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The gauge-natural bilinear brackets on couples of linear vector fields and linear *p*-forms

Abstract. We give complete description of all gauge-natural bilinear operators A transforming pairs of couples of linear vector fields and linear p-forms on a vector bundle E into couples of linear vector fields and linear p-forms on E and satisfying the Jacobi identity in Leibniz form.

1. Introduction. All manifolds considered in the paper are assumed to be Hausdorff, second countable, finite dimensional, without boundary, and smooth (of class $\mathcal{C}^\infty$). Maps between manifolds are assumed to be $\mathcal{C}^\infty$.

A vector field X on a vector bundle E is called linear if $\mathcal{L}_L X = 0$, where $\mathcal{L}$ is the Lie derivative and L is the Euler vector field. A p-form ω on a vector bundle E is called linear if $\mathcal{L}_L\omega = \omega$.

Let $\mathcal{VB}_{m,n}$ be the category of n-rank vector bundles with m-dimensional bases and their vector bundle isomorphism onto images.

A $\mathcal{VB}_{m,n}$-gauge-natural bilinear operator (bracket)

$$A : \Gamma^l\Big(T \oplus \bigwedge^p T^*\Big) \times \Gamma^l\Big(T \oplus \bigwedge^p T^*\Big) \rightsquigarrow \Gamma^l\Big(T \oplus \bigwedge^p T^*\Big)$$

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is invariant with respect to morphisms of $\mathcal{VB}_{m,n}$ family of $\mathbf{R}$-bilinear operators

$$A : \Gamma^l_E\Big(TE \oplus \bigwedge^p T^*E\Big) \times \Gamma^l_E\Big(TE \oplus \bigwedge^p T^*E\Big) \to \Gamma^l_E\Big(TE \oplus \bigwedge^p T^*E\Big)$$

for all $\mathcal{VB}_{m,n}$-objects E, where $\Gamma^l_E(TE \oplus \bigwedge^p T^*E)$ is the space of couples $X \oplus \omega$ of linear vector fields X and linear p-forms ω on E.

The first main result of the article is the following theorem.

Theorem 1.1. *Let m, n, p be positive integers such that $m \geq p + 1$. Any $\mathcal{VB}_{m,n}$-gauge-natural bilinear operator*

$$A : \Gamma^l\Big(T \oplus \bigwedge^p T^*\Big) \times \Gamma^l\Big(T \oplus \bigwedge^p T^*\Big) \rightsquigarrow \Gamma^l\Big(T \oplus \bigwedge^p T^*\Big)$$

is of the form

$$\begin{aligned} A(X^1 \oplus \omega^1, X^2 \oplus \omega^2) &= a[X^1, X^2] \oplus \{b_1 \mathcal{L}_{X^1}\omega^2 + b_2 \mathcal{L}_{X^2}\omega^1 \\ &\quad + b_3 d i_{X^1}\omega^2 + b_4 d i_{X^2}\omega^1 + b_5 \mathcal{L}_{X^1} d i_L \omega^2 + b_6 \mathcal{L}_{X^2} d i_L \omega^1\} \end{aligned} \tag{1}$$

for arbitrary (uniquely determined by A) real numbers $a, b_1, b_2, b_3, b_4, b_5, b_6$, where $[-,-]$ is the usual bracket on vector fields, $\mathcal{L}$ is the Lie derivative, d is the exterior derivative, i is the insertion derivative and L is the Euler vector field.

A $\mathcal{VB}_{m,n}$-gauge-natural $\mathbf{R}$-bilinear operator A satisfies the Jacobi identity in Leibniz form if

$$A(\nu^1, A(\nu^2, \nu^3)) = A(A(\nu^1, \nu^2), \nu^3) + A(\nu^2, A(\nu^1, \nu^3)) \tag{2}$$

for any $\nu^i \in \Gamma^l_E(TE \oplus \bigwedge^p T^*E)$ for $i = 1, 2, 3$.

