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Three algebraic number systems based on the q-addition with applications

Abstract. In the spirit of our earlier articles on q-ω special functions, the purpose of this article is to present many new q-number systems, which are based on the q-addition, which was introduced in our previous articles and books. First, we repeat the concept biring, in order to prepare for the introduction of the q-integers, which extend the q-natural numbers from our previous book. We formally introduce a q-logarithm for the q-exponential function for later use. In order to find q-analogues of the corresponding formulas for the generating functions and q-trigonometric functions, we also introduce q-rational numbers. Then the so-called q-real numbers $\mathbb{R}_{\oplus_q}$, with a norm, a q-deformed real line, and with three inequalities, are defined. The purpose of the more general q-real numbers $\mathbb{R}_q$ is to allow the other q-addition too. The closely related JHC q-real numbers $\mathbb{R}_{\boxplus_q}$ have applications to several q-Euler integrals. This brings us to a vector version of the q-binomial theorem from a previous paper, which is associated with a special case of the q-Lauricella function. New q-trigonometric function formulas are given to show the application of this umbral calculus. Then, some equalities between q-trigonometric zeros and extreme values are proved. Finally, formulas and graphs for q-hyperbolic functions are shown.

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Since our first book [4], where the q-umbral calculus was introduced, there has been a demand to define those alphabets and letters in a more strict form. At the suggestion of colleagues, the Rota paper was q-deformed, but the question arose, what kind of alphabets were used. It is the purpose of this paper to explain this. We want to mention that the more general q-complex numbers already have appeared in [10]. To illustrate the many applications, we give some proofs of multiple q-Euler integrals with these q-numbers in the function argument.

This paper is organized as follows: First a general biring is defined. The purpose of Section 1 is to extend the semiring $\mathbb{N}_{\oplus_q}$ from [4] to the biring of q-integers $\mathbb{Z}_{\oplus_q}$. The purpose of Section 2 is to define two so-called q-rational numbers $\mathbb{Q}_{\oplus_q}$ and $\mathbb{Q}_q$, with many applications. A variation of q-rational numbers, an Abelian group, is introduced in Section 3. In order to find q-analogues of the corresponding formulas for the generating functions, in Subsection 4, we, formally, introduce a q-logarithm for the q-exponential function for later use. In Section 5, we introduce the so-called q-real numbers, with a norm, a q-deformed real line, and with three inequalities. The purpose of Section 6 is to define powers of the q-real numbers $\mathbb{R}_{\oplus_q}$ and of the more general q-real numbers $\mathbb{R}_q$. In Section 7, we introduce the JHC q-real numbers $\mathbb{R}_{\boxplus_q}$ with applications to several q-Euler integrals. In Section 8, we introduce a vector version of the q-binomial theorem, which is associated with a special case of the q-Lauricella function and another type of q-real numbers. In order to give some applications of the new q-numbers, in Section 10 we therefore introduce new q-trigonometric and q-hyperbolic functions together with some graphs. We also show how the multiples of π

are replaced by the q-trigonometric zeros, which will be useful for the definitions of q-tangens. The purpose of Section 10.1 is to state the corresponding new q-hyperbolic formulas together with corresponding graphs.

In this paper we will define the following q-analogues of number systems with inclusions indicated:

$$\mathbb{N}_{\oplus_q} \subset \mathbb{Q}_{\oplus_q} \subset \mathbb{R}_{\oplus_q} \subset \mathbb{R}_q,$$
$$\mathbb{N}_{\oplus_q} \subset \mathbb{Z}_q \subset \mathbb{Q}_q \subset \mathbb{R}_q,$$

where the q-rational numbers $\mathbb{Q}_q$ have $\mathbb{Z}_q$ instead of $\mathbb{N}_{\oplus_q}$ in the numerator. Several variations of these number systems are also defined. The symbol $\sim$ denotes that two letters in the alphabet gives the same result when operating with the linear functional. It defines an equivalence relation on this alphabet [4, p. 99]. An index $\oplus_q$ indicates that only NWA q-addition is used. An index q indicates that both q-additions are used.

From time to time we also use the following duality: Letters with n members can correspond to vectors with n real elements. Some of these number systems are special cases of the following general object:

Definition 1. A q-groupoid $(G_q, \oplus_q, \sim)$ is a set of letters with an associative and commutative mapping $\oplus_q : G_q \times G_q \mapsto G_q$. The associativity can be expressed as follows:

$$(a \oplus_q b) \oplus_q c \sim a \oplus_q (b \oplus_q c), \quad a, b, c \in G_q.$$

Theorem 0.1 (Compare with [15, p. 39]). *In a q-groupoid, all composite q-additions, independent of the brackets, represent the same element. It is denoted by $\oplus_{q,l=0}^{j-1} a_l$, $\{a_l\}_{l=0}^{j-1} \in G_q$.*

Definition 2. A q-module is a function $\mathbb{R}_{\oplus_q} \times \mathbb{R} \mapsto \mathbb{R}_{\oplus_q}$, where we can multiply letters $\alpha \in \mathbb{R}_{\oplus_q}$ with scalars $b \in \mathbb{R}$ to form letters $\gamma \in \mathbb{R}_{\oplus_q}$:

$$\gamma \sim b\alpha. \tag{1}$$

This operation is distributive over the q-addition:

$$b(\alpha \oplus_q \beta) \sim b\alpha \oplus_q b\beta, \quad b \in \mathbb{R},\ \alpha, \beta \in \mathbb{R}_q. \tag{2}$$

The operations (1) and (2), as well as similar formulas for q-modules are used in the rest of the paper without further explanation.

1. The semiring $\mathbb{N}_{\oplus_q}$ and the biring of q-integers $\mathbb{Z}_{\oplus_q}$

In [4, p. 167], we introduced the following concept:

Definition 3. There is a Ward number $\overline{n}_q$

$$\overline{n}_q \sim 1 \oplus_q 1 \oplus_q \ldots \oplus_q 1,$$

where the number of 1 in the RHS is n. Let $(\mathbb{N}_{\oplus_q}, \oplus_q, \odot_q)$ denote the semiring of Ward numbers $\overline{k}_q$, $k \geq 0$ together with a binary operation: $\oplus_q$

is the usual Ward q-addition. The multiplication $\odot_q$ is defined as follows:

$$\overline{n}_q \odot_q \overline{m}_q \sim \overline{nm}_q, \tag{3}$$

where $\sim$ denotes the equivalence in the alphabet.

Theorem 1.1 ([8, p. 5]). *Functional equations for Ward numbers operating on the q-exponential function. First assume that the letters $\overline{m}_q$ and $\overline{n}_q$ are independent, i.e., come from two different functions, when operating with the functional. Furthermore, $mnt < \frac{1}{1-q}$. Then we have*

$$\mathrm{E}_q(\overline{m}_q\overline{n}_q t) = \mathrm{E}_q(\overline{mn}_q t).$$

Furthermore,

$$\mathrm{E}_q(\overline{jm}_q) = \mathrm{E}_q(\overline{j}_q)^m = \mathrm{E}_q(\overline{m}_q)^j = \mathrm{E}_q(\overline{j}_q \odot_q \overline{m}_q).$$

Definition 4. Powers of Ward numbers are defined by

$$\overline{n}_q^m \equiv \overline{n}_q \odot_q \overline{n}_q \odot_q \ldots \odot_q \overline{n}_q,$$

where the number of $\overline{n}_q$ in the RHS is m.

We shall now extend this semiring to a biring. Therefore we first define our general biring. The following definition prepares for the biring in Theorem 1.2. The following definition could be compared with quantum groups, where one works in a non-commutative algebra, with two dual additions. Our second addition corresponds to this coaddition.

Definition 5. Assume that $R \equiv R_1 \cup -(R_1)$, a gradation. A graded commutative biring is a set $(R, \oplus, \boxplus, \odot, 0)$, with two binary operations $\oplus$ and $\odot$ on R, and a dual addition $\boxplus$, a zero 0, (and a unit 1), which satisfy the following axioms. The third and fourth axioms explains this dual addition. For each elements $a, b, c \in R$:

(1) Additive associativity: $(a \oplus b) \oplus c = a \oplus (b \oplus c)$.
(2) Additive commutativity: $a \oplus b = b \oplus a$.
(3) Additive identity: There exists an element $0 \in R$ such that $0 \oplus a = a \oplus 0 = a \boxplus 0 = a$.
(4) Additive inverse: There exists an element $a^\star \in R$ such that $a \boxplus a^\star = a^\star \boxplus a = 0$.
(5) Multiplicative identity: There exists an element $1 \in R$ such that $a \odot 1 = 1 \odot a = a$.
(6) Multiplicative associativity: $(a \odot b) \odot c = a \odot (b \odot c)$.
(7) distributivity: $a \odot (b \oplus c) = a \odot b \oplus a \odot c$.
(8) Multiplicative commutativity: $a \odot b = b \odot a$.

We assume that for b or c equal to $-d$, $d \in R_1$, we may replace $\oplus$ by $\boxplus$.

We can now extend the q-addition with JHC to obtain a graded commutative biring.

