doi: 10.17951/a.2021.75.2.13-29

ANNALES
UNIVERSITATIS MARIAE CURIE-SKŁODOWSKA
LUBLIN – POLONIA

VOL. LXXV, NO. 2, 2021 SECTIO A 13–29

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Generalized perturbed Ostrowski-type inequalities

Abstract. In this paper, we present new perturbed inequalities of Ostrowski-type, for twice differentiable functions with absolutely continuous first derivative and second-order derivative in some L^p-space for $1 \le p \le \infty$.

1. Introduction. In 1938, the Ukrainian mathematician Alexander Markowich Ostrowski (1893–1986) presented a new inequality in [25]. This inequality is now known as Ostrowski's inequality. Many researchers have written papers about generalizations of Ostrowski's inequality in the past few decades, including [1, 10, 12, 21, 22]. Ostrowski's inequality has proved to be a huge and remarkable tool for the enlargement of several branches of mathematics. Inequalities involving integrals, which create bounds in physical quantities, are of great significance in the sense that these kinds of inequalities are not only used in integral approximation theory, operator theory, nonlinear analysis, numerical integration, stochastic analysis, information theory, statistics, and probability theory, but we may also see their applications in various fields such as biological sciences, engineering, and physics. For some recent contributions to the study of Ostrowski's inequality to different subject areas, we refer to [2, 4, 5, 13–18, 23, 24, 26].

In this paper, we give some new perturbed inequalities of Ostrowski type for second-order differentiable mappings, which generalise and refine the

2020 *Mathematics Subject Classification.* Primary 26D10; Secondary 26D20, 26D99.

Key words and phrases. Ostrowski's inequality, perturbed inequality, twice differentiable.

inequalities that are presented in [6–9, 11, 19, 20, 27, 28] and [3, Theorem 20]. One such inequality is extracted from [11, Theorem 4], and it reads as follows.

Lemma 1.1. *Let $\varphi : [\alpha, \beta] \to \mathbb{R}$ be a mapping whose first-order derivative is absolutely continuous in $[\alpha, \beta]$ and $\varphi'' \in \mathrm{L}^\infty(\alpha, \beta)$. Then*

$$\begin{aligned} &\left| \int_\alpha^\beta \varphi(\tau)\mathrm{d}\tau - \frac{1}{2}(\beta-\alpha)\left[\varphi(y) + \frac{\varphi(\alpha)+\varphi(\beta)}{2} - \left(y - \frac{\alpha+\beta}{2}\right)\varphi'(y)\right]\right| \\ &\qquad \le \left[\frac{(\beta-\alpha)^3}{48} + \frac{1}{3}\left|\left(y - \frac{\alpha+\beta}{2}\right)\right|^3\right] \|\varphi''\|_\infty \end{aligned} \tag{1.1}$$

for all $y \in [\alpha, \beta]$.

2. Auxiliary result and notation. Let $p \ge 1$. We introduce the space Φ_p of functions $\varphi : [\alpha, \beta] \to \mathbb{R}$ such that φ' is absolutely continuous in $[\alpha, \beta]$ and $\varphi'' \in \mathrm{L}^p(\alpha, \beta)$. Our main tool is the following identity.

Theorem 2.1. *Suppose*

$$\alpha < \beta, \quad \rho := \frac{\beta-\alpha}{2}, \quad \mu := \frac{\alpha+\beta}{2}, \quad |\kappa| \le \rho, \quad 0 \le \ell \le \rho. \tag{2.1}$$

Put

$$\alpha_1 := \alpha, \quad \alpha_2 := \mu - \ell, \quad \alpha_3 := \mu + \ell, \quad \alpha_4 := \beta \tag{2.2}$$

and define $K : [\alpha, \beta] \to \mathbb{R}$ by

$$K(\tau; \kappa, \ell) = \frac{1}{2}\begin{cases} k_1^2(\tau) & \text{if } \tau \in [\alpha_1, \alpha_2), \\ k_2^2(\tau) & \text{if } \tau \in [\alpha_2, \alpha_3), \\ k_3^2(\tau) & \text{if } \tau \in [\alpha_3, \alpha_4], \end{cases}$$

where

$$k_1(\tau) := \tau - \mu + \kappa, \quad k_2(\tau) := \tau - \mu, \quad k_3(\tau) := \tau - \mu - \kappa.$$

If $\varphi \in \Phi_1$, then the identity

$$\int_\alpha^\beta \varphi(\tau)\mathrm{d}\tau - A(\kappa, \ell) = \int_\alpha^\beta K(\tau; \kappa, \ell)\varphi''(\tau)\mathrm{d}\tau \tag{2.3}$$

holds, where

$$\begin{aligned} A(\kappa, \ell) := &(\rho-\kappa)\left(\varphi(\alpha)+\varphi(\beta)\right) + \frac{(\rho-\kappa)^2}{2}\left(\varphi'(\alpha)-\varphi'(\beta)\right) \\ &+ \kappa\left(\varphi(\mu-\ell)+\varphi(\mu+\ell)\right) \\ &+ \kappa\left(\ell - \frac{\kappa}{2}\right)\left(\varphi'(\mu-\ell)-\varphi'(\mu+\ell)\right). \end{aligned} \tag{2.4}$$

Proof. Note that our assumptions guarantee that

$$\alpha_1 \le \alpha_2 \le \alpha_3 \le \alpha_4$$

and

$$\begin{aligned} -k_1(\alpha_1) &= k_3(\alpha_4) = \rho - \kappa, \\ k_1(\alpha_2) &= -k_3(\alpha_3) = \kappa - \ell, \\ -k_2(\alpha_2) &= k_2(\alpha_3) = \ell. \end{aligned}$$

See Figure 2.1 for a graph of the kernel K. We have

$$k_i' = 1 \quad \text{and} \quad \left(\frac{k_i^2}{2}\right)' = k_i \quad \text{for} \quad i \in \{1,2,3\}.$$

