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Some results on convex meromorphic functions

Abstract. In this paper, we define a function $F : D \times D \times D \to \mathbb{C}$ in terms of f and show that $\operatorname{Re} F > 0$ for all $\zeta, z, w \in D$ if and only if f belongs to the class of convex meromorphic functions.

1. Introduction and preliminaries. Let us denote by $S(p)$ with $0 < p < 1$ the set of univalent functions in the unit disk $D = \{z \in \mathbb{C} : |z| < 1\}$ such that $f(0) = 0, f'(0) = 1$ and $f(p) = \infty$. We denote by K the subset of functions in $S(p)$ which omits a convex set in the extended plane $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, that is, $f \in K$ if and only if the set $\widehat{\mathbb{C}} \setminus f(D) = \{w \in \mathbb{C} : f(z) \neq w\}$ is convex. Functions in K are called convex meromorphic functions. Many people have worked on convex holomorphic functions and the results obtained have already found their place in many books; see, for examle Ruscheweyh and Sheil-Small [6], Sheil-Small [8], Schober [7] and Duren [1]. So far, several works on convex meromorphic functions have appeared in the literature; for further reading see Ohno [5] and the references therein including Yulin and Owa [9]. It should be remarked that the functions that omit a convex set are called concave functions, nowadays. The set K we described above is a subset of concave functions that belong to the set $S(p)$. Therefore, we prefer to call the functions in K convex meromorphic functions. In this paper, we also consider these functions and we think our results in a way unify the earlier results. That is, as Duren [1, p. 250] comments "We shall digress

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briefly to establish some global properties of convex functions. Everything is a consequence of the following proposition ..."; many earlier results will follow from ours. However, a slight modification in the statement and the proof of our Theorem 1 below yields that it is true for all concave functions. To see this, it is enough to assume $p \in D$ and replace the factor $\frac{1-pz}{1-p\zeta}$ by $\frac{1-\overline{p}z}{1-\overline{p}\zeta}$ in Theorem 1.

2. Main theorems.

Theorem 1. *Let $F : D \times D \times D \to \mathbb{C}$ be the function defined by*

$$F(\zeta, z, w) = \frac{z+\zeta}{z-\zeta} - \frac{2\zeta}{z-\zeta}\frac{(p-z)(1-pz)}{(p-\zeta)(1-p\zeta)}\frac{f(w)-f(z)}{f(w)-f(\zeta)}\frac{w-\zeta}{w-z} \tag{2.1}$$

for each f in K. Then, F has a positive real part with $F(0, z, w) = 1$.

Proof. Without loss of generality, we can assume that the function f can be extended to the boundary as a continuous function. The function $F(\zeta, z, w)$ has a holomorphic extension to D^3, that is, all the singularities of F are removable. To see this, it is enough to observe the following three evaluations:

1) $F(\zeta, z, p) = \dfrac{1+p\zeta}{1-p\zeta}$,

2) $F(\zeta, p, w) = \dfrac{p+\zeta}{p-\zeta} - \dfrac{2\zeta}{(p-\zeta)^2}\dfrac{1-p^2}{1-p\zeta}\dfrac{\alpha_f}{f(w)-f(\zeta)}\dfrac{w-\zeta}{w-p}$

where $\alpha_f = Res_{z=p}(f)$ which implies

$$F(\zeta, p, p) = \lim_{w\to p} F(\zeta, p, w) = \frac{1+p\zeta}{1-p\zeta},$$

3) $\lim\limits_{\zeta\to z} F(\zeta, z, w) = 0$.