The second main result of the article is the following theorem

Theorem 1.2. *Let m, n, p be positive integers such that $m \geq p + 2$. Any $\mathcal{VB}_{m,n}$-gauge-natural $\mathbf{R}$-bilinear operator A of the form (1) satisfies the Jacobi identity in Leibniz form if and only if the 7-tuple $(a, b_1, b_2, b_3, b_4, b_5, b_6)$ is from the following list of 7-tuples:*

$$\begin{aligned} &(c,0,0,0,0,c,0),\ (c,0,0,0,0,c,-c)\,, \\ &(c,c,0,0,0,-c,0),\ (c,c,-c,0,0,-c,c)\,, \\ &(c,0,0,0,0,0,0),\ (c,c,0,0,0,0,0)\,, \\ &(c,c,0,0,0,0,-c),\ (c,c,-c,0,0,0,0)\,, \\ &(c,c,-c,0,c-\lambda,0,\lambda),\ (0,0,0,\lambda,\mu,-\lambda,-\mu)\,, \end{aligned} \tag{3}$$

where c, λ, μ are arbitrary real numbers with $c \neq 0$.

The above Theorems 1.1 and 1.2 for $p = 1$ are proved in [2]. If $p = 1$, the most important example of a $\mathcal{VB}_{m,n}$-gauge-natural $\mathbf{R}$-bilinear operator satisfying the Jacobi identity in Leibniz form is the Dorfman–Courant bracket being the restriction of the well-known Courant bracket.

2. The gauge-natural bilinear brackets on couples of linear vector fields and linear p-forms. Let m, n, p be positive integers.

Let $E = (E \to M)$ be a vector bundle from $\mathcal{VB}_{m,n}$.

Applying the tangent and the cotangent functors, we obtain double vector bundles $(TE; E, TM; M)$ and $(T^*E; E, E^*; M)$.

A vector field X on E is called linear if it is a vector bundle map $X : E \to TE$ between $E \to M$ and $TE \to TM$.

Equivalently, a vector field X on E is linear if it has the expression

$$X = \sum_{i=1}^{m} a^i(x^1, \dots, x^m)\frac{\partial}{\partial x^i} + \sum_{j,k=1}^{n} b^k_j(x^1, \dots x^m)y^j \frac{\partial}{\partial y^k}$$

in any local vector bundle trivialization $x^1, \dots, x^m, y^1, \dots, y^n$ on E.

Equivalently, a vector field X on E is linear iff $\mathcal{L}_L X = 0$, where $\mathcal{L}$ denotes the Lie derivative and L is the Euler vector field on E (in vector bundle coordinates $L = \sum_{j=1}^{n} y^j \frac{\partial}{\partial y^j}$).

Equivalently, a vector field X on E is linear if $(a_t)_* X = X$ for any $t > 0$, where $a_t : E \to E$ is the fibre-homothety by t.

A p-form ω on E is called linear if the induced vector bundle morphism

$$\omega^\sharp : \oplus^{k-1} TE \to T^*E$$

over the identity on E is also a vector bundle morphism over a map $\oplus^{k-1}TM \to E^*$ on the other side of the double vector bundle.

Equivalently, a p-form ω on E is linear if it has the expression

$$\omega = \sum a_{i_1,\dots,i_p,j}(x)y^j dx^{i_1} \wedge \dots \wedge dx^{i_p} + \sum b_{i_1,\dots,i_{p-1},j}(x)dy^j \wedge dx^{i_1} \wedge \dots \wedge dx^{i_{p-1}}$$

in any local vector bundle trivialization $x^1, \dots, x^m, y^1, \dots, y^n$ on E.

Equivalently, a p-form ω on E is linear iff $\mathcal{L}_L \omega = \omega$.

Equivalently, a p-form ω on E is linear iff $(a_{\frac{1}{t}})_* \omega = t\omega$ for any $t > 0$.

We have the following definition being modification of the general one from [1].