Using the three formulas

$$\begin{aligned} \int_{\alpha_i}^{\alpha_{i+1}} \frac{k_i^2(\tau)}{2}\varphi''(\tau)\mathrm{d}\tau &= \frac{k_i^2(\tau)}{2}\varphi'(\tau)\Big|_{\alpha_i}^{\alpha_{i+1}} - \int_{\alpha_i}^{\alpha_{i+1}} k_i(\tau)\varphi'(\tau)\mathrm{d}\tau \\ &= \frac{k_i^2(\tau)}{2}\varphi'(\tau)\Big|_{\alpha_i}^{\alpha_{i+1}} - k_i(\tau)\varphi(\tau)\Big|_{\alpha_i}^{\alpha_{i+1}} + \int_{\alpha_i}^{\alpha_{i+1}} k_i'(\tau)\varphi(\tau)\mathrm{d}\tau \\ &= \frac{k_i^2(\tau)}{2}\varphi'(\tau) - k_i(\tau)\varphi(\tau)\Big|_{\alpha_i}^{\alpha_{i+1}} + \int_{\alpha_i}^{\alpha_{i+1}} \varphi(\tau)\mathrm{d}\tau, \quad i \in \{1,2,3\}, \end{aligned}$$

we find

$$\begin{aligned} &\int_\alpha^\beta K(\tau;\gamma,\vartheta)\varphi''(\tau)\mathrm{d}\tau = \sum_{i=1}^3 \int_{\alpha_i}^{\alpha_{i+1}} \frac{k_i^2(\tau)}{2}\varphi''(\tau)\mathrm{d}\tau \\ &= \sum_{i=1}^3 \left(\frac{k_i^2(\tau)}{2}\varphi'(\tau) - k_i(\tau)\varphi(\tau)\Big|_{\alpha_i}^{\alpha_{i+1}} + \int_{\alpha_i}^{\alpha_{i+1}} \varphi(\tau)\mathrm{d}\tau \right) \\ &= \int_\alpha^\beta \varphi(\tau)\mathrm{d}\tau + \frac{(\kappa-\ell)^2}{2}\varphi'(\alpha_2) - (\kappa-\ell)\varphi(\alpha_2) - \frac{(\rho-\kappa)^2}{2}\varphi'(\alpha) \\ &\quad - (\rho-\kappa)\varphi(\alpha) + \frac{\ell^2}{2}\varphi'(\alpha_3) - \ell\varphi(\alpha_3) - \frac{\ell^2}{2}\varphi'(\alpha_2) - \ell\varphi(\alpha_2) \\ &\quad + \frac{(\rho-\kappa)^2}{2}\varphi'(\beta) - (\rho-\kappa)\varphi(\beta) - \frac{(\kappa-\ell)^2}{2}\varphi'(\alpha_3) - (\kappa-\ell)\varphi(\alpha_3) \\ &= \int_\alpha^\beta \varphi(\tau)\mathrm{d}\tau - (\rho-\kappa)\left(\varphi(\alpha)+\varphi(\beta)\right) - \frac{(\rho-\kappa)^2}{2}\left(\varphi'(\alpha)-\varphi'(\beta)\right) \\ &\quad - \kappa\left(\varphi(\alpha_2)+\varphi(\alpha_3)\right) - \frac{\ell^2-(\kappa-\ell)^2}{2}\left(\varphi'(\alpha_2)-\varphi'(\alpha_3)\right) \\ &= \int_\alpha^\beta \varphi(\tau)\mathrm{d}\tau - A(\kappa,\ell), \end{aligned}$$

where we used (2.2) in the last step. This shows (2.3). □

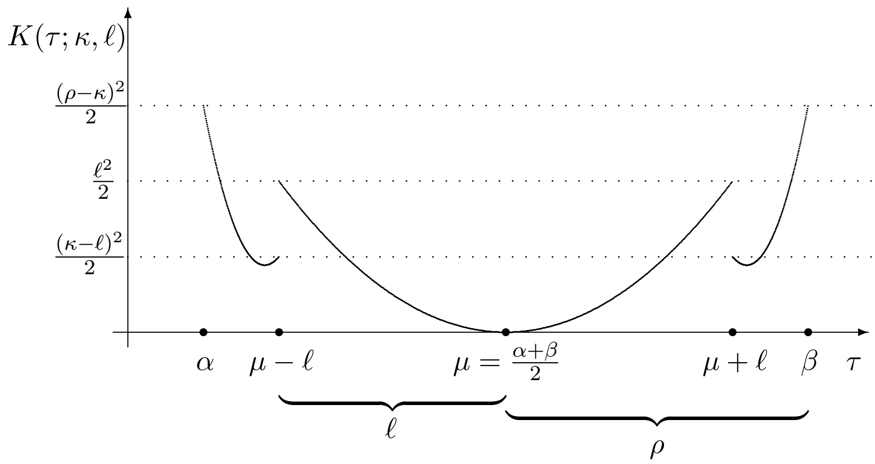

FIGURE 2.1. The kernel K

3. Main results. Our main result is a new perturbed Ostrowski-type inequality, and it reads as follows.

Theorem 3.1. *Assume* $1 \le p \le \infty$, (2.1), *and* (2.4). *If* $\varphi \in \Phi_p$, *then*

$$\left| \int_\alpha^\beta \varphi(\tau)\mathrm{d}\tau - A(\kappa,\ell) \right| \le E_p(\kappa,\ell) \left\|\varphi''\right\|_p , \tag{3.1}$$

where $E_p(\kappa,\ell)$ *is defined by*

$$\begin{cases} \frac{1}{3}\left((\rho-\kappa)^3 + (\kappa-\ell)^3 + \ell^3\right) & \text{if } p=\infty, \\ \frac{1}{2}\left(\frac{(\rho-\kappa)^{\frac{3p-1}{p-1}} + (\kappa-\ell)|\kappa-\ell|^{\frac{2p}{p-1}} + \ell^{\frac{3p-1}{p-1}}}{\frac{3}{2}+\frac{1}{p-1}}\right)^{\frac{p-1}{p}} & \text{if } 1<p<\infty, \\ \frac{1}{2}\left(\max\left\{\rho-\kappa, |\kappa-\ell|, \ell\right\}\right)^2 & \text{if } p=1. \end{cases} \tag{3.2}$$