Definition 6. Let the q-integers $(\mathbb{Z}_q, \oplus_q, \boxplus_q, \odot_q, \overline{0}_q)$ denote $\pm$ the Ward numbers, i.e., $\mathbb{Z}_q \equiv \mathbb{N}_{\oplus_q} \cup -\mathbb{N}_{\oplus_q}$, where there are two inverse q-additions $\oplus_q$ and $\boxplus_q$. $\overline{0}_q$ denotes the zero θ, and $\overline{1}_q$ denotes the multiplicative identity. The dual addition is defined by

$$\overline{n}_q \boxplus_q -\overline{m}_q \sim \overline{n-m}_q, \quad n \geq m. \tag{4}$$

Furthermore, the multiplication $\odot_q$ is defined by (3) and

$$\overline{n}_q \odot_q -\overline{m}_q \sim -\overline{nm}_q. \tag{5}$$

Finally, we define

$$\overline{-m}_q \equiv -\overline{m}_q.$$

Theorem 1.2 (An extension of [4, p. 167]). *Assume that $\mathbb{Z}_q$ is defined by the previous definition. Then $(\mathbb{Z}_q, \oplus_q, \boxplus_q, \odot_q, \overline{0}_q)$ is a graded commutative biring.*

Proof. The proof is achieved for three elements $\overline{n}_q, \overline{m}_q, \overline{k}_q \in \mathbb{N}_{\oplus_q}$ by [4, p. 167]. Instead, choose three elements $\overline{n}_q, -\overline{m}_q, \overline{k}_q \in \mathbb{Z}_{\oplus_q}$. The associativity for $(\mathbb{Z}_{\oplus_q}, \oplus_q, \boxplus_q)$ is shown as follows:

$$(\overline{n}_q \boxplus_q -\overline{m}_q) \oplus_q \overline{k}_q \overset{\text{by}(4)}{\sim} \overline{n-m}_q \oplus_q \overline{k}_q \sim \overline{(n-m)+k}_q, \quad n \geq m.$$

$$\overline{n}_q \oplus_q (\overline{k}_q \boxplus_q -\overline{m}_q) \overset{\text{by}(4)}{\sim} \overline{n}_q \oplus_q \overline{k-m}_q \sim \overline{n+(k-m)}_q, \quad k \geq m.$$

The associativity for $(\mathbb{Z}_{\oplus_q}, \odot_q)$ is shown as follows:

$$(\overline{n}_q \odot_q -\overline{m}_q) \odot_q \overline{k}_q \overset{\text{by}(5)}{\sim} -\overline{nm}_q \odot_q \overline{k}_q \overset{\text{by}(5)}{\sim} -\overline{nmk}_q.$$

$$\overline{n}_q \odot_q (-\overline{m}_q \odot_q \overline{k}_q) \overset{\text{by}(5)}{\sim} \overline{n}_q \odot_q -\overline{mk}_q \overset{\text{by}(5)}{\sim} -\overline{nmk}_q.$$

The identity is $\overline{1}_q$. The commutativity for $(\mathbb{Z}_{\oplus_q}, \odot_q)$ follows from (5).

The distributive law is proved as follows: assume that $k \geq m$, then

$$\overline{n}_q \odot_q (\overline{k}_q \boxplus_q -\overline{m}_q)) \overset{\text{by}(4)}{\sim} \overline{n}_q \odot_q \overline{k-m}_q \sim \overline{n(k-m)}_q,$$

$$(\overline{n}_q \odot_q \overline{k}_q) \boxplus_q (\overline{n}_q \odot_q -\overline{m}_q) \overset{\text{by}(5)}{\sim} \overline{-nm}_q \oplus_q \overline{nk}_q \overset{\text{by}(4)}{\sim} \overline{(nk-nm)}_q.$$

$\square$

Definition 7. We define an order for $\mathbb{Z}_q$ as follows: the statements $\overline{m}_q < \overline{n}_q$ and $\overline{n}_q > \overline{m}_q$ mean that

$$n - m \in \mathbb{N}.$$

The statements $\overline{m}_q \leq \overline{n}_q$ and $\overline{n}_q \geq \overline{m}_q$ mean that

$$\overline{m}_q < \overline{n}_q \vee \overline{m}_q \sim \overline{n}_q.$$

2. The q-rational numbers $\mathbb{Q}_{\oplus_q}$ and $\mathbb{Q}_q$

Definition 8 ([7, p. 4]). Let the q-rational numbers $\mathbb{Q}_{\oplus_q}$ be defined as follows:

$$\mathbb{Q}_{\oplus_q} \equiv \left\{ \frac{\overline{m}_q}{\overline{n}_q}, \quad m \in \mathbb{N} \cup \{0\},\ n \in \mathbb{N},\ m \neq n,\ \frac{\overline{0}_q}{\overline{n}_q} \sim \theta,\ \frac{\overline{n}_q}{\overline{n}_q} \sim 1 \right\},$$

together with a linear functional

$$v,\ \mathbb{R}[x] \times \mathbb{Q}_{\oplus_q} \to \mathbb{R},$$

called the evaluation. If $v(x) = \sum_{k=0}^{n} a_k x^k$, then

$$v\left(\frac{\overline{m}_q}{\overline{n}_q}\right) \equiv \sum_{k=0}^{n} a_k \frac{(\overline{m}_q)^k}{(\overline{n}_q)^k}.$$

Definition 9. Similarly, let the q-rational numbers $\mathbb{Q}_q$ be defined as follows:

$$\mathbb{Q}_q \equiv \left\{ \frac{\overline{m}_q}{\overline{n}_q},\ \overline{m}_q \in \mathbb{Z}_q,\ \overline{n}_q \in \mathbb{N}_{\oplus_q},\ n \neq 0,\ m \neq n,\ \frac{\overline{0}_q}{\overline{n}_q} \sim \theta,\ \frac{\overline{n}_q}{\overline{n}_q} \sim 1 \right\},$$

Theorem 2.1. *We have the following rules for computations with both q-rational numbers, where $m, k \in \mathbb{N} \vee \mathbb{Z}$, $n \in \mathbb{N}$:*

$$\begin{aligned} \frac{\overline{m}_q}{\overline{n}_q} \oplus_q \frac{\overline{k}_q}{\overline{n}_q} &\sim \frac{\overline{m+k}_q}{\overline{n}_q} \\ \frac{\overline{m}_q}{\overline{n}_q} \boxminus_q \frac{\overline{k}_q}{\overline{n}_q} &\sim \frac{\overline{m-k}_q}{\overline{n}_q}, \quad m \geq k. \end{aligned}$$

Proof. Just q-add the numerators, the denominators are the same. □

Definition 10. The notation $\frac{x}{\overline{n}_q}$ means $x\frac{\overline{1}_q}{\overline{n}_q}$ with q-module multiplication according to formula (1).

3. The Abelian group $\mathbb{Q}_{\mathbf{M},\oplus_q}$ and the monoid $\mathbb{Q}_{\mathbf{M},q}$

Definition 11. Let the q-rational numbers $\mathbb{Q}_{\mathrm{M},\oplus_q}$ be defined as follows:

$$\mathbb{Q}_{\mathrm{M},\oplus_q} \equiv \left\{ \frac{M}{N},\ M \equiv \prod_{i=1}^{k} \overline{m_i}_q,\ N \equiv \prod_{i=1}^{l} \overline{n_i}_q,\ m_i, n_i \in \mathbb{N},\ m_i \neq n_i \right\},$$

where equal letters in numerator and denominator can be canceled. Two elements $Q_1, Q_2 \in \mathbb{Q}_{\mathrm{M},\oplus_q}$ can be multiplied by a multiplication $\bigodot$, to form a new element $Q_3 \equiv Q_1 \bigodot Q_2$, where we multiply numerators and denominators in Q_1 and Q_2 and cancel equal elements, like for ordinary rational numbers.

Theorem 3.1. *The set*

$$\left(\mathbb{Q}_{\mathrm{M},\oplus_q}, \bigodot, 1\right)$$

is an Abelian group with the unit element 1.

Proof. First we check that the multiplication $\bigodot$ is well defined. This is clear since each object $Q_3 = Q_1 \bigodot Q_2$ is unique after simplification. The inverse of $\frac{M}{N} \in \mathbb{Q}_{\mathrm{M},\oplus_q}$ is $\frac{N}{M}$. Since no objects are zero, this causes no problem. Finally, the multiplication $\bigodot$ is commutative and associative, since it is a multiplication of letters in the alphabet. $\square$

Similarly,

Definition 12. Let the set $\mathbb{Q}_{\mathrm{M},q}$ be defined as follows:

$$\mathbb{Q}_{\mathrm{M},q} \equiv \left\{ \frac{M}{N},\, M \equiv \prod_{i=1}^{k} \overline{m_i}_q,\, N \equiv \prod_{i=1}^{l} \overline{n_i}_q,\, m_i \in \mathbb{Z},\, n_i \in \mathbb{N},\, m_i \neq 0,\, m_i \neq n_i \right\},$$

where equal letters in numerator and denominator can be canceled.

Two elements $Q_1, Q_2 \in \mathbb{Q}_{\mathrm{M},q}$ can be multiplied by a multiplication $\bigodot$, to form a new element $Q_3 \equiv Q_1 \bigodot Q_2$, where we multiply numerators and denominators in Q_1 and Q_2 and cancel equal elements.

Theorem 3.2. *The set*

$$\left(\mathbb{Q}_{\mathrm{M},q}, \bigodot, 1 \right)$$

is a commutative monoid with the unit element 1.

Proof. First we check that the multiplication $\bigodot$ is well defined. This is clear since each object $Q_3 \sim Q_1 \bigodot Q_2$ is unique after simplification. The multiplication $\bigodot$ is commutative and associative, since it is a usual multiplication of letters in the alphabet. We cannot have inverses, since the elements in numerators and denominators are not the same. This proves that $(\mathbb{Q}_{\mathrm{M},q}, \bigodot, 1)$ is a monoid. $\square$

The alphabets in this section are different from the other alphabets and are not q-groupoids. The letters in these alphabets are not multiplied with the multiplication $\odot_q$.

4. The q-logarithm

The power series

$$\sum_{k=0}^{\infty} \frac{x^k}{\{k\}_q!} \tag{6}$$

converges for $|x| < (1-q)^{-1}$, the function defined by (6) can be continued to an analytic function

$$\mathrm{E}_q :]-\infty, (1-q)^{-1}[\longrightarrow \mathbb{R}.$$

This follows from the product expression [4, 6.151] for $\mathrm{E}_q(x)$. In fact:

Theorem 4.1. *We have* $\mathrm{E}_q(\,]-\infty,(1-q)^{-1}[\,) =]0,\infty[$, *and*

$$\mathrm{E}_q :]-\infty,(1-q)^{-1}[\longrightarrow]0,\infty[$$

is an analytic isomorphism.

Proof. The product representation of E_q, which applies throughout the definition interval, shows that E_q has no zeros. Since $\mathrm{E}_q(0) = 1$, it follows that $\mathrm{E}_q(x) > 0$. It suffices to show that the logarithmic derivative of $\mathrm{E}_q(x)$ is > 0. However, this follows from

$$\mathrm{D}\left(\log \mathrm{E}_q(x)\right) = \sum_{m=0}^{\infty} \frac{(1-q)q^m}{1-x(1-q)q^m} > 0, \quad 0 < q < 1.$$

This implies that E_q is strictly increasing with non-disappearing derivative. We have already seen in [4] that the limits of $\mathrm{E}_q(x)$ for $x \to -\infty$ and $x \to (1-q)^{-1}$ are 0 and ∞. □

Definition 13. The q-logarithm

$$\log_q :]0,\infty[\longrightarrow]-\infty,(1-q)^{-1}[$$

is the inverse function of E_q.

