doi: 10.17951/a.2020.74.2.31-40

ANNALES
UNIVERSITATIS MARIAE CURIE-SKŁODOWSKA
LUBLIN – POLONIA

VOL. LXXIV, NO. 2, 2020 SECTIO A 31–40

SZYMON IGNACIUK and MACIEJ PAROL

Kaplan classes of a certain family of functions

Abstract. We give the complete characterization of members of Kaplan classes of products of power functions with all zeros symmetrically distributed in $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$ and weakly monotonic sequence of powers. In this way we extend Sheil-Small's theorem. We apply the obtained result to study univalence of antiderivative of these products of power functions.

Introduction. Let $\mathcal{H}_d$ be the class of all analytic functions $f : \mathbb{D} \to \mathbb{C}$ normalized by $f(0) = 1$ and such that $f \neq 0$ in $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$. Let $\mathcal{S}$ be the class of all analytic functions $f : \mathbb{D} \to \mathbb{C}$ normalized by $f(0) = f'(0) - 1 = 0$ which are univalent and $\mathcal{C}$ be the class of functions in $\mathcal{S}$ that are close-to-convex. For $\alpha, \beta \geq 0$ the Kaplan class $K(\alpha, \beta)$ is the set of all functions $f \in \mathcal{H}_d$ satisfying one of the two equivalent conditions:

$$\arg f(re^{i\theta_2}) - \arg f(re^{i\theta_1}) \leq \beta\pi - \frac{1}{2}(\alpha - \beta)(\theta_1 - \theta_2)\,, \tag{0.1}$$

$$-\alpha\pi - \frac{1}{2}(\alpha - \beta)(\theta_1 - \theta_2) \leq \arg f(re^{i\theta_2}) - \arg f(re^{i\theta_1})\,. \tag{0.2}$$

for $0 < r < 1$ and $\theta_1 < \theta_2 < \theta_1 + 2\pi$ (see [6, pp. 32–33]).

Let $\mathbb{N}_j := \mathbb{N} \cap [1; j]$ for $j \in \mathbb{N}$ and $\mathbb{R}^+ := (0; +\infty)$. Fix $n \in \mathbb{N}$ and a weakly monotonic sequence $m : \mathbb{N}_n \to \mathbb{R}^+$. Define the functions

$$\mathbb{D} \ni z \mapsto f_k(z) := 1 - z e^{-i\frac{2\pi(k-1)}{n}} \quad \text{for} \quad k \in \mathbb{N}_n \tag{0.3}$$

2010 *Mathematics Subject Classification.* Primary: 30C15; secondary: 30C45, 30C55.

Key words and phrases. Kaplan classes, univalence, close-to-convex functions, critical points.

and

$$\mathbb{D} \ni z \mapsto P_n(z;m) := \prod_{k=1}^{n} f_k^{m_k}(z)\,. \tag{0.4}$$

We denote the class of all such functions $P_n(\cdot;m)$ by $\mathcal{P}_n$. Let us notice that the function $P_n(\cdot;m)$ is a product of power functions with all zeros symmetrically distributed in $\mathbb{T}$. In particular if $m_k \in \mathbb{N}$ for all $k \in \mathbb{N}_n$, then $P_n(\cdot;m)$ is a polynomial of degree $\sum_{k=1}^{n} m_k$ with all zeros symmetrically distributed in $\mathbb{T}$. The functions of the form $\mathbb{D} \ni z \mapsto 1 - z\mathrm{e}^{-\mathrm{i}t}$ for $t \in [0;2\pi)$ play the central role in the univalent functions theory. Due to the result of Royster [5] they are used for example as an extremal functions in many articles (see [1, 4]).

The Kaplan classes were used as the universal tool for establishing many important subclasses of $\mathcal{S}$ (see [6, p. 47]). Complete membership study even for the simplest functions from $\mathcal{H}_d$ was not carried out. For a given function it can be difficult to check if it belongs to any Kaplan class. We deduce from [2, Theorem 1.1] that $f_k \in K(1,0)$ for any $k \in \mathbb{N}_n$. Moreover, Sheil-Small proved the following theorem (see [7, p. 248]).