Proof. We use the notation from Theorem 2.1. By (2.3), the triangle inequality, and the nonnegativity of K, we obtain

$$\begin{aligned} \left| \int_\alpha^\beta \varphi(\tau)\mathrm{d}\tau - A(\kappa,\ell) \right| &= \left| \int_\alpha^\beta K(\tau;\kappa,\ell)\varphi''(\tau)\mathrm{d}\tau \right| \\ &\le \int_\alpha^\beta K(\tau;\kappa,\ell) \left|\varphi''(\tau)\right| \mathrm{d}\tau. \end{aligned} \tag{3.3}$$

First, we consider the case $p=\infty$. By the assumption,

$$\left\|\varphi''\right\|_\infty = \sup_{\alpha\le\tau\le\beta} \left|\varphi''(\tau)\right| < \infty.$$

By (3.3), we have

$$\left|\int_\alpha^\beta \varphi(\tau)\mathrm{d}\tau - A(\kappa,\ell)\right| \le \|\varphi''\|_\infty \int_\alpha^\beta K(\tau;\kappa,\ell)\mathrm{d}\tau \tag{3.4}$$

and

$$\begin{aligned}\int_\alpha^\beta K(\tau;\kappa,\ell)\mathrm{d}\tau &= \sum_{i=1}^{3}\int_{\alpha_i}^{\alpha_{i+1}} \frac{k_i^2(\tau)}{2}\mathrm{d}\tau = \sum_{i=1}^{3}\frac{k_i^3(\tau)}{6}\Big|_{\alpha_i}^{\alpha_{i+1}} \\ &= \frac{1}{6}\left((\kappa-\ell)^3 + (\rho-\kappa)^3 + \ell^3 + \ell^3 + (\rho-\kappa)^3 + (\kappa-\ell)^3\right) \\ &= \frac{1}{3}\left((\rho-\kappa)^3 + (\kappa-\ell)^3 + \ell^3\right) = E_\infty(\kappa,\ell),\end{aligned}$$

which, together with (3.4), proves (3.1) in this case. Next, we consider the case $1 < p < \infty$. By the assumption,

$$\|\varphi''\|_p = \left(\int_\alpha^\beta |\varphi''(\tau)|^p \,\mathrm{d}\tau\right)^{\frac{1}{p}} < \infty.$$

By (3.3) and Hölder's inequality, with $q := p/(p-1)$, we have

$$\left|\int_\alpha^\beta \varphi(\tau)\mathrm{d}\tau - A(\kappa,\ell)\right| \le \|\varphi''\|_p \left(\int_\alpha^\beta K^q(\tau;\kappa,\ell)\mathrm{d}\tau\right)^{\frac{1}{q}} \tag{3.5}$$

and

$$\begin{aligned}&\int_\alpha^\beta K^q(\tau;\kappa,\ell)\mathrm{d}\tau = \sum_{i=1}^{3}\int_{\alpha_i}^{\alpha_{i+1}} \frac{\left(k_i^2(\tau)\right)^q}{2^q}\mathrm{d}\tau \\ &= \frac{\int_\alpha^{\mu-\ell}\left((\tau-\mu+\kappa)^2\right)^q\mathrm{d}\tau + \int_{\mu-\ell}^{\mu+\ell}\left((\tau-\mu)^2\right)^q\mathrm{d}\tau + \int_{\mu+\ell}^{\beta}\left((\tau-\mu-\kappa)^2\right)^q\mathrm{d}\tau}{2^q} \\ &= \frac{1}{2^q}\left\{\int_{\kappa-\rho}^{\kappa-\ell}\left(s^2\right)^q\mathrm{d}s + \int_{-\ell}^{\ell}\left(s^2\right)^q\mathrm{d}s + \int_{\ell-\kappa}^{\rho-\kappa}\left(s^2\right)^q\mathrm{d}s\right\} \\ &= \frac{1}{2^q}\left\{\int_{\kappa-\rho}^{\kappa-\ell}|s|^{2q}\mathrm{d}s + \int_{-\ell}^{0}|s|^{2q}\mathrm{d}s + \int_{0}^{\ell}|s|^{2q}\mathrm{d}s + \int_{\ell-\kappa}^{\rho-\kappa}|s|^{2q}\mathrm{d}s\right\} \\ &= \frac{1}{2^{q-1}}\left\{\int_0^{\ell}|s|^{2q}\mathrm{d}s + \int_{\ell-\kappa}^{\rho-\kappa}|s|^{2q}\mathrm{d}s\right\} \\ &= \frac{1}{2^{q-1}}\left\{\int_0^{\ell}|s|^{2q}\mathrm{d}s + \int_0^{\rho-\kappa}|s|^{2q}\mathrm{d}s + \int_{\ell-\kappa}^{0}|s|^{2q}\mathrm{d}s\right\} \\ &= \frac{1}{2^{q-1}}\left\{\int_0^{\ell}s^{2q}\mathrm{d}s + \int_0^{\rho-\kappa}s^{2q}\mathrm{d}s + \int_0^{\kappa-\ell}|s|^{2q}\mathrm{d}s\right\} \\ &= \frac{1}{\left(q+\frac{1}{2}\right)2^q}\left(\ell^{2q+1} + (\rho-\kappa)^{2q+1} + (\kappa-\ell)|\kappa-\ell|^{2q}\right) = E_p^q(\kappa,\ell),\end{aligned}$$

which, together with (3.5), proves (3.1) in this case. Finally, we consider the case $p = 1$. By the assumption,

$$\left\|\varphi''\right\|_1 = \left(\int_\alpha^\beta \left|\varphi''(\tau)\right| \mathrm{d}\tau\right) < \infty.$$

By (3.3), we have

$$\left|\int_\alpha^\beta \varphi(\tau)\mathrm{d}\tau - A(\kappa,\ell)\right| \le \left\|\varphi''\right\|_1 \sup_{\alpha\le\tau\le\beta} K(\tau;\kappa,\ell) \tag{3.6}$$

and (observe (2.1), see also Figure 2.1)

$$\begin{aligned}\sup_{\alpha\le\tau\le\beta} K(\tau;\kappa,\ell) &= \max\left\{\frac{(\rho-\kappa)^2}{2}, \frac{(\kappa-\ell)^2}{2}, \frac{\ell^2}{2}\right\}\\ &= \frac{1}{2}\left(\max\left\{|\rho-\kappa|, |\kappa-\ell|, |\ell|\right\}\right)^2\\ &= \frac{1}{2}\left(\max\left\{\rho-\kappa, \kappa-\ell, \ell-\kappa, \ell\right\}\right)^2\\ &= \frac{1}{2}\left(\max\left\{\rho-\kappa, \kappa-\ell, \ell\right\}\right)^2\\ &= E_1(\kappa,\ell),\end{aligned}$$

which, together with (3.6), proves (3.1) in this case. □

4. Applications. Now we discuss some special cases in each of the situations $p = \infty$, $p = 2$, $1 < p < \infty$, $p = 1$.