Now, we consider the function

$$F(\zeta, z, w) = \frac{z+\zeta}{z-\zeta} - \frac{2\zeta}{z-\zeta}\frac{(p-z)(1-pz)}{(p-\zeta)(1-p\zeta)}\frac{f(w)-f(z)}{f(w)-f(\zeta)}\frac{w-\zeta}{w-z}.$$

Since

$$\frac{(p-z)(1-pz)/z}{(p-\zeta)(1-p\zeta)/\zeta} = \frac{|1-pz|^2}{|1-p\zeta|^2} > 0,$$

(2.1) can be written as

$$F(\zeta, z, w) = \frac{z+\zeta}{z-\zeta} - \frac{2z}{z-\zeta}\frac{|1-pz|^2}{|1-p\zeta|^2}\frac{f(w)-f(z)}{f(w)-f(\zeta)}\frac{w-\zeta}{w-z} \tag{2.2}$$

for $|z| = |\zeta| = 1$. We set $z = \alpha w$ and $\zeta = \beta w$ in (2.2), where $\alpha = e^{ia}$ and $\beta = e^{ib}$, $0 < a, b < 2\pi$ are distinct constants. Thus we have

$$F(\beta w, \alpha w, w) = \frac{e^{ia}+e^{ib}}{e^{ia}-e^{ib}} - \frac{2e^{ia}}{e^{ia}-e^{ib}}\frac{\left|1-pwe^{ia}\right|^2}{\left|1-pwe^{ib}\right|^2}\frac{f(w)-f(e^{ia}w)}{f(w)-f(e^{ib}w)}\frac{1-e^{ib}}{1-e^{ia}}.$$

Since

$$\frac{e^{ia}+e^{ib}}{e^{ia}-e^{ib}} = -i\frac{\cos\left(\frac{a-b}{2}\right)}{\sin\left(\frac{a-b}{2}\right)}$$

is purely imaginary and

$$\frac{2e^{ia}}{e^{ia}-e^{ib}}\frac{1-e^{ib}}{1-e^{ia}} = -i\frac{\sin\left(\frac{b}{2}\right)}{\sin\left(\frac{a-b}{2}\right)\sin\left(\frac{a}{2}\right)},$$

we have

$$F(\beta w, \alpha w, w) = i\frac{\sin\left(\frac{b}{2}\right)}{\sin\left(\frac{a-b}{2}\right)\sin\left(\frac{a}{2}\right)}\frac{\left|1-pwe^{ia}\right|^2}{|1-pwe^{ib}|^2}\frac{f(w)-f(e^{ia}w)}{f(w)-f(e^{ib}w)} - i\frac{\cos\left(\frac{a-b}{2}\right)}{\sin\left(\frac{a-b}{2}\right)}$$

and thus

$$\operatorname{Re}\{F(\beta w, \alpha w, w)\} = -\frac{\sin\left(\frac{b}{2}\right)}{\sin\left(\frac{a-b}{2}\right)\sin\left(\frac{a}{2}\right)}\frac{\left|1-pwe^{ia}\right|^2}{|1-pwe^{ib}|^2}R\sin\varphi,$$

where $\frac{f(w)-f(e^{ia}w)}{f(w)-f(e^{ib}w)} = Re^{i\varphi}$ for $|w|=1$. Note that for $a<b$, the images of w, $e^{ia}w$ and $e^{ib}w$ under f have the same order on $\partial F(D)$ in positive direction. Therefore in this case $\varphi \in (0,\pi)$. Similarly, $\varphi \in (\pi, 2\pi)$ for $b<a$. It follows that $\operatorname{Re}\{F(\beta w, \alpha w, w)\} > 0$ on $(\partial D)^3$. Furthermore, it is a consequence of the Cauchy integral formula in a polydisk that the absolute value of the holomorphic function $e^{-F(\zeta,z,w)}$ attains its maximum on the distinguished boundary $(\partial D)^3$ of the polydisk D^3 (see, for example, Gunning [3] Theorem 4, p. 6 or Hörmander [4] Section 2.2, p. 25). It follows that $\operatorname{Re}\{F(\zeta,z,w)\} > 0$ throughout the polydisk $D\times D\times D$. Thus we get the desired result. Obviously $F(0,z,w)=1$. □

Now we can prove that the converse of our theorem is also true:

Theorem 2. *If $F: D\times D\times D \to \mathbb{C}$ defined by*

$$F(\zeta,z,w) = \frac{z+\zeta}{z-\zeta} - \frac{2\zeta}{z-\zeta}\frac{(p-z)(1-pz)}{(p-\zeta)(1-p\zeta)}\frac{f(w)-f(z)}{f(w)-f(\zeta)}\frac{w-\zeta}{w-z}$$

has a positive real part, then $f \in K$.