Definition 14. The function $\mathbb{R}_q \mapsto \mathbb{R} : \mathrm{E}_q(x)$ is defined by the formula

$$\mathrm{E}_q(x \oplus_q y) \equiv \mathrm{E}_q(x)\mathrm{E}_q(y).$$

For the definition of the q-real numbers $\mathbb{R}_{\oplus_q}$, see next section. This definition can be extended to any number of variables by induction.

Theorem 4.2. *The q-logarithm $\log_q(x)$ has the following properties ($\log_q(x)$ and $\log_q(y)$ have small real values, $n \in \mathbb{N}$):*

(1) *It is strictly increasing.*

(2) *Under the assumption that the LHS is a q-real number:*

$$\log_q(x) \oplus_q \log_q(y) \cong \log_q(xy).$$

(3) *Under the assumption that the LHS is a q-real number:*

$$\log_q(x) \oplus_q \log_q(x) \oplus_q \cdots \oplus_q \log_q(x) \cong \log_q(x^n).$$

Proof. To prove the second and the third statements, operate with E_q on both sides. □

5. The q-real numbers $\mathbb{R}_{\oplus_q}$ and $\mathbb{R}_q$

The purpose of these q-real numbers is to give a convenient notation for q-additions in formal power series, in particular for q-exponential and q-trigonometric functions. There is a one-to-one correspondence between the convergence regions of the two q-Lauricella functions $\Phi_{\mathrm{A}}^{(n)}$ and $\Phi_{\mathrm{C}}^{(n)}$ ([5], [6]) and the existence of q-real numbers with n letters. In the same way, there is a one-to-one correspondence between the convergence regions of the same multiple functions with one factor $q^{\binom{k}{2}}$ and the same q-real numbers, but with one corresponding JHC q-addition.

Definition 15. Let $I^n \subset \mathbb{R}^n$, $I \equiv (0,1]$ denote the open n-dimensional hypercube. For q fixed, the q-real numbers $\mathbb{R}_{\oplus_q}$ form a subset of the disjoint union of all hypercubes

$$\mathbb{R}_{\oplus_q} \subset \bigcup_{n=1}^{\infty} I^n.$$

For the following definition one could compare with the formula [4, 4.74 p. 110]:

Definition 16. For $\vec{m} \in \mathbb{N}^n$ put

$$|\vec{m}| \equiv m_1 + ... + m_n.$$

For $a = (a_1, ..., a_n) \in I^n$ put

$$(a_1 \oplus_q a_2 \oplus_q \ldots \oplus_q a_n)^k \equiv \sum_{\vec{|m|}=k} \prod_{l=1}^{n} (a_l)^{m_l} \binom{k}{\vec{m}}_q .$$

Conjecture 1. If the function

$$F(k) \equiv (a_1 \oplus_q a_2 \oplus_q \ldots \oplus_q a_n)^k$$

has exactly one absolute maximum in $\mathbb{N}$, then we have $\lim_{k\to\infty} F(k) = 0$.

Definition 17. We have $\vec{a} := (a_1, ..., a_n) \in \mathbb{R}_{\oplus_q}$ exactly when the function $F(k)$ has exactly one absolute maximum.

Remark 1. $I \subset \mathbb{R}_{\oplus_q}$.

Definition 18. We denote this absolute maximum by $|\vec{a}|$ and call it the norm of the q-real number $\vec{a} \in \mathbb{R}_{\oplus_q}$.

We now return to the duality for q-real numbers and set $\vec{\alpha} = \alpha$.

Definition 19. We define a linear ordering on $\mathbb{R}_{\oplus_q}$ as follows: For two different letters α, $\beta \in \mathbb{R}_{\oplus_q}$ we say that $\alpha > \beta$ if $|\alpha| > |\beta|$ and $\alpha < \beta$ if $|\alpha| < |\beta|$. A letter α is called positive if $\mathrm{E}_q(\alpha) > 1$. This implies that for each $q < 1$, we have a q-real line based on this linear ordering.

There is even a more general q-real number, which is defined in formula (7).

Definition 20. For $a = (a_1, ..., a_n) \in I^n$ we put

$$\begin{aligned} F(k) &\equiv (a_1 \oplus_q a_2 \oplus_q \ldots \oplus_q a_n)^k \\ &\equiv \sum_{|\vec{m}|=k} \prod_{l=1}^{n} (a_l)^{m_l} \binom{k}{\vec{m}}_q , \quad \oplus_q \equiv \vee \oplus_q \vee \ominus_q \vee \boxplus_q \vee \boxminus_q . \end{aligned} \tag{7}$$

In formula (7) we have to mulltiply every term $(a_l)^{m_l}$ by $(-1)^{m_l} q^{\binom{m_l}{2}}$ if a minus and/or a $\boxplus_q$ is preceded by a_l in $F(k)$.

Definition 21. We define $\vec{a} := (a_1, ..., a_n) \in \mathbb{R}_q$ exactly when $\lim_{k\to\infty} F(k) = 0$.

Definition 22. If $F(k)$ in (7) has an absolute maximum, we call it the norm of a.

Definition 23. Let

$$\binom{k}{\vec{m}}' \equiv \begin{cases} \binom{k}{\vec{m}}_q & \text{if } \vec{m} \text{ is connected by a } q\text{-addition} \\ \binom{k}{\vec{m}} & \text{otherwise.} \end{cases}$$

If b_q and c_q are q-real numbers with j letters together, we can add them with the operation $\oplus$ to form the letter

$$a_q \equiv b_q \oplus c_q,$$

with kth power given by

$$((b_0 \oplus_q b_1 \oplus_q \dots b_{m-1}) \oplus (c_m \oplus_q c_{m+1} \oplus_q \dots c_{j-1}))^k \equiv \sum_{|m|=k} \prod_{l=0}^{j-1} (a_l)^{m_l} \binom{k}{\vec{m}}',$$

where for each JHC-addition in a_i, we multiply by $q^{\binom{m_i}{2}}$.

We return again to $\mathbb{R}_{\oplus_q}$.

Definition 24 (compare with [4, p. 110]). Given an integer k, the formula

$$m_0 + m_1 + \ldots + m_j = k$$

determines a set $J_{m_0,\ldots,m_j} \in \mathbb{N}^{j+1}$. Then if $f(x)$ is the formal power series $\sum_{l=0}^{\infty} a_l x^l$, its kth NWA-power is given by

$$(8) \quad (\oplus_{q,l=0}^{\infty} a_l x^l)^k \equiv (a_0 \oplus_q a_1 x \oplus_q \ldots)^k \equiv \sum_{|\vec{m}|=k} \prod_{m_l \in J_{m_0,\ldots,m_j}} (a_l x^l)^{m_l} \binom{k}{\vec{m}_q}.$$

Theorem 5.1. *Three inequalities for q-real numbers $\mathbb{R}_{\oplus_q}$:*

$$(9) \qquad \left(\left(\oplus_{q,m=0}^{n} \sqrt{|x_m|}\right)^k\right)^2 > \left(\oplus_{q,m=0}^{n} |x_m|\right)^k,$$

$$(10) \qquad \sum_{|r|=m} \left(\binom{m}{\vec{r}}_q\right)^2 |\vec{x}|^{\vec{r}} < \left(\sqrt{|x_1|} \oplus_q \ldots \oplus_q \sqrt{|x_n|}\right)^{2m}.$$

Note that $\oplus_{q,m=1}^{n}$ means q-addition, while $\sum_{|r|=m}$ means ordinary addition.

$$(11) \qquad |(a \oplus_q b) \oplus_q (c \oplus_q d)| < |(a \oplus_q b) \oplus (c \oplus_q d)|, \quad a, b, c, d > 0.$$

Proof. The inequality (9) is proved as follows: develop both sides and observe that all terms $\{|x_j|^k\}_{j=0}^n$ cancel each other out. The multiple products on the left side are larger than those on the right.

The inequality (10) is proved similarly: The coefficient for

$$\prod_{i=1}^n |x_i|^{r_i} \ \frac{(\{m\}_q!)^2}{(\prod_{i=1}^n \{r_i\}_q!)^2}$$

on the left is smaller than the coefficient for

$$\prod_{i=1}^n |x_i|^{\frac{s_i}{2}} \ \frac{\{2m\}_q!}{\prod_{i=1}^n \{s_i\}_q!}$$

on the right side. There are also extra multiple products on the right hand side. As soon as we have $\oplus_q$ instead of $\oplus$ in formula (11), the result becomes smaller as in the proof of (10). $\square$

We shall give one example of a q-real number, where the norm in (8) is infinite in formula (5). We make a list of function values $M(q)$. To determine the convergence region, it will be sufficient to write the values of $M(q)$ together with parameter values as in the following table:

q	a	b	c	$M(q)(a \oplus_q b \oplus_q c)$
.96	.94	.95	.93	3.37315×10^{28}
.97	.94	.95	.93	7.69869×10^{37}
.98	.94	.95	.93	4.03485×10^{56}
.99	.94	.95	.93	∞

Finally, we have $M(.99)(.93 \oplus_{.99} .95 \boxplus_{.99} .94) \approx 1.77812 \times 10^{85}$.

6. Powers of the q-real numbers $\mathbb{R}_{\oplus_q}$ and $\mathbb{R}_q$

In some cases it is convenient to work with integer powers of q-real numbers.

Definition 25 ([9]). We extend the two q-additions as follows: If we write a letter γ in the form

$$\gamma \equiv (\alpha \oplus_q \beta)^k \vee \gamma \equiv (\alpha \boxplus_q \beta)^k,$$

this means the two linear functionals

$$\gamma^n \equiv (\alpha \oplus_q \beta)^{nk} \vee \gamma^n \equiv (\alpha \boxplus_q \beta)^{nk}.$$

This definition will be used in formulas (12) and (13). Observe that this q-addition will have the correct q-power to enable a proof by the q-Beta method. The formula [4, 6.192] with $x = 1$ will be used to switch to q-shifted factorials in the proofs.