4.1. The case $p = \infty$. We start by restating Theorem 3.1 for the case $p = \infty$.

Theorem 4.1. *Assume* (2.1) *and* (2.4). *If* $\varphi \in \Phi_\infty$, *then*

$$\left|\int_\alpha^\beta \varphi(\tau)\mathrm{d}\tau - A(\kappa,\ell)\right| \le E_\infty(\kappa,\ell)\left\|\varphi''\right\|_\infty, \tag{4.1}$$

where

$$E_\infty(\kappa,\ell) = \frac{(\beta-\alpha)^3}{24} - \kappa(\rho-\ell)(\rho+\ell-\kappa).$$

Proof. The calculation

$$\begin{aligned}E_\infty(\kappa,\ell) &= \frac{1}{3}\left((\rho-\kappa)^3 + (\kappa-\ell)^3 + \ell^3\right)\\ &= \frac{1}{3}\left(\rho^3 - 3\rho^2\kappa + 3\rho\kappa^2 - 3\kappa^2\ell + 3\kappa\ell^2\right)\\ &= \frac{(\beta-\alpha)^3}{24} - \rho^2\kappa + \rho\kappa^2 - \kappa^2\ell + \kappa\ell^2\\ &= \frac{(\beta-\alpha)^3}{24} - \kappa(\rho-\ell)(\rho+\ell-\kappa)\end{aligned}$$

together with Theorem 3.1 completes the proof. □

Setting $\kappa = \rho$ in Theorem 4.1, we obtain the following perturbed form of [7, Theorem 2.1].

Corollary 4.2. *Assume* (2.1). *If* $\varphi \in \Phi_\infty$, *then*

$$\left|\int_\alpha^\beta \varphi(\tau)\mathrm{d}\tau - A(\rho,\ell)\right| \le E_\infty(\rho,\ell)\left\|\varphi''\right\|_\infty,$$

where

$$A(\rho,\ell) = \rho\left(\varphi(\mu-\ell)+\varphi(\mu+\ell)\right) + \rho\left(\ell - \frac{\rho}{2}\right)\left(\varphi'(\mu-\ell) - \varphi'(\mu+\ell)\right) \tag{4.2}$$

and

$$E_\infty(\rho,\ell) = \frac{(\beta-\alpha)^3}{24} - \rho\ell(\rho-\ell).$$

Setting $\kappa = \frac{\rho}{2}$ in Theorem 4.1, we obtain the following result.

Corollary 4.3. *Assume* (2.1). *If* $\varphi \in \Phi_\infty$, *then*

$$\left|\int_\alpha^\beta \varphi(\tau)\mathrm{d}\tau - A\left(\frac{\rho}{2},\ell\right)\right| \le E_\infty\left(\frac{\rho}{2},\ell\right)\left\|\varphi''\right\|_\infty,$$

where

$$\begin{aligned} A\left(\frac{\rho}{2},\ell\right) &= \frac{\rho}{2}\left(\varphi(\alpha)+\varphi(\beta)\right) + \frac{\rho^2}{8}\left(\varphi'(\alpha)-\varphi'(\beta)\right) \\ &\quad + \frac{\rho}{2}\left(\varphi(\mu-\ell)+\varphi(\mu+\ell)\right) \\ &\quad + \frac{\rho}{2}\left(\ell-\frac{\rho}{4}\right)\left(\varphi'(\mu-\ell)-\varphi'(\mu+\ell)\right) \end{aligned} \tag{4.3}$$

and

$$E_\infty\left(\frac{\rho}{2},\ell\right) = \frac{(\beta-\alpha)^3}{96} - \frac{\rho\ell}{2}\left(\frac{\rho}{2}-\ell\right).$$

It may be seen that (4.3) is a perturbation of the left-hand side of (1.1). Further, Corollary 4.3 gives a better estimation than (1.1) for $\ell = 0$. Hence, we can give refinements of Corollary 4.3 by the help of the following result, which follows by setting $\ell = 0$ in Theorem 3.1.

Corollary 4.4. *Assume* (2.1). *If* $\varphi \in \Phi_\infty$, *then*

$$\left|\int_\alpha^\beta \varphi(\tau)\mathrm{d}\tau - A(\kappa,0)\right| \le E_\infty(\kappa,0)\left\|\varphi''\right\|_\infty,$$

where

$$A(\kappa,0) = (\rho-\kappa)\left(\varphi(\alpha)+\varphi(\beta)\right) + \frac{(\rho-\kappa)^2}{2}\left(\varphi'(\alpha)-\varphi'(\beta)\right) + 2\kappa\varphi(\mu) \tag{4.4}$$

and

$$E_\infty(\kappa,0) = \frac{(\beta-\alpha)^3}{24} - \kappa\rho(\rho-\kappa).$$

Some special cases of Corollary 4.4 are given in the following remark.