Definition 2.1. A $\mathcal{VB}_{m,n}$-gauge-natural bilinear operator (bracket)

$$A : \Gamma^l\Big(T \oplus \bigwedge^p T^*\Big) \times \Gamma^l\Big(T \oplus \bigwedge^p T^*\Big) \rightsquigarrow \Gamma^l\Big(T \oplus \bigwedge^p T^*\Big)$$

is a $\mathcal{VB}_{m,n}$-invariant family of $\mathbf{R}$-bilinear operators

$$A : \Gamma^l_E\Big(TE \oplus \bigwedge^p T^*E\Big) \times \Gamma^l_E\Big(TE \oplus \bigwedge^p T^*E\Big) \to \Gamma^l_E\Big(TE \oplus \bigwedge^p T^*E\Big)$$

for all $\mathcal{VB}_{m,n}$-objects E, where $\Gamma^l_E(TE \oplus \bigwedge^p T^*E)$ is the vector space of linear sections of $TE \oplus \bigwedge^p T^*E$.

Remark 2.2. The $\mathcal{VB}_{m,n}$-invariance of A means that if

$$(X^1 \oplus \omega^1, X^2 \oplus \omega^2) \in \Gamma^l_E\Big(TE \oplus \bigwedge^p T^*E\Big) \times \Gamma^l_E\Big(TE \oplus \bigwedge^p T^*E\Big)$$

and

$$(\overline{X}^1 \oplus \overline{\omega}^1, \overline{X}^2 \oplus \overline{\omega}^2) \in \Gamma^l_{\overline{E}}\Big(T\overline{E} \oplus \bigwedge^p T^*\overline{E}\Big) \times \Gamma^l_{\overline{E}}\Big(T\overline{E} \oplus \bigwedge^p T^*\overline{E}\Big)$$

are φ-related by an $\mathcal{VB}_{m,n}$-map $\varphi : E \to \overline{E}$ (i.e., $\overline{X}^i \circ \varphi = T\varphi \circ X^i$ and $\overline{\omega}^i \circ \varphi = \bigwedge^p T^*\varphi \circ \omega^i$ for $i = 1, 2$), then so are $A(X^1 \oplus \omega^1, X^2 \oplus \omega^2)$ and $A(\overline{X}^1 \oplus \overline{\omega}^1, \overline{X}^2 \oplus \overline{\omega}^2)$.

Remark 2.3. Quite similarly, we can define $\mathcal{VB}_{m,n}$-gauge-natural bilinear operators

$$\begin{aligned}
&\Gamma^l(T) \times \Gamma^l(T) \rightsquigarrow \Gamma^l(T)\,,\\
&\Gamma^l(T) \times \Gamma^l(T) \rightsquigarrow \Gamma^l\Big(\bigwedge^p T^*\Big)\,,\\
&\Gamma^l(T) \times \Gamma^l\Big(\bigwedge^p T^*\Big) \rightsquigarrow \Gamma^l(T)\,,\\
&\Gamma^l(T) \times \Gamma^l\Big(\bigwedge^p T^*\Big) \rightsquigarrow \Gamma^l\Big(\bigwedge^p T^*\Big)\,,\\
&\Gamma^l\Big(\bigwedge^p T^*\Big) \times \Gamma^l(T) \rightsquigarrow \Gamma^l(T)\,,\\
&\Gamma^l\Big(\bigwedge^p T^*\Big) \times \Gamma^l(T) \rightsquigarrow \Gamma^l\Big(\bigwedge^p T^*\Big)\,,\\
&\Gamma^l\Big(\bigwedge^p T^*\Big) \times \Gamma^l\Big(\bigwedge^p T^*\Big) \rightsquigarrow \Gamma^l(T)\,,\\
&\Gamma^l\Big(\bigwedge^p T^*\Big) \times \Gamma^l\Big(\bigwedge^p T^*\Big) \rightsquigarrow \Gamma^l\Big(\bigwedge^p T^*\Big)\,.
\end{aligned}$$

For example, a $\mathcal{VB}_{m,n}$-gauge-natural bilinear operator

$$A : \Gamma^l(T) \times \Gamma^l\Big(\bigwedge^p T^*\Big) \rightsquigarrow \Gamma^l(T)$$

is a $\mathcal{VB}_{m,n}$-invariant family of $\mathbf{R}$-bilinear operators

$$A : \Gamma^l_E(TE) \times \Gamma^l_E\Big(\bigwedge^p T^*E\Big) \to \Gamma^l_E(TE)$$

for all $\mathcal{VB}_{m,n}$-objects E, where $\Gamma^l_E(TE)$ is the space of linear vector fields on E and $\Gamma^l_E(\bigwedge^p T^*E)$ is the space of linear p-forms on E.