Theorem 6.1 (Almost a q-analogue of [16, p. 104 (4)], [9, (27)]). *Let $k, s \in \mathbb{N}$. If $k \vee s = 1$, change $2k \vee 2s$ to $k \vee s$.*

$$\begin{aligned}(12)\quad &\mathrm{B}_q(\alpha,\beta)\ {}_{p+2(k+s)}\phi_{r+2(k+s)}\left[\begin{array}{c} c_1,\cdots,c_p,\triangle(q;k;\alpha),\triangle(q;s;\beta) \\ d_1,\cdots,d_r,\triangle(q;k+s;\alpha+\beta)\end{array}\middle| q;t\right] \\ &= \int_0^1 x^{\alpha-1}(xq;q)_{\beta-1}\ {}_p\phi_r(c_1,\cdots,c_p;d_1,\cdots,d_r|q;tx^k(1\boxminus_q xq^\beta)^s)d_q(x).\end{aligned}$$

Theorem 6.2 (A q-analogue of [13, p. 36 2.1.3.7]), [9, (69)]).

$$\begin{aligned}&\int_0^1 x^{a-1}(qx;q)_{b-1}\Phi^{C:D;D'}_{C:D-1;D-1}\left[\begin{array}{c}(c):(d);(d')\\(c'):(g);(g')\end{array}\middle| q;rx^k,s(1\boxminus_q q^b x)^k\right]d_q(x) \\ (13)\quad &= \mathrm{B}_q(a,b)\Phi^{C+2k:D+2k;D'+2k}_{C+2k:D+2k-1;D'+2k-1} \\ &\left[\begin{array}{c}(c),2k\infty:(d),\triangle(q;k;a);(d'),\triangle(q;k;b)\\(c'),\triangle(q;k;a+b):(g),2k\infty;(g'),2k\infty\end{array}\middle| q;r,s\right].\end{aligned}$$

Proof. Start with the left hand side and change the order between summation and q-integration.

$$\begin{aligned}&\sum_{m,n=0}^{\infty}\frac{\langle (c);q\rangle_{m+n}\langle (d);q\rangle_m\langle (d');q\rangle_n r^m s^n}{\langle (c');q\rangle_{m+n}\langle 1,(g);q\rangle_m\langle 1,(g)');q\rangle_n}\int_0^1 x^{a+mk-1}(qx;q)_{b+kn-1} \\ &= \sum_{m,n=0}^{\infty}\frac{\langle (c);q\rangle_{m+n}\langle (d);q\rangle_m\langle (d');q\rangle_n r^m s^n}{\langle (c');q\rangle_{m+n}\langle 1,(g);q\rangle_m\langle 1,(g');q\rangle_n}\Gamma_q\left[\begin{array}{c}a+km,b+kn\\a+b+k(m+n)\end{array}\right] \\ &= \sum_{m,n=0}^{\infty}\frac{\langle (c);q\rangle_{m+n}\langle (d),\triangle(q;k;a);q\rangle_m\langle (d'),\triangle(q;k;b);q\rangle_n r^m s^n}{\langle (c'),\triangle(q;k;a+b);q\rangle_{m+n}\langle 1,(g);q\rangle_m\langle 1,(g');q\rangle_n} \\ &\times\Gamma_q\left[\begin{array}{c}a,b\\a+b\end{array}\right] = \mathrm{RHS}.\end{aligned}$$

□

7. The JHC q-real numbers $\mathbb{R}_{\boxplus_q}$

We now come to the other q-real numbers, which allow an arbitrary exponent. The following sections are organized as follows: We shall define a specific symmetric q-real number, exemplified by several q-Euler integrals.

Definition 26. Assume that $\vec{m} \equiv (m_1,\ldots,m_n)$, $m \equiv m_1+\ldots+m_n$ and $a \in \mathbb{R}^\star$. The vector q-multinomial-coefficient $\binom{a}{\vec{m}}_q^\star$ is defined by the symmetric expression

$$(14)\qquad \binom{a}{\vec{m}}_q^\star \equiv \frac{\langle -a;q\rangle_m(-1)^m q^{-\binom{\vec{m}}{2}+am}}{\langle 1;q\rangle_{m_1}\langle 1;q\rangle_{m_2}\ldots\langle 1;q\rangle_{m_n}}.$$

Definition 27. Let the JHC q-real numbers $\mathbb{R}_{\boxplus_q}$ with $n+1$ letters be defined as follows:

$$\mathbb{R}_{\boxplus_q} \equiv \{1 \boxminus_q q^a x_1 \boxminus_q \ldots \boxminus_q q^a x_n\}, \tag{15}$$

$\{x_k\}_1^n \in \mathbb{R}$, $a \in \mathbb{R}^\star$, $|x_k| < 1$, $0 < q < 1$. When any x_k is negative, we replace $\boxminus_q$ by $\boxplus_q$. This means that the JHC q-real numbers in (15) are functions of $n+1$ real numbers $\{x_k\}_1^n$, a.

The following formula applies for a q-deformed hypercube of length 1 in $\mathbb{R}^n$. Note that formulas (16) and (17) are symmetric in the x_i.

Definition 28. Assuming that the right hand side converges, and $a \in \mathbb{R}^\star$:

$$(1 \boxminus_q q^a x_1 \boxminus_q \ldots \boxminus_q q^a x_n)^{-a} \equiv \sum_{m_1,\ldots,m_n=0}^{\infty} \prod_{j=1}^{n} (-x_j)^{m_j} \binom{-a}{\vec{m}}_q^\star q^{\binom{\vec{m}}{2}+am}. \tag{16}$$

The q-real number in (15) only exists when the series (16) or (17) converges.

We have shown that the two first q-real numbers have the expected properties, i.e., q-dependent convergence regions with larger convergence regions with more JHC-additions. We have defined a unique norm for each q-real number, with three inequalities. Even if the norms of two q-real numbers are the same, this does not imply that these numbers are equal. The construction of the other q-real number $\mathbb{R}_{\boxplus_q}$ is more complicated, since the exponents are somehow connected to the respective numbers. On the other hand, real 'exponents' are allowed in the latter case.

8. A vector version of the q-binomial theorem

In this section, which was first published in [9], we give a practical example of $\mathbb{R}_{\boxplus_q}$. First we present a new function.

Definition 29. Let S_r denote the additional poles of Γ_q, vertical if q is real and slanting if q is complex. If $\vec{x}$ and $\vec{y}$ have dimension n, the vector q-Beta function, a function $(\mathbb{C}\setminus(\{\mathbb{Z} \leq 0\} \cup S_r))^{2n} \times \mathbb{C} \mapsto \mathbb{C}$, is defined as follows:

$$\mathrm{B}_q(\vec{x}, \vec{y}) \equiv \frac{\Gamma_q(\vec{x})\Gamma_q(\vec{y})}{\Gamma_q(\vec{x}+\vec{y})}.$$

In accordance with [4, (4.75) p. 110], we can now prove the following vector version of the q-binomial theorem. The following formula applies to a q-deformed hyper-rhombus of length 1 in $\mathbb{R}^n$.

Corollary 8.1. *A generalization of the q-binomial theorem*

$$(1 \boxminus_q q^a x_1 \boxminus_q \ldots \boxminus_q q^a x_n)^{-a} = \sum_{\vec{m}=\vec{0}}^{\vec{\infty}} \frac{\langle a; q\rangle_m \vec{x}^{\vec{m}}}{\langle \vec{1}; q\rangle_{\vec{m}}}, \quad a \in \mathbb{R}^\star. \tag{17}$$

Proof. Use formulas (14) and (16), the terms with factors $q^{-\binom{\vec{m}}{2}+am}$ cancel each other. $\square$

Theorem 8.2. *Two dual multiplication formulas for the simplest $\mathbb{R}_{\boxplus_q}$:*

$$(1 \boxminus_q q^a x)^{-a}(1 \boxminus_q x)^{-b} = (1 \boxminus_q q^{a+b} x q^{-b})^{-a-b}.$$
$$(1 \boxminus_q q^a x)^{-a}(1 \boxminus_q x q^{a+b})^{-b} = (1 \boxminus_q q^{a+b} x)^{-a-b}.$$

Proof. Use the two q-Vandermonde formulas. □

There are several q-Taylor formulas, some of them very similar, and some with q-integral remainder term. All of these formulas can be generalized to n variables, where the summation indices and the variables are written in the same form, but with vectors. The formula (17) is a very simple example of such a vector q-Taylor formula.

Definition 30. Let $f(x) \in \mathbb{R}[[x]]$. The multiple q-integral I_n of order n is defined by

$$\begin{aligned} I_n &\equiv \int_0^{a_1} \dots (n) \dots \int_0^{a_n} f(a_1, \dots, a_n, q)\ d_q(x_1) \dots\ d_q(x_n) \\ &\equiv \prod_{i=1}^{n} (a_i(1-q)) \sum_{\vec{m}=\vec{0}}^{\vec{\infty}} f(x_1 q^{m_1}, \dots, x_n q^{m_n}, q) q^m, \end{aligned}$$

$0 < |q| < 1$, $\{a_i\}_{i=1}^n \in \mathbb{R}$.

The next formula [9, (22), p. 183] is a q-analogue of Lauricella [14, p. 145], [12, p. 48 2.3.3]).

Theorem 8.3. *A q-integral represention of the first q-Lauricella function by q-beta functions:*

$$\begin{aligned} &\mathrm{B}_q(\vec{b}, \overrightarrow{c-b}) \Phi_{\mathrm{A}}^{(n)}(a, \vec{b}; \vec{c}|q; \vec{x}) \\ &\quad = \int_0^1 \dots (n) \dots \int_0^1 {u_1}^{b_1 - 1} \dots {u_n}^{b_n - 1} (q u_1; q)_{c_1 - b_1 - 1} \\ &\qquad \dots (q u_n; q)_{c_n - b_n - 1} (1 \boxminus_q q^a u_1 x_1 \boxminus_q \dots \boxminus_q q^a u_n x_n)^{-a}\ d_q(u_1) \dots\ d_q(u_n). \end{aligned}$$

Proof. We use the q-Beta integral.

$$\begin{aligned} \text{LHS} &\overset{\text{bydef}}{=} \sum_{m_1, \dots, m_n = 0}^{\infty} \frac{\langle a; q \rangle_m \vec{x}^{\vec{m}}}{\langle \vec{1}; q \rangle_{\vec{m}}} \Gamma_q \begin{bmatrix} b_1 + m_1, c_1 - b_1, \dots, b_n + m_n, c_n - b_n \\ c_1 + m_1, \dots, c_n + m_n \end{bmatrix} \\ &= \int_{\vec{0}}^{\vec{1}} (q\vec{u}; q)_{\vec{c} - \vec{b} - \vec{1}} \sum_{m_1, \dots, m_n = 0}^{\infty} \frac{\langle a; q \rangle_m \vec{u}^{\vec{m}} \vec{x}^{\vec{m}}}{\langle \vec{1}; q \rangle_{\vec{m}}} \vec{u}^{\vec{b} - \vec{1}}\ d_q(\vec{u}) \overset{\text{by(17)}}{=} \text{RHS}. \end{aligned}$$

□

9. Corresponding q-Euler integrals

Next we find a q-analogue of a multiple integral formula for a q-Kampé de Fériet function.