Remark 4.5. Using Corollary 4.4 for $\kappa = \rho$, we get the classical midpoint inequality

$$\left| \int_\alpha^\beta \varphi(\tau)\mathrm{d}\tau - (\beta - \alpha)\varphi(\mu) \right| \le \frac{(\beta - \alpha)^3}{24} \left\| \varphi'' \right\|_\infty . \tag{4.5}$$

Using Corollary 4.4 for $\kappa = 0$, we obtain

$$\begin{aligned} &\left| \int_\alpha^\beta \varphi(\tau)\mathrm{d}\tau - \frac{\beta - \alpha}{2} \left(\varphi(\alpha) + \varphi(\beta) \right) + \frac{(\beta - \alpha)^2}{8} \left(\varphi'(\beta) - \varphi'(\alpha) \right) \right| \\ &\qquad \le \frac{(\beta - \alpha)^3}{24} \left\| \varphi'' \right\|_\infty . \end{aligned} \tag{4.6}$$

Inequality (4.6) is a perturbed trapezoid inequality, and it is easy to see that it is better than the classical trapezoid inequality (which has 12 in the denominator). Further, (4.6) is better than the perturbed trapezoid inequalities given in [7, 19]. Next, using Corollary 4.4 for $\kappa = \frac{\rho}{2}$, we get

$$\begin{aligned} &\left| \int_\alpha^\beta \varphi(\tau)\mathrm{d}\tau - \frac{\beta - \alpha}{4} \left(\varphi(\alpha) + 2\varphi(\mu) + \varphi(\beta) \right) + \frac{(\beta - \alpha)^2}{32} \left(\varphi'(\beta) - \varphi'(\alpha) \right) \right| \\ &\qquad \le \frac{(\beta - \alpha)^3}{96} \left\| \varphi'' \right\|_\infty . \end{aligned} \tag{4.7}$$

Inequality (4.7) is a new perturbed averaged trapezoid midpoint rule, and is better than the simple average midpoint-trapezoid inequality in [19]. Finally, using Corollary 4.4 for $\kappa = \frac{2\rho}{3}$, we obtain

$$\begin{aligned} &\left| \int_\alpha^\beta \varphi(\tau)\mathrm{d}\tau - \frac{\beta - \alpha}{6} \left(\varphi(\alpha) + 4\varphi(\mu) + \varphi(\beta) \right) + \frac{(\beta - \alpha)^2}{72} \left(\varphi'(\beta) - \varphi'(\alpha) \right) \right| \\ &\qquad \le \frac{(\beta - \alpha)^3}{72} \left\| \varphi'' \right\|_\infty . \end{aligned} \tag{4.8}$$

Inequality (4.8) is Simpson's inequality with a new perturbed variant. However, the simple Simpson inequality given in [19] is better than (4.8).

A special situation occurs when all three terms in the expression for E_∞ in (3.2) are equal, which happens when

$$\kappa = \frac{2\rho}{3} \quad \text{and} \quad \ell = \frac{\rho}{3}, \quad \text{i.e.,} \quad \rho - \kappa = \kappa - \ell = \ell = \frac{\rho}{3}. \tag{4.9}$$

See Figure 4.1 for a graph of the kernel K in this special situation.

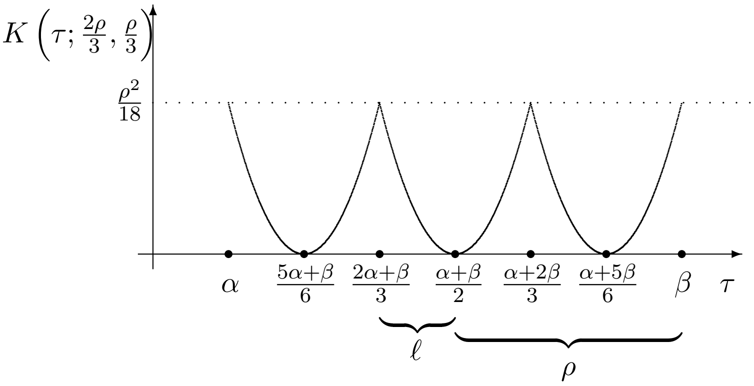

FIGURE 4.1. The special kernel K

Applying Theorem 4.1 for this case, we obtain the following result.

Theorem 4.6. *If $\varphi \in \Phi_\infty$, then*

$$\left|\int_\alpha^\beta \varphi(\tau)\mathrm{d}\tau - A\left(\frac{2\rho}{3},\frac{\rho}{3}\right)\right| \le \frac{(\beta-\alpha)^3}{216}\left\|\varphi''\right\|_\infty,$$

where

$$A\left(\frac{2\rho}{3},\frac{\rho}{3}\right) = \frac{\beta-\alpha}{6}\left(\varphi(\alpha)+2\varphi\left(\frac{2\alpha+\beta}{3}\right)+2\varphi\left(\frac{\alpha+2\beta}{3}\right)+\varphi(\beta)\right) + \frac{(\beta-\alpha)^2}{72}\left(\varphi'(\alpha)-\varphi'(\beta)\right). \tag{4.10}$$

4.2. The case $p = 2$. We start by restating Theorem 3.1 for the case $p = 2$.

Theorem 4.7. *Assume* (2.1) *and* (2.4). *If $\varphi \in \Phi_2$, then*

$$\left|\int_\alpha^\beta \varphi(\tau)\mathrm{d}\tau - A(\kappa,\ell)\right| \le E_2(\kappa,\ell)\left\|\varphi''\right\|_2, \tag{4.11}$$

where

$$E_2(\kappa,\ell) = \sqrt{\frac{(\beta-\alpha)^5}{320} - \frac{1}{2}\kappa(\rho-\ell)(\rho+\ell-\kappa)\left(\kappa^2-(\rho+\ell)\kappa+\rho^2+\ell^2\right)}.$$

Proof. The calculation

$$E_2^2(\kappa,\ell) = \frac{1}{10}\left((\rho-\kappa)^5+(\kappa-\ell)^5+\ell^5\right)$$
$$= \frac{1}{10}\left(\rho^5-5\rho^4\kappa+10\rho^3\kappa^2-10\rho^2\kappa^3+5\rho\kappa^4-5\kappa^4\ell+10\kappa^3\ell^2-10\kappa^2\ell^3+5\kappa\ell^4\right)$$
$$= \frac{(\beta-\alpha)^5}{320} - \frac{1}{2}\left(\rho^4\kappa-2\rho^3\kappa^2+2\rho^2\kappa^3-\rho\kappa^4+\kappa^4\ell-2\kappa^3\ell^2+2\kappa^2\ell^3-\kappa\ell^4\right)$$

$$= \frac{(\beta-\alpha)^5}{320} - \frac{1}{2}\kappa(\rho-\ell)(\rho+\ell-\kappa)\left(\kappa^2-(\rho+\ell)\kappa+\rho^2+\ell^2\right)$$

together with Theorem 3.1 completes the proof. □

Setting $\kappa = \rho$ in Theorem 4.7, we obtain the following result.