Example 2.4. The usual bracket $[X, Y]$ of (linear) vector fields X and Y is again a linear vector field. Thus we have the corresponding $\mathcal{VB}_{m,n}$-gauge-natural bilinear operator

$$[-,-] : \Gamma^l(T) \times \Gamma^l(T) \rightsquigarrow \Gamma^l(T)\,.$$

Example 2.5. The Lie derivative $\mathcal{L}_X\omega$ of a linear p-form ω with respect to a linear vector field X is again a linear p-form. Thus we have the corresponding $\mathcal{VB}_{m,n}$-gauge-natural bilinear operator

$$\mathcal{L} : \Gamma^l(T) \times \Gamma^l\Big(\bigwedge^p T^*\Big) \rightsquigarrow \Gamma^l\Big(\bigwedge^p T^*\Big)\,.$$

Example 2.6. Let ω be a linear p-form and X be a linear vector field on a vector bundle E. Then $i_X d\omega$, where d denotes the exterior derivative and $i_{(-)}$ denotes the insertion derivative, is again a linear p-form. Thus we have the corresponding $\mathcal{VB}_{m,n}$-gauge-natural bilinear operator

$$i_{(-)}d : \Gamma^l(T) \times \Gamma^l\Big(\bigwedge^p T^*\Big) \rightsquigarrow \Gamma^l\Big(\bigwedge^p T^*\Big)\,.$$

Example 2.7. Let ω be a linear p-form and X be a linear vector field on a vector bundle E and L denotes the Euler vector field on E. Then $\mathcal{L}_X d i_L \omega$, is again a linear p-form. Thus we have the corresponding $\mathcal{VB}_{m,n}$-gauge-natural bilinear operator

$$\mathcal{L} d i_L : \Gamma^l(T) \times \Gamma^l\Big(\bigwedge^p T^*\Big) \rightsquigarrow \Gamma^l\Big(\bigwedge^p T^*\Big)\,.$$

Example 2.8. The bracket

$$[[X^1 \oplus \omega^1, X^2 \oplus \omega^2]] := [X^1, X^2] \oplus (\mathcal{L}_{X^1}\omega^2 - i_{X^2} d\omega^1)$$

is a $\mathcal{VB}_{m,n}$-gauge-natural bilinear operator in the sense of Definition 2.1.

Lemma 2.9. *Any $\mathcal{VB}_{m,n}$-gauge-natural bilinear operator A in the sense of Definition 2.1 can be considered (in obvious way) as the system $A = (A^1, A^2, \ldots, A^8)$ of $\mathcal{VB}_{m,n}$-gauge natural bilinear operators*

$$A^1 : \Gamma^l(T) \times \Gamma^l(T) \rightsquigarrow \Gamma^l(T)\,,$$

$$A^2 : \Gamma^l(T) \times \Gamma^l(T) \rightsquigarrow \Gamma^l\Big(\bigwedge^p T^*\Big)\,,$$

$$A^3 : \Gamma^l(T) \times \Gamma^l\Big(\bigwedge^p T^*\Big) \rightsquigarrow \Gamma^l(T)\,,$$

$$A^4 : \Gamma^l(T) \times \Gamma^l\Big(\bigwedge^p T^*\Big) \rightsquigarrow \Gamma^l\Big(\bigwedge^p T^*\Big)\,,$$

$$A^5 : \Gamma^l\Big(\bigwedge^p T^*\Big) \times \Gamma^l(T) \rightsquigarrow \Gamma^l(T)\,,$$

$$A^6 : \Gamma^l\Big(\bigwedge^p T^*\Big) \times \Gamma^l(T) \rightsquigarrow \Gamma^l\Big(\bigwedge^p T^*\Big),$$
$$A^7 : \Gamma^l\Big(\bigwedge^p T^*\Big) \times \Gamma^l\Big(\bigwedge^p T^*\Big) \rightsquigarrow \Gamma^l(T),$$
$$A^8 : \Gamma^l\Big(\bigwedge^p T^*\Big) \times \Gamma^l\Big(\bigwedge^p T^*\Big) \rightsquigarrow \Gamma^l\Big(\bigwedge^p T^*\Big).$$

Proof. The lemma is obvious. □

In the rest of the present section we prove Theorem 1.1, i.e., the following.