First we repeat the definition.

Definition 31 ([4, p. 368]). The vectors

$$(a), (b), (g_i), (h_i), (a'), (b'), (g'_i), (h'_i)$$

have dimensions $A, B, G, H, A', B', G', H'$. Let

$$1 + B + B' + H + H' - A - A' - G - G' \geq 0.$$

Then the generalized q-Kampé de Fériet function is defined by

$$\Phi^{A+A':G+G'}_{B+B':H+H'}\left[\begin{array}{c}(\hat{a}) : (\hat{g_1}); \ldots ; (\hat{g_n}) \\ (\hat{b}) : (\hat{h_1}); \ldots ; (\hat{h_n})\end{array}\middle| \vec{q}; \vec{x} \middle\| \begin{array}{c}(a') : (g'_1); \ldots ; (g'_n) \\ (b') : (h'_1); \ldots ; (h'_n)\end{array}\right]$$

$$\equiv \sum_{\vec{m}} \frac{\langle(\hat{a}); q_0\rangle_m (a')(q_0, m) \prod_{j=1}^{n} (\langle(\hat{g_j}); q_j\rangle_{m_j} ((g'_j)(q_j, m_j) x_j^{m_j})}{\langle(\hat{b}); q_0\rangle_m (b')(q_0, m) \prod_{j=1}^{n} (\langle(\hat{h_j}); q_j\rangle_{m_j} (h'_j)(q_j, m_j)\langle 1; q_j\rangle_{m_j})}$$

$$\times (-1)^{\sum_{j=1}^{n} m_j (1+H+H'-G-G'+B+B'-A-A')}$$

$$\times \mathrm{QE}\left((B + B' - A - A')\binom{m}{2}, q_0\right)$$

$$\times \prod_{j=1}^{n} \mathrm{QE}\left((1 + H + H' - G - G')\binom{m_j}{2}, q_j\right),$$

where

$$\hat{a} \equiv a \vee \tilde{a} \vee \widetilde{\tfrac{m}{n} a} \vee_k \widetilde{a} \vee \triangle(q; l; \lambda).$$

It is assumed that there are no zero factors in the denominator. We assume that $(a')(q_0, m)$, $(g'_j)(q_j, m_j)$, $(b')(q_0, m)$, $(h'_j)(q_j, m_j)$ contain factors of the form $\langle a(\hat{k}); q\rangle_k$, $(s; q)_k, (s(k); q)_k$ or $\mathrm{QE}\,(f(\vec{m}))$.

Theorem 9.1 (A q-analogue of [1, p. 154]). *Put*

$$C \equiv \Gamma_q\left[\begin{array}{c}\delta_1, \ldots, \delta_n, \delta'_1, \ldots, \delta'_n \\ \beta_1, \ldots, \beta_n, \beta'_1, \ldots, \beta'_n, \delta_1 - \beta_1, \ldots, \delta_n - \beta_n, \delta'_1 - \beta'_1, \ldots, \delta'_n - \beta'_n\end{array}\right].$$

Then

$$\Phi^{1:n+1}_{1:n}\left[\begin{array}{c}\alpha : \beta_1, \ldots, \beta_n, \infty; \beta'_1, \ldots, \beta'_n, \infty \\ \infty : \delta_1, \ldots, \delta_n; \delta'_1, \ldots, \delta'_n\end{array}\middle| q; x, y\right]$$

$$= C \int_0^1 \ldots (2n) \ldots \int_0^1 (1 \boxminus_q q^{\alpha} u_1 \ldots u_n x \boxminus_q q^{\alpha} v_1 \ldots v_n y)^{-\alpha}$$

$$\prod_{i=1}^{n}\left[u_i^{\beta_i - 1} v_i^{\beta'_i - 1} (q u_i; q)_{\delta_i - \beta_i - 1} (q v_i; q)_{\delta'_i - \beta'_i - 1}\, d_q(u_i)\, d_q(v_i)\right].$$

Proof.

$$\text{LHS} = \sum_{m_1,m_2=0}^{\infty} \frac{\langle \alpha;q\rangle_{m_1+m_2}\langle \vec{\beta};q\rangle_{m_1}\langle \vec{\beta'};q\rangle_{m_2}}{\langle \vec{\delta},1;q\rangle_{m_1}\langle \vec{\delta'},1;q\rangle_{m_2}} x^{m_1}y^{m_2}$$

$$\overset{\text{by [4, 1.46]}}{=} C \sum_{m_1,m_2=0}^{\infty} \frac{\langle \alpha;q\rangle_{m_1+m_2}}{\langle 1;q\rangle_{m_1}\langle 1;q\rangle_{m_2}} x^{m_1}y^{m_2}\Gamma_q\left[\begin{matrix}\beta_1+m_1,\ldots,\beta_n+m_1\\ \delta_1+m_1,\ldots,\delta_n+m_1\end{matrix}\right]$$

$$\Gamma_q\left[\begin{matrix}\beta'_1+m_2,\ldots,\beta'_n+m_2,\delta_1-\beta_1,\ldots,\delta_n-\beta_n,\delta'_1-\beta'_1,\ldots,\delta'_n-\beta'_n\\ \delta'_1+m_2,\ldots,\delta'_n+m_2\end{matrix}\right]$$

$$\overset{\text{by } 2n\times \text{ [4, 7.55]}}{=} C\int_0^1 \ldots (2n) \ldots \int_0^1$$

$$\sum_{m_1,m_2=0}^{\infty} \frac{\langle \alpha;q\rangle_{m_1+m_2}}{\langle 1;q\rangle_{m_1}\langle 1;q\rangle_{m_2}} (u_1\ldots u_n x)^{m_1}(v_1\ldots v_n y)^{m_2}$$

$$\prod_{i=1}^{n}\left[u_i^{\beta_i-1}v_i^{\beta'_i-1}(qu_i;q)_{\delta_i-\beta_i-1}(qv_i;q)_{\delta'_i-\beta'_i-1}\,d_q(u_i)\,d_q(v_i)\right] \overset{\text{by (17)}}{=} \text{RHS}.$$

□

Definition 32. The triple q-Saran function Φ_M is defined by

$$\Phi_M \equiv \sum_{m,n,p=0}^{\infty} \frac{\langle \beta_2;q\rangle_n\langle \beta_1;q\rangle_{m+p}\langle \alpha_1;q\rangle_m\langle \alpha_2;q\rangle_{n+p}}{\langle 1,\gamma_1;q\rangle_m\langle 1;q\rangle_n\langle 1;q\rangle_p\langle \gamma_2;q\rangle_{n+p}} x^m y^n z^p.$$

Theorem 9.2. *Here is a q-analogue of Saran* [17, 2.13]*:*

$$\Phi_M = \Gamma_q\left[\begin{matrix}\gamma_1,\gamma_2\\ \alpha_1,\alpha_2,\gamma_1-\alpha_1,\gamma_2-\alpha_2\end{matrix}\right]\int_0^1\int_0^1 u^{\alpha_1-1}(qu;q)_{\gamma_1-\alpha_1-1}v^{\alpha_2-1}$$

$$(qv;q)_{\gamma_2-\alpha_2-1}\frac{1}{(vy;q)_{\beta_2}}(1\boxminus_q q^{\beta_1}ux\boxminus_q q^{\beta_1}vz)^{-\beta_1}\,d_q(u)\,d_q(v).$$

Proof.

$$\text{LHS} = \sum_{\vec{m}=\vec{0}}^{\vec{\infty}} \frac{\langle \beta_2;q\rangle_n\langle \beta_1;q\rangle_{m+p}\langle \alpha_1;q\rangle_m\langle \alpha_2;q\rangle_{n+p}}{\langle 1,\gamma_1;q\rangle_m\langle 1;q\rangle_n\langle 1;q\rangle_p\langle \gamma_2;q\rangle_{n+p}} x^m y^n z^p$$

$$\overset{\text{by [4, 1.46]}}{=} \sum_{\vec{m}=\vec{0}}^{\vec{\infty}} \frac{\langle \beta_2;q\rangle_n\langle \beta_1;q\rangle_{m+p}}{\langle 1;q\rangle_m\langle 1;q\rangle_n\langle 1;q\rangle_p} x^m y^n z^p\,\Gamma_q\left[\begin{matrix}\gamma_1,\gamma_2,\alpha_1+m,\alpha_2+n+p\\ \alpha_1,\alpha_2,\gamma_1+m,\gamma_2+n+p\end{matrix}\right]$$

$$\overset{\text{by [4, 7.55]}}{=} \Gamma_q\left[\begin{matrix}\gamma_1,\gamma_2\\ \alpha_1,\alpha_2,\gamma_1-\alpha_1,\gamma_2-\alpha_2\end{matrix}\right]\int_0^1\int_0^1 u^{\alpha_1-1}(qu;q)_{\gamma_1-\alpha_1-1}v^{\alpha_2-1}$$

$$(qv;q)_{\gamma_2-\alpha_2-1}\sum_{\vec{m}=\vec{0}}^{\vec{\infty}} \frac{\langle \beta_2;q\rangle_n\langle \beta_1;q\rangle_{m+p}}{\langle 1;q\rangle_m\langle 1;q\rangle_n\langle 1;q\rangle_p}(ux)^m(vy)^n(vz)^p \overset{\text{by [4, 7.27], (17))}}{=} \text{RHS}.$$

□

Theorem 9.3. *Here is a q-analogue of Winter* [18, p. 42]*. Put*

$$C \equiv \Gamma_q \begin{bmatrix} b_1, b_2, b_3 \\ a_2, a_3, a_4, b_1 - a_2, b_2 - a_3, b_3 - a_4 \end{bmatrix}, \; d_q \equiv d_q(s)\, d_q(t)\, d_q(u).$$

Then ${}_4\phi_3$ has the triple integral representation

$$\begin{aligned} &{}_4\phi_3 \left[\begin{matrix} a_1, a_2, a_3, a_4 \\ b_1, b_2, b_3 \end{matrix} \middle| q; x \right] \\ &= C \int_{\vec{0}}^{\vec{1}} \frac{s^{a_2-1} t^{a_3-1} u^{a_4-1}}{(xstu; q)_{a_1}} (qs; q)_{b_1-a_2-1} (qt; q)_{b_2-a_3-1} (qu; q)_{b_3-a_4-1}\, d_q. \end{aligned}$$

Proof.

$$\begin{aligned} \text{RHS} &\overset{\text{by [4, 7.27]}}{=} C \int_{\vec{0}}^{\vec{1}} \sum_{n=0}^{\infty} \frac{\langle a_1; q\rangle_n}{\langle 1; q\rangle_n} (xstu)^n \\ &\times s^{a_2-1} t^{a_3-1} u^{a_4-1} (qs; q)_{b_1-a_2-1} (qt; q)_{b_2-a_3-1} (qu : q)_{b_3-a_4-1}\, d_q \\ &\overset{\text{by [4, 7.54]}}{=} C \sum_{n=0}^{\infty} \frac{\langle a_1; q\rangle_n}{\langle 1; q\rangle_n} \Gamma_q \begin{bmatrix} a_2 + n, a_3 + n, a_4 + n, b_1 - a_2, b_2 - a_3, b_3 - a_4 \\ b_1 + n_2, b_2 + n_3, b_3 + n \end{bmatrix} \\ &= \text{LHS}. \end{aligned}$$

□

This formula can easily be generalized to a q-analogue of Erdélyi [3, p. 268].