Theorem A (Sheil-Small). *For any polynomial $Q \in \mathcal{H}_d$ of the degree $n \in \mathbb{N} \setminus \{1\}$ with all zeros in $\mathbb{T}$, if λ is the minimal arclength between two consecutive zeros of Q, then $Q \in K(1, 2\pi/\lambda - n + 1)$.*

Theorem A can also be deduced from [3], where Jahangiri obtained a certain gap condition for polynomials with all zeros in $\mathbb{T}$. In [2], we extended the Jahangiri's result for all $\alpha, \beta \geq 0$ and effectively determined complete membership to Kaplan classes of polynomials with all zeros in $\mathbb{T}$. In this article, we extend the above results by describing complete membership to Kaplan classes of functions from the class $\mathcal{P}_n$ for all $n \in \mathbb{N}$. To this end we recall some properties of Kaplan classes (see [7, p. 245]).

Lemma B. *For all $\alpha_1, \alpha_2, \beta_1, \beta_2 \geq 0$ and $t > 0$ the following conditions hold:*

$$f \in K(\alpha_1,\beta_1) \text{ and } g \in K(\alpha_2,\beta_2) \Rightarrow fg \in K(\alpha_1+\alpha_2, \beta_1+\beta_2)\,,$$
$$f \in K(\alpha_1,\beta_1) \Rightarrow f^0 \in K(0,0)\,,$$
$$f \in K(\alpha_1,\beta_1) \iff f^t \in K(t\alpha_1, t\beta_1)\,,$$
$$f \in K(\alpha_1,\beta_1) \iff f^{-1} \in K(\beta_1, \alpha_1)\,.$$

1. Main theorems. Assume that $m_0 := 0$. For all $j \in \mathbb{N}$ and $k \in \mathbb{N}_n$ we define

$$t_j := \frac{2\pi(j-1)}{n}, \quad s := \sum_{l=1}^{n} m_l\,,$$

$$a_k := -\frac{n-k}{k}, \quad b_k := -s + \frac{n}{k}\sum_{l=n-k}^{n} m_l\,,$$

$$x_k := \sum_{l=n-k}^{n} m_l - k m_{n-k}\,, \quad y_k := (n-k)m_{n-k} - \sum_{l=1}^{n-k-1} m_l\,,$$

$$\begin{aligned}
\Pi_0 &:= \{(x,y) \in \mathbb{R}^2 : x \geq m_n\}\,,\\
\Pi_k &:= \{(x,y) \in \mathbb{R}^2 : y \geq a_k x + b_k\}\,,\\
\Pi_0' &:= \{(x,y) \in \mathbb{R}^2 : 0 \leq x < m_n\}\,,\\
\Pi_k' &:= \{(x,y) \in \mathbb{R}^2 : 0 \leq x,\ 0 \leq y < a_k x + b_k\}\,,\\
\Pi &:= \bigcap_{l=0}^{n} \Pi_l\,.
\end{aligned}$$

Now we give the complete characterization of membership of $P_n(\cdot;m)$ to Kaplan classes.

Theorem 1.1. *If $m : \mathbb{N}_n \to \mathbb{R}^+$ is weakly monotonic, then for all $\alpha, \beta \geq 0$, $P_n(\cdot;m) \in K(\alpha,\beta)$ if and only if $(\alpha,\beta) \in \Pi$.*

Proof. Without loss of generality we assume that m is a nondecreasing sequence. Since $\prod_{k=1}^{n} f_k(z) = 1 - z^n$ and $1 - z^n$ has positive real part in $\mathbb{D}$, we have

$$\prod_{k=1}^{n} f_k \in K(1,1)\,. \tag{1.1}$$

First we prove that $P_n(\cdot;m) \in K(x_k, y_k)$ for $k \in \mathbb{N}_n$. Fix $k \in \mathbb{N}_n$. Therefore,

$$\begin{aligned}
P_n(\cdot;m) &= \prod_{l=1}^{n} f_l^{m_{n-k}} \prod_{l=1}^{n} f_l^{m_l - m_{n-k}}\\
&= \prod_{l=1}^{n} f_l^{m_{n-k}} \prod_{l=1}^{n-k-1} f_l^{m_l - m_{n-k}} \prod_{l=n-k+1}^{n} f_l^{m_l - m_{n-k}}\\
&= \prod_{l=1}^{n} f_l^{m_{n-k}} \prod_{l=1}^{n-k-1} \left(\frac{1}{f_l}\right)^{m_{n-k} - m_l} \prod_{l=n-k+1}^{n} f_l^{m_l - m_{n-k}}\,.
\end{aligned}$$