Theorem 2.10. *Let m and $n \geq 1$ and $p \geq 1$ be natural numbers such that $m \geq p+1$. Any $\mathcal{VB}_{m,n}$-gauge-natural bilinear operator*

$$A : \Gamma^l\Big(T \oplus \bigwedge^p T^*\Big) \times \Gamma^l\Big(T \oplus \bigwedge^p T^*\Big) \rightsquigarrow \Gamma^l\Big(T \oplus \bigwedge^p T^*\Big)$$

is of the form

$$\begin{aligned} A(X^1 \oplus \omega^1, X^2 \oplus \omega^2) = a[X^1, X^2] \oplus \{ & b_1\mathcal{L}_{X^1}\omega^2 + b_2\mathcal{L}_{X^2}\omega^1 + b_3 d i_{X^1}\omega^2 \\ & + b_4 d i_{X^2}\omega^1 + b_5\mathcal{L}_{X^1} d i_L \omega^2 + b_6 \mathcal{L}_{X^2} d i_L \omega^1\} \end{aligned} \tag{4}$$

for arbitrary (uniquely determined by A) real numbers $a, b_1, b_2, b_3, b_4, b_5, b_6$.

Proof. Because of Lemma 2.9, our theorem is a immediate consequence of Lemmas 2.11–2.18, below. □

Lemma 2.11. *Let $m \geq 2$ and $n \geq 1$ be integers. Any $\mathcal{VB}_{m,n}$-gauge-natural bilinear operator*

$$A^1 : \Gamma^l(T) \times \Gamma^l(T) \rightsquigarrow \Gamma^l(T)$$

is the constant multiple of the usual bracket $[-,-]$ on (linear) vector fields.

Proof. It is Proposition 2.15 in [2]. □

Lemma 2.12. *Let m, n, p be positive integers. Any $\mathcal{VB}_{m,n}$-gauge-natural (not necessarily bilinear) operator*

$$A^2 : \Gamma^l(T) \times \Gamma^l(T) \rightsquigarrow \Gamma^l\Big(\bigwedge^p T^*\Big)$$

is 0.

Proof. Using the invariance of A^2 with respect to the fiber homotheties we get $A^2(X, X_1) = tA^2(X, X_1)$ for any linear vector fields X and X_1 on a $\mathcal{VB}_{m,n}$-object E and any $t > 0$. Then $A^2 = 0$. □

Lemma 2.13. *Let m, n, p be positive integers. Any $\mathcal{VB}_{m,n}$-gauge-natural bilinear operator*

$$A^3 : \Gamma^l(T) \times \Gamma^l\Big(\bigwedge^p T^*\Big) \rightsquigarrow \Gamma^l(T)$$

is 0.

Proof. Using the invariance of A with respect to the fiber homotheties we get $A^3(X, t\omega) = A^3(X, \omega)$ for any linear vector field X and any linear p-form ω on a $\mathcal{VB}_{m,n}$-object E and any $t > 0$. Then, by the bi-linearity of A^3, we get $tA^3(X, \omega) = A^3(X, \omega)$, and then $A^3(X, \omega) = 0$, i.e., $A^3 = 0$. □

Lemma 2.14. *Let m and $n \geq 1$ and $p \geq 1$ be natural numbers such that $m \geq p + 1$. Any $\mathcal{VB}_{m,n}$-gauge-natural bilinear operator*

$$A^4 : \Gamma^l(T) \times \Gamma^l\Big(\bigwedge^p T^*\Big) \rightsquigarrow \Gamma^l\Big(\bigwedge^p T^*\Big)$$

is of the form

$$A^4(X, \omega) = \lambda_1 \mathcal{L}_X \omega + \lambda_2 i_X d\omega + \lambda_3 \mathcal{L}_X d i_L \omega$$

for the (uniquely determined by A^4) real numbers $\lambda_1, \lambda_2, \lambda_3$.