Theorem 9.4. *Let $\vec{a}, \vec{b}$ be vectors of length n. Then the function ${}_{n+1}\phi_n$ has the n-fold integral representation*

$$\begin{aligned} &{}_{n+1}\phi_n \left[\begin{matrix} a, a_1, \ldots, a_n \\ b_1, b_2, \ldots, b_n \end{matrix} \middle| q; x \right] \\ &= \Gamma_q \begin{bmatrix} \vec{b} \\ \vec{a}, b \vec{-} a \end{bmatrix} \int_{\vec{0}}^{\vec{1}} \frac{\vec{x}^{a \vec{-} 1}}{(xu_1 \ldots u_n; q)_a} (q\vec{u}; q)_{b - \vec{a} - 1}\, d_q(u_1)\ \ldots d_q(u_n). \end{aligned}$$

10. Applications of the new q-numbers

We show how the new numbers can be applied to q-addition formulas for q-trigonometric and hyperbolic functions. Relying on the definitions in [4], in the following two sections, we present q-tangens and cotangens functions together with some fundamental theorems about zeros of q-sines and cosines. These zeros are important since they replace multiples of π in these trigonometric definitions. If the series expansion is not defined, we can use the corresponding q-addition formula to define the q-addition for the product expansion. In the following, the notation $\oplus_q$ in connection with $\pm$ and $\mp$ always means $\oplus_q \vee \ominus_q$. Likewise, the notation $\boxplus$ in connection with $\pm$ and $\mp$ always means $\boxplus \vee \boxminus$.

The following definitions can be found in [4].

Definition 33. If $|q| > 1 \vee 0 < |q| < 1$, $|x| < |1-q|^{-1}$,

$$\mathrm{Sin}_q(x) \equiv \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{\{2n+1\}_q!},$$

$$\mathrm{Cos}_q(x) \equiv \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{\{2n\}_q!}.$$

The complementary q-trigonometric functions are defined by

$$\mathrm{Sin}_{\frac{1}{q}}(x) \equiv \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{\{2n+1\}_q!} \mathrm{QE}\left(\binom{2n+1}{2}\right), \quad x \in \mathbb{C},\ |q| < 1,$$

$$\mathrm{Cos}_{\frac{1}{q}}(x) \equiv \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{\{2n\}_q!} \mathrm{QE}\left(\binom{2n}{2}\right), \quad x \in \mathbb{C},\ |q| < 1.$$

The following theorem for powers and products of q-trigonometric functions is easily proved.

Theorem 10.1.

$$\mathrm{Cos}_q^2(x) - \mathrm{Sin}_q^2(x) = \mathrm{Cos}_q(\overline{2}_q x).$$

$$\mathrm{Cos}_q(x)\mathrm{Sin}_q(x) = \mathrm{Sin}_q(\overline{2}_q x).$$

$$\mathrm{Cos}_q(x)\mathrm{Cos}_{\frac{1}{q}}(x) + \mathrm{Sin}_q(x)\mathrm{Sin}_{\frac{1}{q}}(x) = 1. \tag{18}$$

$$\mathrm{Cos}_q^3(x) = \frac{1}{4}(3\mathrm{Cos}_q(x)\mathrm{Cos}_q(x \ominus_q x) + \mathrm{Cos}_q(\overline{3}_q x)).$$

$$\mathrm{Sin}_q^3(x) = \frac{1}{4}(3\mathrm{Sin}_q(x)\mathrm{Cos}_q(x \ominus_q x) - \mathrm{Sin}_q(\overline{3}_q x)).$$

The following example illustrates the q-analogues of the chain rule. One can easily check the validity of the formulas for $q = 1$. For the corresponding formulas with q-hyperbolic functions, take away the minus signs.

Example 1. For the q-derivatives of $\mathrm{Cos}_q(\overline{2}_q x)$ and $\mathrm{Sin}_q(\overline{2}_q x)$ we obtain

$$\mathrm{D}_q^m \mathrm{Cos}_q(\overline{2}_q x) = \begin{cases} \sum_{k=0}^{\infty} \frac{(\overline{2}_q)^{2k+m}(-1)^{k+f(m)}}{\{2k\}_q!} x^{2k}, & m \text{ even} \\ \sum_{k=0}^{\infty} \frac{(\overline{2}_q)^{2k+m+1}(-1)^{k+f(m)}}{\{2k+1\}_q!} x^{2k+1}, & m \text{ odd}, \end{cases}$$

where

$$f(m) = \begin{cases} 1 & \text{if } m \equiv 1 \bmod 4 \\ 1 & \text{if } m \equiv 2 \bmod 4 \\ 0 & \text{if } m \equiv 3 \bmod 4 \\ 0 & \text{if } m \equiv 0 \bmod 4. \end{cases}$$

$$\mathrm{D}_q^m \mathrm{Sin}_q(\overline{2}_q x) = \begin{cases} \sum_{k=0}^{\infty} \frac{(\overline{2}_q)^{2k+m+1}(-1)^{k+f(m)}}{\{2k+1\}_q!} x^{2k+1}, & m \text{ even} \\ \sum_{k=0}^{\infty} \frac{(\overline{2}_q)^{2k+m}(-1)^{k+f(m)}}{\{2k\}_q!} x^{2k}, & m \text{ odd}, \end{cases}$$

where

$$f(m) = \begin{cases} 0 & \text{if } m \equiv 1 \bmod 4 \\ 1 & \text{if } m \equiv 2 \bmod 4 \\ 1 & \text{if } m \equiv 3 \bmod 4 \\ 0 & \text{if } m \equiv 0 \bmod 4. \end{cases}$$

Definition 34. Denote the kth zero of $\mathrm{Sin}_q(x)$, $x > 0$ by $\xi(q,k)$. Denote the kth zero of $\mathrm{Cos}_q(x)$, $x > 0$ by $\tau(q,k)$.

Theorem 10.2. *First equality between q-trigonometric zeros:*

$$\xi(q,k) = \xi\left(\frac{1}{q}, k\right), \quad k > 0. \tag{19}$$

Theorem 10.3. *Second equality between q-trigonometric zeros:*

$$\tau(q,k) = \tau\left(\frac{1}{q}, k\right), \quad k > 0. \tag{20}$$

Proof. We prove (19). Use

$$\mathrm{E}_q(ix)\mathrm{E}_{\frac{1}{q}}(-ix) = 1, \tag{21}$$

and put $x = \xi(q,k)$. This implies $\mathrm{Sin}_{\frac{1}{q}}(\xi(q,k)) = 0$, since the right hand side is real. Equation (20) is proved in a similar way. □

Theorem 10.4.

$$\mathrm{Sin}_q(\tau(q,k))\mathrm{Sin}_{\frac{1}{q}}(\tau(q,k)) = 1, \quad k > 0.$$
$$\mathrm{Cos}_q(\xi(q,k))\mathrm{Cos}_{\frac{1}{q}}(\xi(q,k)) = 1, \quad k > 0.$$

Proof. Put $x = \xi(q,k)$ in (21). □

Theorem 10.5. *The function $f(x) : x \mapsto \mathrm{Sin}_q(x)\mathrm{Sin}_{\frac{1}{q}}(x)$ has extreme values for $x = \tau(q,k)$, $k > 0$.*

The function $g(x) : x \mapsto \mathrm{Cos}_q(x)\mathrm{Cos}_{\frac{1}{q}}(x)$ has extreme values for $x = \xi(q,k)$, $k > 0$. These extreme values are both 1.

Proof. We prove the first statement. Differentiate formula (18) with respect to x and put $x = \tau(q,k)$. The second term on the left is zero, as well as the right hand side. This proves the first part. The second part follows from Theorem 10.4. □

Remark 2. Numerical computations show that both functions $f(x)$ and $g(x)$ are positive, which means that

$$\mathrm{Sin}_q(x),\ \mathrm{Sin}_{\frac{1}{q}}(x) \text{ and } \mathrm{Cos}_q(x),\ \mathrm{Cos}_{\frac{1}{q}}(x)$$

have the same signs for a fixed value of x, respectively. They have the same zeros by Theorems 10.2 and 10.3. This means that the extreme values in Theorem 10.5 are maxima.

Remark 3. Theorem 10.5 does not mean that the maxima and minima of

$$\mathrm{Sin}_q(x),\ \mathrm{Sin}_{\frac{1}{q}}(x) \text{ and } \mathrm{Cos}_q(x),\ \mathrm{Cos}_{\frac{1}{q}}(x)$$

occur for $x = \tau(q,k)$ and $x = \xi(q,k)$, respectively.

On the basis of Theorem 10.4 and numerical computations we make a guess:

Conjecture 2. The function $f(x) = \mathrm{Sin}_q(x) \vee \mathrm{Cos}_q(x)$ for fixed $q < 1$ oscillates between strictly decreasing positive maximum values and strictly increasing negative minimum values as function of $x > 0$.

Conjecture 3. The function $f(x) = \mathrm{Sin}_{\frac{1}{q}}(x) \vee \mathrm{Cos}_{\frac{1}{q}}(x)$ for fixed $q < 1$ oscillates between strictly increasing positive maximum values and strictly decreasing negative minimum values as function of $x > 0$.