Corollary 4.8. *Assume* (2.1). *If* $\varphi \in \Phi_2$, *then*

$$\left|\int_\alpha^\beta \varphi(\tau)\mathrm{d}\tau - A(\rho,\ell)\right| \le E_2(\rho,\ell)\left\|\varphi''\right\|_2,$$

where $A(\rho,\ell)$ *is given in* (4.2) *and*

$$E_2(\rho,\ell) = \sqrt{\frac{(\beta-\alpha)^5}{320} - \frac{1}{2}\rho\ell(\rho-\ell)\left(\rho^2+\ell^2-\rho\ell\right)}.$$

It is easy to see that the trapezoid inequality in Corollary 4.8 is better than the classical inequality in [19].

Setting $\kappa = \frac{\rho}{2}$ in Theorem 4.7, we obtain the following result.

Corollary 4.9. *Assume* (2.1). *If* $\varphi \in \Phi_2$, *then*

$$\left|\int_\alpha^\beta \varphi(\tau)\mathrm{d}\tau - A\left(\frac{\rho}{2},\ell\right)\right| \le E_2\left(\frac{\rho}{2},\ell\right)\left\|\varphi''\right\|_2,$$

where $A\left(\frac{\rho}{2},\ell\right)$ *is given in* (4.3) *and*

$$E_2\left(\frac{\rho}{2},\ell\right) = \sqrt{\frac{(\beta-\alpha)^5}{5120} - \frac{\rho\ell}{4}\left(\frac{\rho}{2}-\ell\right)\left(\frac{\rho^2}{4}-\frac{\rho\ell}{2}+\ell^2\right)}.$$

Setting $\ell = 0$ in Theorem 4.7, we get the following variant of [29, Corollary 2.5].

Corollary 4.10. *Assume* (2.1). *If* $\varphi \in \Phi_2$, *then*

$$\left|\int_\alpha^\beta \varphi(\tau)\mathrm{d}\tau - A(\kappa,0)\right| \le E_2(\kappa,0)\left\|\varphi''\right\|_2,$$

where $A(\kappa,0)$ *is given in* (4.4) *and*

$$E_2(\kappa,0) = \sqrt{\frac{(\beta-\alpha)^5}{320} - \frac{1}{2}\kappa\rho(\rho-\kappa)\left(\kappa^2-\rho\kappa+\rho^2\right)}.$$

Some special cases of Corollary 4.10 are given in the following remark.

Remark 4.11. Using Corollary 4.10 for $\kappa = \rho$, we get

$$\left|\int_\alpha^\beta \varphi(\tau)\mathrm{d}\tau - (\beta-\alpha)\varphi(\mu)\right| \le \frac{(\beta-\alpha)^{\frac{5}{2}}}{8\sqrt{5}}\left\|\varphi''\right\|_2. \tag{4.12}$$

Using Corollary 4.10 for $\kappa = 0$, we obtain

$$\begin{aligned} &\left| \int_\alpha^\beta \varphi(\tau)\mathrm{d}\tau - \frac{\beta-\alpha}{2}\left(\varphi(\alpha)+\varphi(\beta)\right) + \frac{(\beta-\alpha)^2}{8}\left(\varphi'(\beta)-\varphi'(\alpha)\right) \right| \\ &\qquad \leq \frac{(\beta-\alpha)^{\frac{5}{2}}}{8\sqrt{5}} \left\|\varphi''\right\|_2 . \end{aligned} \tag{4.13}$$

Inequality (4.13) is a perturbed trapezoid inequality, and it is easy to see that it is better than the inequalities of this type as recognized in [7, 19]. Next, using Corollary 4.10 for $\kappa = \frac{\rho}{2}$, we get

$$\begin{aligned} &\left| \int_\alpha^\beta \varphi(\tau)\mathrm{d}\tau - \frac{\beta-\alpha}{4}\left(\varphi(\alpha)+2\varphi(\mu)+\varphi(\beta)\right) + \frac{(\beta-\alpha)^2}{32}\left(\varphi'(\beta)-\varphi'(\alpha)\right) \right| \\ &\qquad \leq \frac{(\beta-\alpha)^{\frac{5}{2}}}{32\sqrt{5}} \left\|\varphi''\right\|_2 . \end{aligned} \tag{4.14}$$

Inequality (4.14) is a new perturbed averaged trapezoid midpoint rule, and it is easy to see that it is better than the inequalities of this type as recognized in [7, 19]. Finally, using Corollary 4.10 for $\kappa = \frac{2\rho}{3}$, we obtain

$$\begin{aligned} &\left| \int_\alpha^\beta \varphi(\tau)\mathrm{d}\tau - \frac{\beta-\alpha}{6}\left(\varphi(\alpha)+4\varphi(\mu)+\varphi(\beta)\right) + \frac{(\beta-\alpha)^2}{72}\left(\varphi'(\beta)-\varphi'(\alpha)\right) \right| \\ &\qquad \leq \frac{\sqrt{11}(\beta-\alpha)^{\frac{5}{2}}}{72\sqrt{5}} \left\|\varphi''\right\|_2 . \end{aligned} \tag{4.15}$$

Inequality (4.15) is a new perturbed variant of Simpson's inequality, and it is better than the one presented in [19].

Now, we apply Theorem 4.7 in the case (4.9) to obtain the following result.

Theorem 4.12. *If $\varphi \in \Phi_2$, then*

$$\left| \int_\alpha^\beta \varphi(\tau)\mathrm{d}\tau - A\left(\frac{2\rho}{3}, \frac{\rho}{3}\right) \right| \leq \frac{(\beta-\alpha)^{\frac{5}{2}}}{72\sqrt{5}} \left\|\varphi''\right\|_2 ,$$

where $A\left(\frac{2\rho}{3}, \frac{\rho}{3}\right)$ is given in (4.10).