Proof. Clearly, A^4 is determined by the values $i_{X_1} \dots i_{X_p} A^4(X, \omega)_{|u} \in \mathbf{R}$ for all $\mathcal{VB}_{m,n}$ objects $\pi : E \to M$, all points $u \in E_x$, all $x \in M$, all vectors $X_1, \dots, X_p \in T_u E$ and all linear vector fields X on E, where i_Y is the insertion derivative.

Since $m \geq p + 1$ and $n \geq 1$, we can assume that $u \neq 0$ and that $T\pi \circ X_{|u}, T\pi(X_1), \dots, T\pi(X_p)$ are linearly independent. Then, using the $\mathcal{VB}_{m,n}$-invariance of A^4 and the vector bundle version of the Frobenius theorem, we can write $E = \mathbf{R}^{m,n}$ and $M = \mathbf{R}^m$ and $x = 0$ and $u = e_1 = (1, 0, \dots, 0) \in \mathbf{R}^n$ and $X = \frac{\partial}{\partial x^1}$ and $X_1 = \frac{\partial}{\partial x^2}_{|u}$ and ... and $X_p = \frac{\partial}{\partial x^{p+1}}_{|u}$. Further, by the linearity of $A^4(\frac{\partial}{\partial x^1}, \omega)$ in ω and the linearity of ω, using the Peetre theorem, we may additionally assume that $\omega = x^\alpha y^k dx^{i_1} \wedge \dots \wedge dx^{i_p}$ or $\omega = x^\beta dy^l \wedge dx^{j_1} \wedge \dots \wedge dx^{j_{p-1}}$, where $\alpha = (\alpha^1, \dots, \alpha^m) \in (\mathbf{N} \cup \{0\})^m$ and $\beta = (\beta^1, \dots, \beta^m) \in (\mathbf{N} \cup \{0\})^m$ and $1 \leq i_1 < \dots < i_p \leq m$ and $1 \leq j_1 < \dots < j_{p-1} \leq m$ and $k, l \in \{1, \dots, n\}$, where (of course) $x^\alpha = (x^1)^{\alpha^1} \cdot \ldots \cdot (x^m)^{\alpha^m}$.

In the case of

$$\omega = x^\alpha y^k dx^{i_1} \wedge \dots \wedge dx^{i_p}$$

(as above), by the invariance of A with respect to the homotheties

$$\left(\frac{1}{\tau^1} x^1, \dots, \frac{1}{\tau^m} x^m, y^1, \frac{1}{t} y^2, \dots, \frac{1}{t} y^n\right)$$

for positive numbers $\tau^1, \dots, \tau^m$ and t and the bi-linearity of A^4, we get

$$t^{1-\delta^k_1} \tau^\alpha \tau^{i_1} \cdot \ldots \cdot \tau^{i_p} i_{X_1} \dots i_{X_p} A^4(X, \omega)_{|u} = \tau^1 \cdot \ldots \cdot \tau^{p+1} i_{X_1} \dots i_{X_p} A^4(X, \omega)_{|u} ,$$

where δ^k_1 is the Cronecker delta. So, if $i_{X_1} \dots i_{X_p} A^4(X, \omega)_{|u} \neq 0$, then

$$\omega = y^1 x^i dx^1 \wedge \dots \wedge \widehat{dx^i} \wedge \dots \wedge dx^{p+1}$$

for some $i = 1, \dots, p + 1$, where $\widehat{a}$ means that a is dropped.

Similarly, in the case of

$$\omega = x^\beta dy^l \wedge dx^{j_1} \wedge \ldots \wedge dx^{j_{p-1}} ,$$

if $i_{X_1} \ldots i_{X_p} A^4(X,\omega)_{|u} \neq 0$, then

$$\omega = x^{l_1} x^{l_2} dy^1 \wedge dx^1 \wedge \ldots \wedge \widehat{dx^{l_1}} \wedge \ldots \wedge \widehat{dx^{l_2}} \wedge \ldots \wedge dx^{p+1}$$

for some l_1 and l_2 with $1 \leq l_1 < l_2 \leq p+1$.