Definition 35. The q-tangens and cotangens for $x > 0$ are defined by

$$\mathrm{Tan}_q(x) \equiv \frac{\mathrm{Sin}_q(x)}{\mathrm{Cos}_q(x)}, \quad x \neq \tau(q,k).$$
$$\mathrm{Cot}_q(x) \equiv \frac{\mathrm{Cos}_q(x)}{\mathrm{Sin}_q(x)}, \quad x \neq \xi(q,k).$$

These two functions are odd.

Theorem 10.6. *Formulas for q-Tangens and Cotangens:*

$$\mathrm{Tan}_q(x \oplus_q y) = \frac{\mathrm{Tan}_q(x) \pm \mathrm{Tan}_q(y)}{1 \mp \mathrm{Tan}_q(x)\mathrm{Tan}_q(y)}.$$
$$\mathrm{Cot}_q(x \oplus_q y) = \frac{\mathrm{Cot}_q(x)\mathrm{Cot}_q(y) \mp 1}{\mathrm{Cot}_q(x) \pm \mathrm{Cot}_q(y)}.$$
$$\mathrm{Tan}_q(\overline{2}_q x) = \frac{2\mathrm{Tan}_q(x)}{1 - \mathrm{Tan}_q^2(x)}.$$
$$\mathrm{Cot}_q(\overline{2}_q x) = \frac{\mathrm{Cot}_q^2(x) - 1}{2\mathrm{Cot}_q(x)}.$$

Again, the following formulas closely resemble the ordinary ones.

Theorem 10.7. *More q-differences for quotient q-trigonometric functions.*

$$\mathrm{D}_q\mathrm{Tan}_q(x) = \frac{\mathrm{Cos}_q(x \ominus_q x)}{\mathrm{Cos}_q(x)\ \mathrm{Cos}_q(qx)}, \tag{22}$$

$$\mathrm{D}_q\mathrm{Cot}_q(x) = -\frac{\mathrm{Cos}_q(x \ominus_q x)}{\mathrm{Sin}_q(x)\ \mathrm{Sin}_q(qx)}, \tag{23}$$

$$\mathrm{D}_q\left(\frac{1}{\mathrm{Sin}_q(x)}\right) = -\frac{\mathrm{Cos}_q(x)}{\mathrm{Sin}_q(x)\ \mathrm{Sin}_q(qx)}, \tag{24}$$

$$\mathrm{D}_q\left(\frac{1}{\mathrm{Cos}_q(x)}\right) = \frac{\mathrm{Sin}_q(x)}{\mathrm{Cos}_q(x)\ \mathrm{Cos}_q(qx)}. \tag{25}$$

Proof. Use formulas [4, 6.49, 6.170 and 6.171]. □

Remark 4. The numerators in formulas (22) and (23) are q-analogues of 1.

We shall now prove some new q-analogues of trigonometric formulas. The q-rational numbers will be used when we divide the function argument by '2'.

Theorem 10.8. *There are the following formulas with squares of q-trigonometric functions.*

$$\mathrm{Cos}_q^2(x) = \frac{1}{2}\left(\mathrm{Cos}_q(x \ominus_q x) + \mathrm{Cos}_q(\overline{2}_q x)\right), \tag{26}$$

$$\mathrm{Sin}_q^2(x) = \frac{1}{2}\left(\mathrm{Cos}_q\left(x \ominus_q x\right) - \mathrm{Cos}_q\left(\overline{2}_q x\right)\right), \tag{27}$$

$$\mathrm{Sin}_q^2(x) + \mathrm{Cos}_q^2(x) = \mathrm{Cos}_q(x \ominus_q x). \tag{28}$$

Proof. To prove (26), use [4, 6.236] twice with $y = -x$ and $y = x$ and add the results. □

Corollary 10.9. *Formulas with 'half' function argument.*

$$\mathrm{Cos}_q\left(\frac{x}{\overline{2}_q}\right) = \pm\sqrt{\frac{\mathrm{Cos}_q(\frac{x\ominus_q x}{\overline{2}_q}) + \mathrm{Cos}_q(x)}{2}}, \tag{29}$$

$$\mathrm{Sin}_q\left(\frac{x}{\overline{2}_q}\right) = \pm\sqrt{\frac{\mathrm{Cos}_q(\frac{x\ominus_q x}{\overline{2}_q}) - \mathrm{Cos}_q(x)}{2}}, \tag{30}$$

$$\mathrm{Tan}_q\left(\frac{x}{\overline{2}_q}\right) = \pm\sqrt{\frac{\mathrm{Cos}_q(\frac{x\ominus_q x}{\overline{2}_q}) - \mathrm{Cos}_q(x)}{\mathrm{Cos}_q(\frac{x\ominus_q x}{\overline{2}_q}) + \mathrm{Cos}_q(x)}}, \tag{31}$$

$$\mathrm{Cot}_q\left(\frac{x}{\overline{2}_q}\right) = \pm\sqrt{\frac{\mathrm{Cos}_q(\frac{x\ominus_q x}{\overline{2}_q}) + \mathrm{Cos}_q(x)}{\mathrm{Cos}_q(\frac{x\ominus_q x}{\overline{2}_q}) - \mathrm{Cos}_q(x)}}, \quad x \neq 0. \tag{32}$$

Proof. To prove (29), take the square root of formula (26) and replace x by $\frac{x}{\overline{2}_q}$. □

10.1. New q-hyperbolic formulas. After having considered q-trigonometric function formulas, it is easy to switch to q-hyperbolic functions.

Definition 36. The corresponding q-hyperbolic functions are [4, p. 229 f]:

$$\mathrm{Sinh}_q(x) \equiv \sum_{k=0}^{\infty} \frac{x^{2k+1}}{\{2k+1\}_q!}, \quad |(1-q)x| < 1.$$

$$\mathrm{Cosh}_q(x) \equiv \sum_{k=0}^{\infty} \frac{x^{2k}}{\{2k\}_q!}, \quad |(1-q)x| < 1.$$

$$\mathrm{Sinh}_{\frac{1}{q}}(x) \equiv \sum_{k=0}^{\infty} \frac{x^{2k+1}}{\{2k+1\}_q!} q^{\binom{2k+1}{2}}.$$

$$\mathrm{Cosh}_{\frac{1}{q}}(x) \equiv \sum_{k=0}^{\infty} \frac{x^{2k}}{\{2k\}_q!} q^{\binom{2k}{2}}.$$

$$\mathrm{Tanh}_q(x) \equiv \frac{\mathrm{Sinh}_q(x)}{\mathrm{Cosh}_q(x)}, \quad |(1-q)x| < 1.$$

$$\mathrm{Coth}_q(x) \equiv \frac{\mathrm{Cosh}_q(x)}{\mathrm{Sinh}_q(x)}, \quad |(1-q)x| < 1, \; x \neq 0.$$

$$\mathrm{Tanh}_{\frac{1}{q}}(x) \equiv \frac{\mathrm{Sinh}_{\frac{1}{q}}(x)}{\mathrm{Cosh}_{\frac{1}{q}}(x)}.$$

$$\mathrm{Coth}_{\frac{1}{q}}(x) \equiv \frac{\mathrm{Cosh}_{\frac{1}{q}}(x)}{\mathrm{Sinh}_{\frac{1}{q}}(x)}, \quad x \neq 0.$$

We have chosen not to use the names for inverse ratios of hyperbolic functions.

Our next aim is to show pictures of q-hyperbolic functions, which resemble the four basic graphs for hyperbolic functions and their inverse ratios. Each of these pictures contain two functions, just like in the elementary textbooks. We choose five examples; in order to show the similarity with the q-hyperbolic functions from [4, p. 229 f.], we begin with five graphs of the latter functions. Everywhere we have $q = .9$. Figures (1)–(5) show [$\mathrm{Sinh}_q(x)$, $\mathrm{Cosh}_q(x)$], [$\mathrm{Sinh}_{\frac{1}{q}}(x)$, $\mathrm{Cosh}_{\frac{1}{q}}(x)$], [$\mathrm{Cosh}_q(x)$, $(\mathrm{Cosh}_q(x))^{-1}$], [$\mathrm{Sinh}_q(x)$, $(\mathrm{Sinh}_q(x))^{-1}$] and [$\mathrm{Tanh}_q(x)$, $\mathrm{Coth}_q(x)$], respectively.