By (1.1) and Lemma B, we get

$$\prod_{l=1}^{n} f_l^{m_{n-k}} \in K(m_{n-k}, m_{n-k}),$$
$$(1/f_l)^{m_{n-k}-m_l} \in K(0, m_{n-k}-m_l) \text{ for } l \in \mathbb{N}_{n-k-1}$$

and

$$f_l^{m_l-m_{n-k}} \in K(m_l-m_{n-k}, 0) \text{ for } l \in \mathbb{N}_n \setminus \mathbb{N}_{n-k}.$$

Then

$$P_n(\cdot;m) \in K\left(m_{n-k} + \sum_{l=n-k+1}^{n}(m_l - m_{n-k}), m_{n-k} + \sum_{l=1}^{n-k-1}(m_{n-k}-m_l)\right)$$

and as a consequence

$$P_n(\cdot;m) \in K(x_k, y_k). \tag{1.2}$$

By Lemma B, we obtain $f \in \Pi$.

Now we prove the second part of the theorem. Fix $k \in \mathbb{N}_{n-1}$. Consider the left side of inequality (0.1) with $\mathbb{N} \ni j \mapsto \theta_1(j) := -2\pi/n + 1/j$, $\mathbb{N} \ni j \mapsto \theta_2(j) := 2\pi - 2\pi(k+1)/n - 1/j$ and $\mathbb{N} \ni j \mapsto r_j := 1 - 1/j^2$. Therefore,

$$\begin{aligned}
&\arg(P_n(r_j e^{i\theta_2}; m)) - \arg(P_n(r_j e^{i\theta_1}; m)) \\
&= \sum_{l=1}^{n} m_l \left(\arctan\left(\frac{-r_j \sin\left(\theta_2(j) - \frac{2\pi}{n}(l-1)\right)}{1 - r_j\cos\left(\theta_2(j) - \frac{2\pi}{n}(l-1)\right)}\right)\right. \\
&\qquad\qquad \left. - \arctan\left(\frac{-r_j \sin\left(\theta_1(j) - \frac{2\pi}{n}(l-1)\right)}{1 - r_j\cos\left(\theta_1(j) - \frac{2\pi}{n}(l-1)\right)}\right)\right) \\
&= \sum_{l=1}^{n} m_l \left(\arctan\left(\frac{r_j \sin\left(\frac{2\pi}{n}(k+l) + \frac{1}{j}\right)}{1 - r_j\cos\left(\frac{2\pi}{n}(k+l) + \frac{1}{j}\right)}\right)\right. \\
&\qquad\qquad \left. - \arctan\left(\frac{r_j \sin\left(\frac{2\pi l}{n} - \frac{1}{j}\right)}{1 - r_j\cos\left(\frac{2\pi l}{n} - \frac{1}{j}\right)}\right)\right) \\
&= \sum_{l=1}^{n-k-1} m_l \left(\arctan\left(\frac{r_j \sin\left(\frac{2\pi}{n}(k+l) + \frac{1}{j}\right)}{1 - r_j\cos\left(\frac{2\pi}{n}(k+l) + \frac{1}{j}\right)}\right)\right. \\
&\qquad\qquad \left. - \arctan\left(\frac{r_j \sin\left(\frac{2\pi l}{n} - \frac{1}{j}\right)}{1 - r_j\cos\left(\frac{2\pi l}{n} - \frac{1}{j}\right)}\right)\right)
\end{aligned}$$

$$+\sum_{l=n-k+1}^{n-1} m_l \left(\arctan\left(\frac{r_j \sin\left(\frac{2\pi}{n}(k+l)+\frac{1}{j}\right)}{1-r_j\cos\left(\frac{2\pi}{n}(k+l)+\frac{1}{j}\right)}\right)\right.$$

$$\left.-\arctan\left(\frac{r_j \sin\left(\frac{2\pi l}{n}-\frac{1}{j}\right)}{1-r_j\cos\left(\frac{2\pi l}{n}-\frac{1}{j}\right)}\right)\right)$$