4.3. The case $1 < p < \infty$. For the general case $1 < p < \infty$, we offer three special cases of Theorem 3.1. First, setting $\kappa = \rho$ in Theorem 3.1, we obtain the following result.

Corollary 4.13. *Let $1 < p < \infty$. Assume* (2.1). *If $\varphi \in \Phi_p$, then*

$$\left| \int_\alpha^\beta \varphi(\tau)\mathrm{d}\tau - A(\rho, \ell) \right| \leq E_p(\rho, \ell) \left\|\varphi''\right\|_p ,$$

where $A(\rho,\ell)$ *is given in* (4.2) *and*

$$E_p(\rho,\ell)=\frac{1}{2}\left(\frac{(\rho-\ell)^{3+\frac{2}{p-1}}+\ell^{3+\frac{2}{p-1}}}{\frac{3}{2}+\frac{1}{p-1}}\right)^{1-\frac{1}{p}}.$$

Setting $\ell=0$ in Theorem 3.1, we get the following generalization of [29, Corollary 2.5].

Corollary 4.14. *Let* $1<p<\infty$. *Assume* (2.1). *If* $\varphi\in\Phi_p$, *then*

$$\left|\int_\alpha^\beta \varphi(\tau)\mathrm{d}\tau-A(\kappa,0)\right|\le E_p(\kappa,0)\left\|\varphi''\right\|_p,$$

where $A(\kappa,0)$ *is given in* (4.4) *and*

$$E_p(\kappa,0)=\frac{1}{2}\left(\frac{(\rho-\kappa)^{3+\frac{2}{p-1}}+\kappa^{3+\frac{2}{p-1}}}{\frac{3}{2}+\frac{1}{p-1}}\right)^{1-\frac{1}{p}}.$$

Some special cases of Corollary 4.14 are given in the following generalizations of the inequalities presented in Remark 4.11.

Remark 4.15. Using Corollary 4.14 for $\kappa=\rho$, we get

$$\left|\int_\alpha^\beta \varphi(\tau)\mathrm{d}\tau-(\beta-\alpha)\varphi(\mu)\right|\le\frac{(\beta-\alpha)^{3-\frac{1}{p}}}{8\left(3+\frac{2}{p-1}\right)^{1-\frac{1}{p}}}\left\|\varphi''\right\|_p. \tag{4.16}$$

Using Corollary 4.14 for $\kappa=0$, we obtain

$$\begin{aligned}\left|\int_\alpha^\beta \varphi(\tau)\mathrm{d}\tau-\frac{\beta-\alpha}{2}\left(\varphi(\alpha)+\varphi(\beta)\right)+\frac{(\beta-\alpha)^2}{8}\left(\varphi'(\beta)-\varphi'(\alpha)\right)\right|\\ \le\frac{(\beta-\alpha)^{3-\frac{1}{p}}}{8\left(3+\frac{2}{p-1}\right)^{1-\frac{1}{p}}}\left\|\varphi''\right\|_p.\end{aligned} \tag{4.17}$$

Using Corollary 4.14 for $\kappa=\frac{\rho}{2}$, we get

$$\begin{aligned}\left|\int_\alpha^\beta \varphi(\tau)\mathrm{d}\tau-\frac{\beta-\alpha}{4}\left(\varphi(\alpha)+2\varphi(\mu)+\varphi(\beta)\right)+\frac{(\beta-\alpha)^2}{32}\left(\varphi'(\beta)-\varphi'(\alpha)\right)\right|\\ \le\frac{(\beta-\alpha)^{3-\frac{1}{p}}}{32\left(3+\frac{2}{p-1}\right)^{1-\frac{1}{p}}}\left\|\varphi''\right\|_p.\end{aligned} \tag{4.18}$$

Finally, using Corollary 4.14 for $\kappa = \frac{2\rho}{3}$, we obtain

$$\begin{aligned}&\left|\int_\alpha^\beta \varphi(\tau)\mathrm{d}\tau - \frac{\beta-\alpha}{6}(\varphi(\alpha)+4\varphi(\mu)+\varphi(\beta)) + \frac{(\beta-\alpha)^2}{72}\left(\varphi'(\beta)-\varphi'(\alpha)\right)\right| \\ &\qquad\le \frac{(\beta-\alpha)^{3-\frac{1}{p}}}{72}\left(\frac{1+2^{3+\frac{2}{p-1}}}{3\left(3+\frac{2}{p-1}\right)}\right)^{1-\frac{1}{p}}\left\|\varphi''\right\|_p . \end{aligned} \tag{4.19}$$

Now, we apply Theorem 3.1 in the case (4.9) to obtain the following result.

Theorem 4.16. *Let $1 < p < \infty$. If $\varphi \in \Phi_p$, then*

$$\left|\int_\alpha^\beta \varphi(\tau)\mathrm{d}\tau - A\left(\frac{2\rho}{3}, \frac{\rho}{3}\right)\right| \le \frac{(\beta-\alpha)^{3-\frac{1}{p}}}{72\left(3+\frac{2}{p-1}\right)^{1-\frac{1}{p}}}\left\|\varphi''\right\|_p ,$$

where $A\left(\frac{2\rho}{3}, \frac{\rho}{3}\right)$ is given in (4.10).

4.4. The case $p = 1$. There are only three possibilities for E_1, i.e., the supremum of K over $[\alpha, \beta]$, namely (see also Figure 2.1), a half of the square of the largest one of $\rho - \kappa$, $\kappa - \ell$, and ℓ. Theorem 3.1 for the case $p = 1$ can be specified as follows.