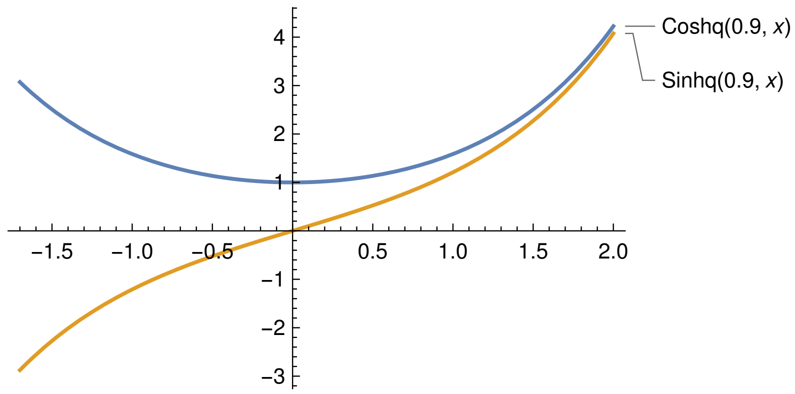

FIGURE 1. $\mathrm{Sinh}_{.9}(x)$, $\mathrm{Cosh}_{.9}(x)$

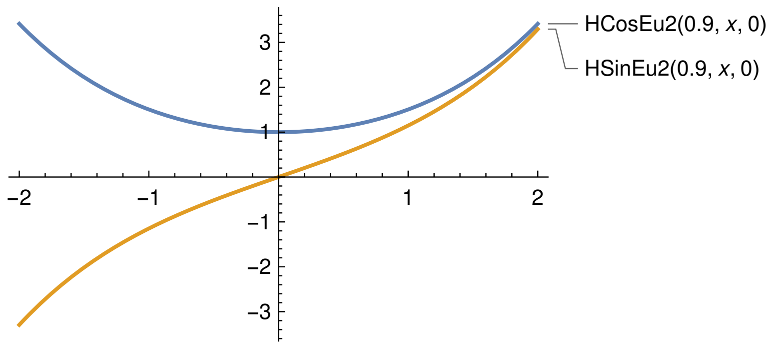

FIGURE 2. $\mathrm{Sinh}_{\frac{1}{.9}}(x)$, $\mathrm{Cosh}_{\frac{1}{.9}}(x)$

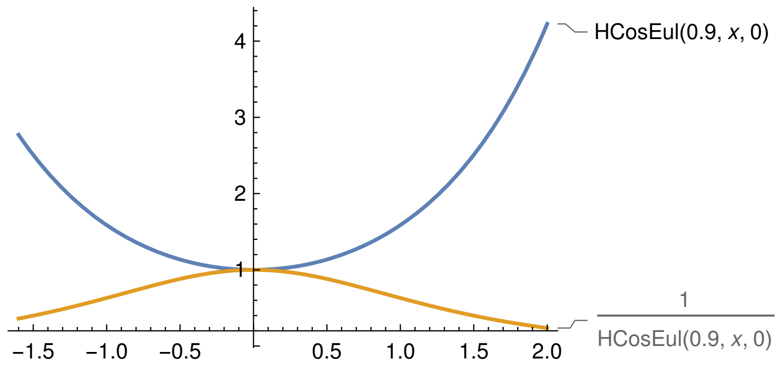

FIGURE 3. $\mathrm{Cosh}_{.9}(x)$, $(\mathrm{Cosh}_{.9}(x))^{-1}$

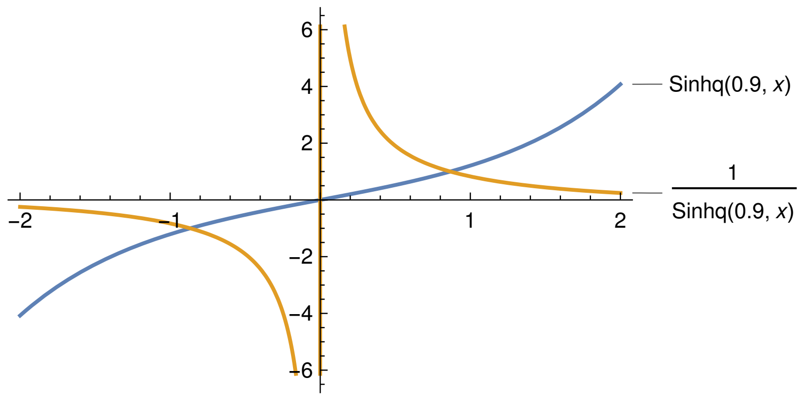

FIGURE 4. $\mathrm{Sinh}_{.9}(x),\ (\mathrm{Sinh}_{.9}(x))^{-1}$

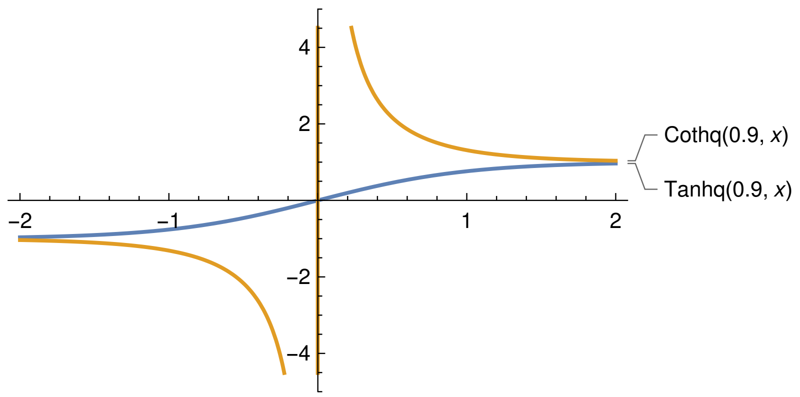

FIGURE 5. $\mathrm{Tanh}_{.9}(x),\ \mathrm{Coth}_{.9}(x)$

The next theorem will be used to prove q-addition theorems for tanh and cotanh-functions.

Theorem 10.10. *Eight q-addition theorems for q-hyperbolic functions from our first book* [4, p. 230]*:*

$$
\begin{aligned}
\mathrm{Cosh}_q(x)\mathrm{Cosh}_q(y) \pm \mathrm{Sinh}_q(x)\mathrm{Sinh}_q(y) &= \mathrm{Cosh}_q(x \oplus_q y),\\
\mathrm{Sinh}_q(x)\mathrm{Cosh}_q(y) \pm \mathrm{Sinh}_q(y)\mathrm{Cosh}_q(x) &= \mathrm{Sinh}_q(x \oplus_q y),\\
\mathrm{Cosh}_q(x)\mathrm{Cosh}_{\frac{1}{q}}(y) \pm \mathrm{Sinh}_q(x)\mathrm{Sinh}_{\frac{1}{q}}(y) &= \mathrm{Cosh}_q(x \boxplus_q y),\\
\mathrm{Sinh}_q(x)\mathrm{Cosh}_{\frac{1}{q}}(y) \pm \mathrm{Sinh}_{\frac{1}{q}}(y)\mathrm{Cosh}_q(x) &= \mathrm{Sinh}_q(x \boxplus_q y).
\end{aligned}
$$

Theorem 10.11. *Formulas with q-addition and Ward number for q-Tanghyp and Cothyp:*

$$
\mathrm{Tanh}_q(x \oplus_q y) = \frac{\mathrm{Tanh}_q(x) \pm \mathrm{Tanh}_q(y)}{1 \pm \mathrm{Tanh}_q(x)\mathrm{Tanh}_q(y)},
$$

$$
\begin{aligned}
\mathrm{Coth}_q(x \oplus_q y) &= \frac{1 \pm \mathrm{Coth}_q(x)\mathrm{Coth}_q(y)}{\mathrm{Coth}_q(x) \pm \mathrm{Coth}_q(y)}, \quad (x,y) \neq (0,0),\\
\mathrm{Tanh}_q(\overline{2}_q x) &= \frac{2\mathrm{Tanh}_q(x)}{1+\mathrm{Tanh}_q^2(x)},\\
\mathrm{Coth}_q(\overline{2}_q x) &= \frac{\mathrm{Coth}_q^2(x)+1}{2\mathrm{Coth}_q(x)}.
\end{aligned}
$$

Corollary 10.12. *Basic formulas for q-hyperbolic functions:*

$$
\begin{gathered}
\mathrm{Cosh}_q^2(x) + \mathrm{Sinh}_q^2(x) = \mathrm{Cosh}_q(\overline{2}_q x).\\
\mathrm{Cosh}_q(x)\mathrm{Sinh}_q(x) = \mathrm{Sinh}_q(\overline{2}_q x).\\
\mathrm{Cosh}_q(x)\mathrm{Cosh}_{\frac{1}{q}}(x) - \mathrm{Sinh}_q(x)\mathrm{Sinh}_{\frac{1}{q}}(x) = 1.
\end{gathered}
$$

Theorem 10.13. *Two q-differences for quotient q-hyperbolic functions:*

$$
\mathrm{D}_q\mathrm{Tanh}_q(x) = \frac{\mathrm{Cosh}_q(x \ominus_q x)}{\mathrm{Cosh}_q(x)\ \epsilon\,\mathrm{Cos}_q(x)}, \tag{33}
$$

$$
\mathrm{D}_q\mathrm{Coth}_q(x) = -\frac{\mathrm{Cosh}_q(x \ominus_q x)}{\mathrm{Sinh}_q(x)\ \epsilon\,\mathrm{Sinh}_q(x)}. \tag{34}
$$

Remark 5. The numerators in formulas (33) and (34) are q-analogues of 1.

Theorem 10.14. *Formulas with squares of q-hyperbolic functions:*

$$
\begin{aligned}
\mathrm{Cosh}_q^2(x) &= \frac{1}{2}\left(\mathrm{Cosh}_q(x \ominus_q x) + \mathrm{Cosh}_q(\overline{2}_q x)\right),\\
\mathrm{Sinh}_q^2(x) &= \frac{1}{2}\left(\mathrm{Cosh}_q\left(\overline{2}_q x\right) - \mathrm{Cosh}_q\left(x \ominus_q x\right)\right),\\
\mathrm{Cosh}_q^2(x) - \mathrm{Sinh}_q^2(x) &= \mathrm{Cosh}_q(x \ominus_q x).
\end{aligned}
$$

Corollary 10.15. *Formulas with 'half' function argument:*

$$
\begin{aligned}
\mathrm{Cosh}_q\left(\frac{x}{\overline{2}_q}\right) &= \sqrt{\frac{\mathrm{Cosh}_q(\frac{x \ominus_q x}{\overline{2}_q}) + \mathrm{Cosh}_q(x)}{2}},\\
\mathrm{Sinh}_q\left(\frac{x}{\overline{2}_q}\right) &= \pm\sqrt{\frac{\mathrm{Cosh}_q(x) - \mathrm{Cosh}_q(\frac{x \ominus_q x}{\overline{2}_q})}{2}},\\
\mathrm{Tanh}_q\left(\frac{x}{\overline{2}_q}\right) &= \pm\sqrt{\frac{\mathrm{Cosh}_q(x) - \mathrm{Cosh}_q(\frac{x \ominus_q x}{\overline{2}_q})}{\mathrm{Cosh}_q(\frac{x \ominus_q x}{\overline{2}_q}) + \mathrm{Cosh}_q(x)}},\\
\mathrm{Coth}_q\left(\frac{x}{\overline{2}_q}\right) &= \pm\sqrt{\frac{\mathrm{Cosh}_q(\frac{x \ominus_q x}{\overline{2}_q}) + \mathrm{Cosh}_q(x)}{\mathrm{Cosh}_q(x) - \mathrm{Cosh}_q(\frac{x \ominus_q x}{\overline{2}_q})}}, \quad x \neq 0.
\end{aligned}
$$

11. Conclusion

We have defined three types of q-real numbers. The two first, $\mathbb{R}_{\oplus_q}$ and $\mathbb{R}_q$ do not have a specific form, but are quite general. The third one, $\mathbb{R}_{\boxplus_q}$, has a specific form, which only consists of nonassociative JHC q-additions. Of course we have $\mathbb{R}_{\boxplus_q} \subset \mathbb{R}_q$, although these number systems are quite different. Several of the results in the last section on q-trigonometric and hyperbolic functions were previously published in more general q,ω-form in our paper [11]. Since the graphs of $\mathrm{Tan}_q(x)$ and $\mathrm{Cot}_q(x)$ closely resemble the graphs of the corresponding q,ω-functions, we have skipped these two graphs.

12. Acknowledgements

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