$$+m_{n-k}\left(\arctan\left(\frac{\left(1-\frac{1}{j^2}\right)\sin\left(\frac{1}{j}\right)}{1-\left(1-\frac{1}{j^2}\right)\cos\left(\frac{1}{j}\right)}\right)\right.$$

$$\left.+\arctan\left(\frac{r_j \sin\left(\frac{2\pi k}{n}+\frac{1}{j}\right)}{1-r_j\cos\left(\frac{2\pi k}{n}+\frac{1}{j}\right)}\right)\right)$$

$$+m_n\left(\arctan\left(\frac{r_j \sin\left(\frac{2\pi k}{n}+\frac{1}{j}\right)}{1-r_j\cos\left(\frac{2\pi k}{n}+\frac{1}{j}\right)}\right)\right.$$

$$\left.+\arctan\left(\frac{\left(1-\frac{1}{j^2}\right)\sin\left(\frac{1}{j}\right)}{1-\left(1-\frac{1}{j^2}\right)\cos\left(\frac{1}{j}\right)}\right)\right)$$

and as a consequence

$$\lim_{j\to+\infty}(\arg(P_n(r_j e^{i\theta_2};m))-\arg(P_n(r_j e^{i\theta_1};m)))$$

$$=\sum_{l=1}^{n-k-1} m_l\left(\arctan\left(\frac{\sin\left(\frac{2\pi}{n}(k+l)\right)}{1-\cos\left(\frac{2\pi}{n}(k+l)\right)}\right)-\arctan\left(\frac{\sin\left(\frac{2\pi l}{n}\right)}{1-\cos\left(\frac{2\pi l}{n}\right)}\right)\right)$$

$$+\sum_{l=n-k+1}^{n-1} m_l\left(\arctan\left(\frac{\sin\left(\frac{2\pi}{n}(k+l)\right)}{1-\cos\left(\frac{2\pi}{n}(k+l)\right)}\right)-\arctan\left(\frac{\sin\left(\frac{2\pi l}{n}\right)}{1-\cos\left(\frac{2\pi l}{n}\right)}\right)\right)$$

$$+m_{n-k}\left(\frac{\pi}{2}+\arctan\left(\frac{\sin\left(\frac{2\pi k}{n}\right)}{1-\cos\left(\frac{2\pi k}{n}\right)}\right)\right)$$

$$+m_n\left(\arctan\left(\frac{\sin\left(\frac{2\pi k}{n}\right)}{1-\cos\left(\frac{2\pi k}{n}\right)}\right)+\frac{\pi}{2}\right).$$

By the trigonometric identity:

$$\frac{\sin x}{1-\cos x}=\tan\left(\frac{\pi}{2}-\frac{x}{2}\right)\quad\text{for}\quad x\in\mathbb{R}\setminus\bigcup_{j\in\mathbb{Z}}\{2j\pi\}$$

we get

$$\begin{aligned}
&\lim_{j\to+\infty}(\arg(P_n(r_j\mathrm{e}^{\mathrm{i}\theta_2};m))-\arg(P_n(r_j\mathrm{e}^{\mathrm{i}\theta_1};m)))\\
&=\sum_{l=1}^{n-k-1} m_l\left(\arctan\left(\tan\left(\frac{\pi}{2}-\frac{\pi}{n}(k+l)\right)\right)-\arctan\left(\tan\left(\frac{\pi}{2}-\frac{\pi l}{n}\right)\right)\right)\\
&\quad+\sum_{l=n-k+1}^{n-1} m_l\left(\arctan\left(\tan\left(\frac{\pi}{2}-\frac{\pi}{n}(k+l)\right)\right)-\arctan\left(\tan\left(\frac{\pi}{2}-\frac{\pi l}{n}\right)\right)\right)\\
&\quad+m_{n-k}\left(\frac{\pi}{2}+\arctan\left(\tan\left(\frac{\pi}{2}-\frac{\pi k}{n}\right)\right)\right)\\
&\quad+m_n\left(\arctan\left(\tan\left(\frac{\pi}{2}-\frac{\pi k}{n}\right)\right)+\frac{\pi}{2}\right).
\end{aligned}$$