Theorem 4.17. *Assume* (2.1) *and* (2.4). *If $\varphi \in \Phi_1$, then*

$$\left|\int_\alpha^\beta \varphi(\tau)\mathrm{d}\tau - A(\kappa,\ell)\right| \le E_1(\kappa,\ell)\left\|\varphi''\right\|_1 , \tag{4.20}$$

where

$$E_1(\kappa,\ell) = \frac{1}{2}\begin{cases}(\rho-\kappa)^2 & \text{if } \kappa \le \frac{3\rho-\ell-|3\ell-\rho|}{4}, \\ (\kappa-\ell)^2 & \text{if } \kappa \ge \frac{\rho+5\ell+|3\ell-\rho|}{4}, \\ \ell^2 & \text{otherwise.}\end{cases}$$

Proof. We have $\max\{\rho-\kappa, \kappa-\ell, \ell\} = \rho-\kappa$ if and only if

$$\kappa - \ell \le \rho - \kappa \quad \text{and} \quad \ell \le \rho - \kappa,$$

i.e.,

$$\kappa \le \min\left\{\frac{\rho+\ell}{2}, \rho-\ell\right\}.$$

We have $\max\{\rho-\kappa, \kappa-\ell, \ell\} = \kappa-\ell$ if and only if

$$\rho - \kappa \le \kappa - \ell \quad \text{and} \quad \ell \le \kappa - \ell,$$

i.e.,

$$\kappa \ge \max\left\{\frac{\rho+\ell}{2}, 2\ell\right\}.$$

In all other cases, we have $\max\{\rho-\kappa,\kappa-\ell,\ell\}=\ell$. Now noting

$$\max\{x,y\}=\frac{x+y+|x-y|}{2} \quad \text{and} \quad \min\{x,y\}=\frac{x+y-|x-y|}{2}$$

$(x,y\in\mathbb{R})$ and applying Theorem 3.1 completes the proof. □

Setting $\kappa=\rho$ in Theorem 4.17, we obtain the following result.

Corollary 4.18. *Assume* (2.1). *If* $\varphi\in\Phi_1$, *then*

$$\left|\int_\alpha^\beta \varphi(\tau)\mathrm{d}\tau - A(\rho,\ell)\right| \le \left(\frac{\rho}{2}+\left|\ell-\frac{\rho}{2}\right|\right)\left\|\varphi''\right\|_1,$$

where $A(\rho,\ell)$ *is given in* (4.2).

Setting $\kappa=\frac{\rho}{2}$ in Theorem 4.17, we obtain the following inequality.

Corollary 4.19. *Assume* (2.1). *If* $\varphi\in\Phi_1$, *then*

$$\left|\int_\alpha^\beta \varphi(\tau)\mathrm{d}\tau - A\left(\frac{\rho}{2},\ell\right)\right| \le \frac{1}{2}\left(\ell+\frac{\rho}{2}+\left|\ell-\frac{\rho}{2}\right|\right)\left\|\varphi''\right\|_1,$$

where $A\left(\frac{\rho}{2},\ell\right)$ *is given in* (4.3).

Setting $\ell=0$ in Theorem 4.17, we get the following result.

Corollary 4.20. *Assume* (2.1). *If* $\varphi\in\Phi_1$, *then*

$$\left|\int_\alpha^\beta \varphi(\tau)\mathrm{d}\tau - A(\kappa,0)\right| \le \left(\frac{\rho}{2}+\left|\kappa-\frac{\rho}{2}\right|\right)\left\|\varphi''\right\|_1,$$

where $A(\kappa,0)$ *is given in* (4.4).

Some special cases of Corollary 4.20 are given in the following remark.

Remark 4.21. Using Corollary 4.20 for $\kappa=\rho$, we get

$$\left|\int_\alpha^\beta \varphi(\tau)\mathrm{d}\tau - (\beta-\alpha)\varphi(\mu)\right| \le \frac{(\beta-\alpha)^2}{8}\left\|\varphi''\right\|_1. \tag{4.21}$$

Using Corollary 4.20 for $\kappa=0$, we obtain

$$\begin{aligned}\left|\int_\alpha^\beta \varphi(\tau)\mathrm{d}\tau - \frac{\beta-\alpha}{2}\left(\varphi(\alpha)+\varphi(\beta)\right)+\frac{(\beta-\alpha)^2}{8}\left(\varphi'(\beta)-\varphi'(\alpha)\right)\right| \\ \le \frac{(\beta-\alpha)^2}{8}\left\|\varphi''\right\|_1.\end{aligned} \tag{4.22}$$

Next, using Corollary 4.20 for $\kappa=\frac{\rho}{2}$, we get

$$\begin{aligned}\left|\int_\alpha^\beta \varphi(\tau)\mathrm{d}\tau - \frac{\beta-\alpha}{4}\left(\varphi(\alpha)+2\varphi(\mu)+\varphi(\beta)\right)+\frac{(\beta-\alpha)^2}{32}\left(\varphi'(\beta)-\varphi'(\alpha)\right)\right| \\ \le \frac{(\beta-\alpha)^2}{32}\left\|\varphi''\right\|_1.\end{aligned} \tag{4.23}$$

Finally, using Corollary 4.20 for $\kappa = \frac{2\rho}{3}$, we obtain

$$\left|\int_\alpha^\beta \varphi(\tau)\mathrm{d}\tau - \frac{\beta-\alpha}{6}\left(\varphi(\alpha)+4\varphi(\mu)+\varphi(\beta)\right)+\frac{(\beta-\alpha)^2}{72}\left(\varphi'(\beta)-\varphi'(\alpha)\right)\right| \leq \frac{(\beta-\alpha)^2}{18}\left\|\varphi''\right\|_1. \tag{4.24}$$

Applying Theorem 4.17 for the case (4.9), we obtain the following result.

Theorem 4.22. *If $\varphi \in \Phi_1$, then*

$$\left|\int_\alpha^\beta \varphi(\tau)\mathrm{d}\tau - A\left(\frac{2\rho}{3}, \frac{\rho}{3}\right)\right| \leq \frac{(\beta-\alpha)^2}{72}\left\|\varphi''\right\|_1.$$

where $A\left(\frac{2\rho}{3}, \frac{\rho}{3}\right)$ is given in (4.10).

5. Conclusion. In this article, generalisations with refinements of Ostrowski-type inequalities for second-order differentiable functions are proved. As special cases, we present perturbed midpoint inequalities versions, Simpson's, averaged trapezoid-midpoint type and trapezoid, which refine the results of [6, 7, 9, 11, 19, 20, 28, 29] and also recapture the results of [7, 22] in perturbed form.

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Received December 13, 2021