Proof. Observe that

$$\begin{aligned}
F(\zeta,z,z) &= \frac{z+\zeta}{z-\zeta} - 2\zeta\frac{(p-z)(1-pz)}{(p-\zeta)(1-p\zeta)}\frac{f'(z)}{f(z)-f(\zeta)} \\
&= \frac{z+\zeta}{z-\zeta} - \frac{2\zeta f'(z)}{f(z)-f(\zeta)} - \left[\frac{(p-z)(1-pz)}{(p-\zeta)(1-p\zeta)} - 1\right]\frac{2\zeta f'(z)}{f(z)-f(\zeta)} \\
&= \frac{z+\zeta}{z-\zeta} - \frac{2\zeta f'(z)}{f(z)-f(\zeta)} + \frac{2\,(z-\zeta)\left[1+p^2-p\,(z+\zeta)\right]}{(p-\zeta)(1-p\zeta)}\frac{\zeta f'(z)}{f(z)-f(\zeta)} \\
&= \frac{(z+\zeta)\,(f(z)-f(\zeta)) - 2\zeta f'(z)\,(z-\zeta)}{(z-\zeta)\,(f(z)-f(\zeta))} \\
&\quad + \frac{2\,(z-\zeta)\left[1+p^2-p\,(z+\zeta)\right]}{(p-\zeta)(1-p\zeta)}\frac{\zeta f'(z)}{f(z)-f(\zeta)},
\end{aligned}$$

so for $z \to \zeta$, we have $\frac{0}{0}$ and applying L'Hospital's rule, we find

$$\begin{aligned}
\lim_{z\to\zeta} F(\zeta,z,z) &= \lim_{z\to\zeta}\frac{f(z)-f(\zeta)+(z+\zeta)f'(z)-2\zeta f''(z)(z-\zeta)-2\zeta f'(z)}{f(z)-f(\zeta)+(z-\zeta)f'(z)} \\
&\quad + 2\zeta\frac{1+p^2-2p\zeta}{(p-\zeta)(1-p\zeta)} \\
&= 1 + 2\zeta\frac{1+p^2-2p\zeta}{(p-\zeta)(1-p\zeta)} - 2\zeta\lim_{z\to\zeta}\frac{(z-\zeta)\,f''(z)}{f(z)-f(\zeta)+(z-\zeta)\,f'(z)} \\
&= 1 - \frac{\zeta f''(\zeta)}{f'(\zeta)} + 2\zeta\frac{1+p^2-2p\zeta}{(p-\zeta)(1-p\zeta)}.
\end{aligned}$$

A simple calculation gives (compare, for example Miller [2, formula (1)])

$$\begin{aligned}
\lim_{z\to\zeta} F(\zeta,z,z) &= -1 - \frac{\zeta f''(\zeta)}{f'(\zeta)} + 2 + 2\zeta\frac{1+p^2-2p\zeta}{(p-\zeta)(1-p\zeta)} \\
&= 2p\frac{1-\zeta^2}{(p-\zeta)(1-p\zeta)} - \left\{1+\frac{\zeta f''(\zeta)}{f'(\zeta)}\right\}.
\end{aligned}$$