Since

$$\begin{aligned}
&\frac{\pi}{2}-\frac{\pi}{n}(k+l)\in\left(-\frac{\pi}{2};\frac{\pi}{2}\right)\quad\text{for } l\in\mathbb{N}_{n-k-1},\\
&\frac{\pi}{2}-\frac{\pi}{n}(k+l)\in\left(-\frac{3\pi}{2};-\frac{\pi}{2}\right)\quad\text{for } l\in\mathbb{N}_{n-1}\setminus\mathbb{N}_{n-k}
\end{aligned}$$

and

$$\frac{\pi}{2}-\frac{\pi l}{n}\in\left(-\frac{\pi}{2};\frac{\pi}{2}\right)\quad\text{for } l\in\mathbb{N}_{n-1},$$

we have

$$\begin{aligned}
&\lim_{j\to+\infty}(\arg(P_n(r_j\mathrm{e}^{\mathrm{i}\theta_2};m))-\arg(P_n(r_j\mathrm{e}^{\mathrm{i}\theta_1};m)))\\
&=\sum_{l=1}^{n-k-1} m_l\left(\frac{\pi}{2}-\frac{\pi}{n}(k+l)-\frac{\pi}{2}+\frac{\pi l}{n}\right)\\
&\quad+\sum_{l=n-k+1}^{n-1} m_l\left(\frac{3\pi}{2}-\frac{\pi}{n}(k+l)-\frac{\pi}{2}+\frac{\pi l}{n}\right)\\
&\quad+m_{n-k}\left(\frac{\pi}{2}+\frac{\pi}{2}-\frac{\pi k}{n}\right)+m_n\left(\frac{\pi}{2}-\frac{\pi k}{n}+\frac{\pi}{2}\right)\\
&=-\frac{\pi k s}{n}+\pi\sum_{l=n-k}^{n} m_l\,.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\lim_{j\to+\infty}\left(\beta\pi+\frac{1}{2}(\alpha-\beta)(\theta_2(j)-\theta_1(j))\right)&=\beta\pi+(\alpha-\beta)\left(\pi-\frac{\pi k}{n}\right)\\
&=\beta\pi\frac{k}{n}+\alpha\pi\frac{n-k}{n}\,,
\end{aligned}$$

from which we deduce that inequality (0.1) does not hold for

$$\beta < -\frac{n-k}{k}\alpha - s + \frac{n}{k}\sum_{l=n-k}^{n} m_l$$

and as a consequence $P_n(\cdot;m) \notin K(\alpha,\beta)$ for $(\alpha,\beta) \in \Pi'_k$. Hence

$$P_n(\cdot;m) \notin \bigcup_{k=1}^{n-1} \Pi'_k\,. \tag{1.3}$$

Now we prove that $P_n(\cdot;m) \notin \Pi'_0$. Consider the right side of inequality (0.2) with $\mathbb{N} \ni j \mapsto \theta_1(j) := 2\pi(n-1)/n+1/j$, $\mathbb{N} \ni j \mapsto \theta_2(j) := 2\pi(n-1)/n-1/j$ and $\mathbb{N} \ni j \mapsto r_j := 1-1/j^2$. Therefore,

$$\begin{aligned}
&\arg(P_n(r_j e^{i\theta_2};m)) - \arg(P_n(r_j e^{i\theta_1};m))\\
&= -2m_n \arctan\left(\frac{\left(1-\frac{1}{j^2}\right)\sin\left(\frac{1}{j}\right)}{1-\left(1-\frac{1}{j^2}\right)\cos\left(\frac{1}{j}\right)}\right)\\
&\quad+\sum_{l=1}^{n-1} m_l\left(\arctan\left(\frac{-r_j\sin\left(\frac{2\pi l}{n}-\frac{1}{j}\right)}{1-r_j\cos\left(\frac{2\pi l}{n}-\frac{1}{j}\right)}\right)\right.\\
&\qquad\qquad\left.-\arctan\left(\frac{-r_j\sin\left(\frac{2\pi l}{n}+\frac{1}{j}\right)}{1-r_j\cos\left(\frac{2\pi l}{n}+\frac{1}{j}\right)}\right)\right)
\end{aligned}$$

and as a consequence

$$\lim_{j\to+\infty}\big(\arg\big(P_n\big(r_j e^{i\theta_2};m\big)\big) - \arg\big(P_n\big(r_j e^{i\theta_1};m\big)\big)\big) = -m_n\pi\,.$$

