$\begin{aligned} M(P, R) \leq & \left(\frac{R^2 + 2|\zeta|Rk + k^2}{1 + 2|\zeta|k + k^2}\right)^{\frac{n}{2}} |P(z_0)| \\ & - \left\{\left(\frac{R^2 + 2|\zeta|Rk + k^2}{1 + 2|\zeta|k + k^2}\right)^{\frac{n}{2}} - 1\right\}|\alpha|m, \end{aligned}$$

which is equivalent to (2.1) and this completes the proof of Lemma 4. □

The following two lemmas are due to Jain [4].

Lemma 5. *Let $P(z)$ be a polynomial of degree n having all its zeros in $|z| \leq 1$. If $T(z)$ is a polynomial of degree at most n such that*

$$|T(z)| \leq |P(z)|, \ \textit{for } |z| = 1,$$

then for $0 \le s < n$,

$$\left|T^{(s)}(z)\right| \le \left|P^{(s)}(z)\right|, \ \textit{for } |z| \ge 1.$$

Lemma 6. *If* $P(z)$ *is a polynomial of degree at most* n, *then for* $0 \le s < n$,

$$\left|P^{(s)}(z)\right| + \left|Q^{(s)}(z)\right| \le \left\{\left|\frac{d^s}{dz^s}(1)\right| + \left|\frac{d^s}{dz^s}(z^n)\right|\right\} M(P,1), \ \textit{for } |z| \ge 1,$$

where $Q(z) = z^n \overline{P(\frac{1}{\bar{z}})}$.

3. Proof of Theorem.

Proof of Theorem 1. Since $P(z) \neq 0$ in $|z| < k$, $k \ge 1$, the polynomial $T(z) = P(kz)$ has no zeros in $|z| < 1$. Let $H(z) = z^n \overline{T(\frac{1}{\bar{z}})}$, then

$$|T(z)| \le |H(z)|, \ \text{for } |z| = 1,$$

and $H(z)$ has all its zeros in $|z| \le 1$. Therefore, applying Lemma 5 to the polynomials $T(z)$ and $H(z)$, we get for $0 \le s < n$ and $r \ge 1$,

$$\left|T^{(s)}(re^{i\theta})\right| \le \left|H^{(s)}(re^{i\theta})\right|, \ 0 \le \theta < 2\pi. \tag{3.1}$$

Also, by Lemma 6, we have for $r \ge 1$ and $0 \le s < n$,

$$\left|T^{(s)}(re^{i\theta})\right| + \left|H^{(s)}(re^{i\theta})\right| \le \left\{\frac{d^s}{dr^s}(r^n + 1)\right\} M(T,1), \ 0 \le \theta < 2\pi,$$

which on combining with (3.1) gives

$$\left|T^{(s)}(re^{i\theta})\right| \le \frac{1}{2}\left\{\frac{d^s}{dr^s}(r^n + 1)\right\} M(T,1),$$

which further implies

$$\left|P^{(s)}(kre^{i\theta})\right| \le \frac{1}{2k^s}\left\{\frac{d^s}{dr^s}(r^n + 1)\right\} M(P,k). \tag{3.2}$$

The above inequality (3.2) in conjunction with Lemma 4 gives

$$\begin{aligned}\left|P^{(s)}(kre^{i\theta})\right| \le \frac{1}{2k^s}\left\{\frac{d^s}{dr^s}(r^n + 1)\right\}\Bigg[&\left(\frac{k^2 + 2|\zeta|k^2 + k^2}{1 + 2|\zeta|k + k^2}\right)^{\frac{n}{2}} M(P,1) \\ &- \left\{\left(\frac{k^2 + 2|\zeta|k^2 + k^2}{1 + 2|\zeta|k + k^2}\right)^{\frac{n}{2}} - 1\right\} tm\Bigg].\end{aligned} \tag{3.3}$$

Now taking $r = \frac{R}{k} \geq 1$ in (3.3), we get

$$\big|P^{(s)}(Re^{i\theta})\big| \leq \frac{1}{2}\Big\{\frac{d^s}{dR^s}(R^n + k^n)\Big\}\Bigg[\bigg(\frac{2+2|\zeta|}{1+2|\zeta|k+k^2}\bigg)^{\frac{n}{2}} M(P,1) - \Bigg\{\bigg(\frac{2+2|\zeta|}{1+2|\zeta|k+k^2}\bigg)^{\frac{n}{2}} - \frac{1}{k^n}\Bigg\}tm\Bigg],$$

which is equivalent to (1.4). □

Remark 2. From Lemma 3, we have

$$\frac{k|a_1|}{n(|a_0| - tm)} \leq 1,$$

and since, as mentioned earlier $\Big(\frac{R^2+2Rkx+k^2}{1+2kx+k^2}\Big)$ is an increasing function of x in [0, 1], it follows from (2.7) that for every α with $|\alpha| \leq 1$ and $1 \leq R \leq k^2$,

$$\max_{|z|=R} \big|P(z) - \alpha m\big| \leq \bigg(\frac{R+k}{1+k}\bigg)^n \max_{|z|=1} \big|P(z) - \alpha m\big|. \tag{3.4}$$

Now proceeding similarly as in the proof of Lemma 4, it follows from (3.4) that

$$M(P,R) \leq \bigg(\frac{R+k}{1+k}\bigg)^n M(P,1) - \Bigg\{\bigg(\frac{R+k}{1+k}\bigg)^n - 1\Bigg\}tm, \tag{3.5}$$

where $0 \leq t \leq 1$ and $1 \leq R \leq k^2$. If we use inequality (3.5) (for $t = 1$) instead of Lemma 4 in the proof of Theorem 1, we get the following refinement of Theorem A.

Corollary 2. *If* $P(z) = \sum_{\nu=0}^{n} a_\nu z^\nu \neq 0$ *in* $|z| < k$, $k \geq 1$ *and* $m = \min_{|z|=k} |P(z)|$, *then for* $0 \leq s < n$,

$$M\big(P^{(s)}, R\big) \leq \frac{1}{2}\Big\{\frac{d^s}{dR^s}(R^n + k^n)\Big\}\Bigg[\Big(\frac{2}{1+k}\Big)^n M(P,1) - \Bigg\{\Big(\frac{2}{1+k}\Big)^n - \frac{1}{k^n}\Bigg\}m\Bigg], \ \textit{for } R \geq k.$$

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