Since $\frac{2p(1-\zeta^2)}{(p-\zeta)(1-p\zeta)}$ is purely imaginary on the boundary $|\zeta| = 1$, we have

$$\operatorname{Re}\left\{1+\frac{\zeta f''(\zeta)}{f'(\zeta)}\right\} < 0.$$

Thus the set of omitted values of f is convex. □

3. A set of useful corollaries. As we pointed out above, the function

$$F(\zeta,z,w) = \frac{z+\zeta}{z-\zeta} - \frac{2\zeta}{z-\zeta}\frac{p-z}{p-\zeta}\frac{1-pz}{1-p\zeta}\frac{f(z)-f(w)}{f(\zeta)-f(w)}\frac{\zeta-w}{z-w}$$

is a holomorphic function for $(\zeta,z,w) \in D \times D \times D = D^3$. Theorems 1 and 2 show that $\operatorname{Re} F(\zeta,z,w) > 0$ if and only if f is a convex meromorphic function in $S(p)$. To simplify, we define the function $H : D^2 \to \mathbb{C}$ by

$$H(z,w) = \frac{(p-z)(1-pz)}{p} \frac{f(z)-f(w)}{z-w} \frac{w}{f(w)},$$

then

$$F(\zeta,z,w) = \frac{z+\zeta}{z-\zeta} - \frac{2\zeta}{z-\zeta} \frac{H(z,w)}{H(\zeta,w)}.$$

Note that $\operatorname{Re} F(\zeta,z,w) > 0$ implies $|F-1| < |F+1|$, hence we obtain

$$\left| \frac{z+\zeta}{z-\zeta} - \frac{2\zeta}{z-\zeta} \frac{H(z,w)}{H(\zeta,w)} - 1 \right| < \left| \frac{z+\zeta}{z-\zeta} - \frac{2\zeta}{z-\zeta} \frac{H(z,w)}{H(\zeta,w)} + 1 \right|,$$

and after some simplification in the last inequality, we have

$$|H(z,w) - H(\zeta,w)| < |\zeta H(z,w) - zH(\zeta,w)|.$$

To summarize, we display

$$\operatorname{Re} F(\zeta,z,w) > 0 \Leftrightarrow |H(z,w) - H(\zeta,w)| < |\zeta H(z,w) - zH(\zeta,w)|.$$

Now, we state several corollaries for convex meromorphic functions f in K.

Corollary 1. *The function $H(z,w)$ maps the domain $D \times D$ onto the unit disk centered at 1, i.e., $|H(z,w) - 1| < 1$.*

Proof. Choosing $\zeta = 0$ in the inequality

$$|H(z,w) - H(\zeta,w)| < |\zeta H(z,w) - zH(\zeta,w)|,$$

we obtain $|H(z,w) - 1| < |z|$. □

Corollary 2. *The function $H(z,w)/z$, considered as a function of z, is univalent in D.*

Proof. Suppose

$$\frac{H(z,w)}{z} = \frac{H(\zeta,w)}{\zeta},$$

then the inequality

$$|H(z,w) - H(\zeta,w)| < |\zeta H(z,w) - zH(\zeta,w)|$$

implies $H(z,w) = H(\zeta,w)$. By Corollary 1, both $H(z,w)$ and $H(\zeta,w)$ are different from zero. Dividing both sides of the equality

$$\frac{H(z,w)}{z} = \frac{H(\zeta,w)}{\zeta}$$

by $H(z,w) = H(\zeta,w)$, we get $z = \zeta$. □

Corollary 3. *We have the following inequalities*

$$\left| \frac{(p-z)(1-pz)}{p} \frac{f(z)}{z} - 1 \right| < |z| \quad \textit{and} \quad \left| \frac{(p-z)(1-pz)}{p} \frac{zf'(z)}{f(z)} - 1 \right| < |z|.$$

Proof. In Corollary 1, putting $w = 0$ and $w = z$, respectively, we obtain the desired inequalities. □

Corollary 4. *We have*

$$\left| -\frac{1-p^2}{p^2}\alpha_f - 1 \right| < p,$$

where α_f is the residue of f at $z = p$.