On the other hand, we have

$$\lim_{j\to+\infty}\left(-\alpha\pi + \frac{1}{2}(\alpha-\beta)(\theta_2(j)-\theta_1(j))\right) = -\alpha\pi\,,$$

from which we deduce that inequality (0.2) does not hold for $\alpha < m_n$ and as a consequence $P_n(\cdot;m) \notin K(\alpha,\beta)$ for $(\alpha,\beta) \in \Pi'_0$. From this and (1.3) we obtain

$$P_n(\cdot;m) \notin \bigcup_{k=0}^{n} \Pi'_k\,. \qquad \square$$

By Theorem A, if $m_k = 1$ for all $k \in \mathbb{N}_n$, then $P_n(\cdot;m) \in K(1,1)$. Theorem 1.1 is an extension of Theorem A for functions from the class $\mathcal{P}_n$. Moreover, in the first part of the proof of Theorem 1.1 we obtain nontrivial, interesting factorization of $P_n(\cdot;m)$ (cf. [7, p. 246]).

Remark 1.2. Let us notice that for a nondecreasing sequence $m : \mathbb{N}_n \to \mathbb{R}^+$ points (x_k, y_k) for $k \in \mathbb{N}_n$ are all vertices of the set Π. Analogously we can effectively determine vertices of Π if m is nonincreasing.

Let $\varphi_q' := (P_n(\cdot; m))^q$ for any $q \in \mathbb{R}$ such that $\varphi_q(0) = 0$. The complete characterization of functions $P_n(\cdot; m)$ belonging to Kaplan classes obtained in Theorem 1.1 can be used to study univalence of φ_q.

Theorem 1.3. *If $m : \mathbb{N}_n \to \mathbb{R}^+$ is nondecreasing sequence, then for any $n \in \mathbb{N}$, $k \in \mathbb{N}_{n-1}$ and $q \geq 0$ the following implications hold:*

$$s \geq nm_{n-1} - 2m_n \Longrightarrow \left(\varphi_q \in \mathcal{C} \iff q \in \left[0; \frac{1}{m_n}\right] \right), \tag{1.4}$$

$$\begin{aligned} &s \in \left[(n+2k)m_{n-k-1} - 2\sum_{l=n-k}^{n} m_l; (n+2k)m_{n-k} - 2\sum_{l=n-k}^{n} m_l \right) \\ &\Longrightarrow \left(\varphi_q \in \mathcal{C} \iff q \in \left[0; \frac{n+2k}{n\sum_{l=n-k}^{n} m_l - ks}\right] \right). \end{aligned} \tag{1.5}$$

Proof. Let m be a nondecreasing sequence. Fix $q \geq 0$. First we prove (1.4). If $s \geq nm_{n-1} - 2m_n$, then $y_1 \leq 3x_1$. This and Theorem 1.1 imply that $P_n(\cdot; m) \in K(m_n, 3m_n)$ and for any $\alpha \in [0; m_n)$, $P_n(\cdot; m) \notin K(\alpha, 3\alpha)$. Therefore, $(P_n(\cdot; m))^q \in K(1, 3)$ if and only if $q \in [0; 1/m_n]$.

Now we prove (1.5). Fix $k \in \mathbb{N}_{n-1}$. Assume that

$$s \in \left[(n+2k)m_{n-k-1} - 2\sum_{l=n-k}^{n} m_l; (n+2k)m_{n-k} - 2\sum_{l=n-k}^{n} m_l \right).$$

Then

$$\begin{cases} y_l > 3x_l & \text{for } l \in \mathbb{N}_k\,, \\ y_l \leq 3x_l & \text{for } l \in \mathbb{N}_n \setminus \mathbb{N}_k\,. \end{cases}$$

This and Theorem 1.1 imply that

$$P_n(\cdot; m) \in K\left(\frac{n}{n+2k} \sum_{l=n-k}^{n} m_l - ks, \frac{3n}{n+2k} \sum_{l=n-k}^{n} m_l - ks \right)$$

and for any

$$\alpha \in \left[0; \frac{n}{n+2k} \sum_{l=n-k}^{n} m_l - ks \right),$$

$P_n(\cdot; m) \notin K(\alpha, 3\alpha)$, which leads to (1.5). □

Theorem 1.4. *If $m : \mathbb{N}_n \to \mathbb{R}^+$ is a nondecreasing sequence, then for any $n \in \mathbb{N}$, $k \in \mathbb{N}_{n-1}$ and $q < 0$ the following implications hold:*

$$s \geq \frac{2}{3}m_n + nm_{n-1} \Longrightarrow \left(\varphi_q \in \mathcal{C} \iff q \in \left[-\frac{3}{m_n}; 0 \right) \right), \tag{1.6}$$

$$s \in \left[\left(n - \frac{2}{3}k \right) m_{n-k-1} + \frac{2}{3} \sum_{l=n-k}^{n} m_l ; \left(n - \frac{2}{3}k \right) m_{n-k} + \frac{2}{3} \sum_{l=n-k}^{n} m_l \right)$$

(1.7)

$$\Longrightarrow \left(\varphi_q \in \mathcal{C} \iff q \in \left[\frac{3n-2k}{ks - n\sum_{l=n-k}^{n} m_l} ; 0 \right) \right).$$

Proof. Let m be a nondecreasing sequence. Fix $q < 0$. First we prove (1.6). If $s \geq 2/3m_n + nm_{n-1}$, then $3y_1 \leq x_1$. This and Theorem 1.1 imply that $P_n(\cdot; m) \in K(m_n, 1/3m_n)$ and for any $\alpha \in [0; m_n)$, $P_n(\cdot; m) \notin K(\alpha, 1/3\alpha)$. Therefore, $(P_n(\cdot; m)) \in K(1,3)$ if and only if $q \in [-3/m_n; 0)$.

Now we prove (1.7). Fix $k \in \mathbb{N}_{n-1}$. Assume that

$$s \in \left[\left(n - \frac{2}{3}k \right) m_{n-k-1} + \frac{2}{3} \sum_{l=n-k}^{n} m_l ; \left(n - \frac{2}{3}k \right) m_{n-k} + \frac{2}{3} \sum_{l=n-k}^{n} m_l \right).$$

Then

$$\begin{cases} 3y_l > x_l & \text{for } l \in \mathbb{N}_k \,, \\ 3y_l \leq x_l & \text{for } l \in \mathbb{N}_n \setminus \mathbb{N}_k \,. \end{cases}$$

This and Theorem 1.1 imply that

$$P_n(\cdot; m) \in K \left(\frac{3n}{3n-2k} \sum_{l=n-k}^{n} m_l - ks, \frac{n}{3n-2k} \sum_{l=n-k}^{n} m_l - ks \right)$$

and for any

$$\alpha \in \left[0; \frac{n}{3n-2k} \sum_{l=n-k}^{n} m_l - ks \right),$$

$P_n(\cdot; m) \notin K(3\alpha, \alpha)$, which leads to (1.7). □

References

[1] Goodman, A. W., *Univalent functions. Vol. II*, Mariner Pub. Co., Inc., Tampa, Florida, 1983.

[2] Ignaciuk, S., Parol, M., *Zeros of complex polynomials and Kaplan classes*, Anal. Math. **46** (2020), 769–779.

[3] Jahangiri, M., *A gap condition for the zeroes of certain polynomials in Kaplan classes* $K(\alpha, \beta)$, Mathematika **34** (1987), 53–63.

[4] Kim, Y. J., Merkes, E. P., *On certain convex sets in the space of locally schlicht functions*, Trans. Amer. Math. Soc. **196** (1974), 217–224.

[5] Royster, W. C., *On the univalence of a certain integral*, Michigan Math. J. **12** (1965), 385–387.

[6] Ruscheweyh, S., *Convolutions in Geometric Function Theory*, Séminaire de Math. Sup. **83**, Presses de l'Université de Montréal, Montréal, 1982.

[7] Sheil-Small, T., *Complex Polynomials*, Cambridge University Press, Cambridge, 2002.

Szymon Ignaciuk
Department of Applied Mathematics and Computer Science
University of Life Sciences in Lublin
ul. Głęboka 28
20-612 Lublin
Poland
e-mail: szymon.ignaciuk@up.lublin.pl

Maciej Parol
Department of Mathematical Analysis
The John Paul II Catholic University of Lublin
ul. Konstantynów 1 H
20-708 Lublin
Poland
e-mail: mparol@kul.lublin.pl

Received July 10, 2020