Proof. We get this inequality by letting $z \to p$ in the first inequality of Corollary 3. □

Remark 1. Note that $-\frac{p^2}{1-p^2}$ is the residue of $k(z) = \frac{pz}{(p-z)(1-pz)}$. Therefore, the inequality in Corollary 4 can be stated as $\left|\frac{\alpha_f}{\alpha_k} - 1\right| < p$.

Corollary 5. *We have*

$$\left| \frac{zH'(z,w)}{H(z,w)} \right| < \left| \frac{zH'(z,w)}{H(z,w)} - 1 \right|,$$

where $H'(z,w) = \frac{\partial H(z,w)}{\partial z}$.

Proof. We divide both sides of the inequality

$$|H(z,w) - H(\zeta,w)| < |\zeta H(z,w) - zH(\zeta,w)|$$

by $|z - \zeta|$ and let $\zeta \to z$; hence, we obtain

$$\left|H'(z,w)\right| < \left|zH'(z,w) - H(z,w)\right|.$$

Corollary follows by dividing both sides by $|H(z,w)|$ and multiplying the left hand side by $|z|$. □

Remark 2. Note that Corollary 5 implies

$$\mathrm{Re}\left\{ \frac{zH'(z,w)}{H(z,w)} \right\} < \frac{1}{2},$$

or

$$\begin{aligned}
&\mathrm{Re}\left\{ 1 - 2\frac{zH'(z,w)}{H(z,w)} \right\} \\
&\quad = \mathrm{Re}\left\{ 1 - 2z\left[-\frac{1}{p-z} - \frac{p}{1-pz} + \frac{f'(z)}{f(z)-f(w)} - \frac{1}{z-w} \right] \right\} \\
&\quad = \mathrm{Re}\left\{ 1 + 2\left[\frac{z}{p-z} + \frac{pz}{1-pz} + \frac{z}{z-w} - \frac{zf'(z)}{f(z)-f(w)} \right] \right\} > 0.
\end{aligned}$$

From this we obtain Theorem 2 by letting $w \to z$.

Corollary 6. *The function $z/H^2(z,w)$ is a holomorphic starlike function as a function of z and therefore it is univalent.*

Proof. We define a function h by

$$1 - 2\frac{zH'(z,w)}{H(z,w)} =: \frac{zh'(z,w)}{h(z,w)}.$$

Dividing both sides by z and then integrating both sides with respect to z, we find $h(z,w) = z/H^2(z,w)$. □

Corollary 7. *The integral representation*

$$\log H(z,w) = \int_{|\eta|=1} \log(1-\eta z) d\mu$$

holds, where μ is a probability measure on ∂D with $\log 1 = 0$.

Proof. It follows from Corollary 5 that there exists a probability measure μ on ∂D such that

$$1 - 2\frac{zH'(z,w)}{H(z,w)} = \int_{|\eta|=1} \frac{1+\eta z}{1-\eta z} d\mu.$$

Subtracting 1 from both sides and dividing by $-2z$, we obtain

$$\frac{H'(z,w)}{H(z,w)} = \int_{|\eta|=1} \frac{-\eta}{1-\eta z} d\mu.$$

Integrating both sides with respect to z and noting that the integral constant is zero, we arrive at

$$\log H(z,w) = \int_{|\eta|=1} \log(1-\eta z) d\mu.$$

□

4. Conclusions. In this paper we focused on the main results and their implications that are listed in a series of corollaries. Of course, each of these results has implications about the coefficients of the convex meromorphic functions. For example, the absolute values of the coefficients of the function

$$H(z,w) = 1 + c_1(w)z + c_2(w)z^2 + \dots$$

are bounded by 1, i.e.,

$$|c_1(w)| = \left| \frac{1}{w} - \frac{1}{f(w)} - \left(p + \frac{1}{p}\right) \right| < 1.$$

Here we note that many coefficient inequalities for concave functions can be found in the references. We think that applications of our results to coefficient inequalities will be the subject of another paper.

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