Let $3 \leq i \leq p+1$. Using the invariance of A^4 with respect to the $\mathcal{VB}_{m,n}$-map sending coordinate x^i into x^2 (and vice-versa) and preserving other coordinates, one can easily see that $i_{X_1} \ldots i_{X_p} A^4(X,\omega)_{|u}$ with

$$\omega = y^1 x^i dx^1 \wedge \ldots \wedge \widehat{dx^i} \wedge \ldots \wedge dx^{p+1}$$

is equal (modulo signum) to $i_{X_1} \ldots i_{X_p} A(X,\omega)_{|u}$ with $\omega = y^1 x^2 dx^1 \wedge dx^3 \wedge \ldots \wedge dx^{p+1}$.

Let $3 \leq l_2 \leq p+1$. Using the invariance of A^4 with respect to the $\mathcal{VB}_{m,n}$-map sending coordinate x^{l_2} into x^2 (and vice-versa) and preserving other coordinates, one can see that $i_{X_1} \ldots i_{X_p} A^4(X,\omega)_{|u}$ with

$$\omega = x^1 x^{l_2} dy^1 \wedge dx^2 \wedge \ldots \wedge \widehat{dx^{l_2}} \wedge \ldots \wedge dx^{p+1}$$

is equal (modulo signum) to $i_{X_1} \ldots i_{X_p} A^4(X,\omega)_{|u}$ with $\omega = x^1 x^2 dy^1 \wedge dx^3 \wedge \ldots \wedge dx^{p+1}$.

Let $2 \leq l_1 < l_2 \leq m$. Then

$$\varphi := (x^1, \ldots, x^{l_1 - 1}, x^{l_1} + x^{l_1} x^{l_2}, x^{l_1+1}, \ldots, x^m, y^1, \ldots, y^n)^{-1}$$

is a $\mathcal{VB}_{m,n}$-map over some neighborhood of $0 \in \mathbf{R}^m$ and it preserves $X, X_1, \ldots, X_p, u$ and sends

$$\tilde{\omega} := x^{l_1} dy^1 \wedge dx^1 \wedge \ldots \wedge \widehat{dx^{l_1}} \wedge \ldots \wedge \widehat{dx^{l_2}} \wedge \ldots \wedge dx^{p+1}$$

into

$$\tilde{\omega} + x^{l_1} x^{l_2} dy^1 \wedge dx^1 \wedge \ldots \wedge \widehat{dx^{l_1}} \wedge \ldots \wedge \widehat{dx^{l_2}} \wedge \ldots \wedge dx^{p+1} .$$

Then using the invariance of A^4 with respect to φ, from $i_{X_1} \ldots i_{X_p} A^4(X,\tilde{\omega})_{|u} = 0$, we get $i_{X_1} \ldots i_{X_p} A^4(X,\omega)_{|u} = 0$ for

$$\omega := x^{l_1} x^{l_2} dy^1 \wedge dx^1 \wedge \ldots \wedge \widehat{dx^{l_1}} \wedge \ldots \wedge \widehat{dx^{l_2}} \wedge \ldots \wedge dx^{p+1} .$$

Summing up, we have shown that A^4 is determined by three real numbers $i_{X_1} \ldots i_{X_p} A^4(X,\omega_a)_{|u}$ for $a = 1,2,3$, where $\omega_1 := y^1 x^1 dx^2 \wedge \ldots \wedge dx^{p+1}$ and $\omega_2 := y^1 x^2 dx^1 \wedge dx^3 \wedge \ldots \wedge dx^{p+1}$ and $\omega_3 := x^1 x^2 dy^1 \wedge dx^3 \wedge \ldots \wedge dx^{p+1}$ ($\omega_1 := y^1 x^1 dx^2$ and $\omega_2 := y^1 x^2 dx^1$ and $\omega_3 := x^1 x^2 dy^1$ if $p = 1$) and where $u = e_1 = (1,0,\ldots,0) \in \mathbf{R}^n$ and $X = \frac{\partial}{\partial x^1}$ and $X_1 = \frac{\partial}{\partial x^2}_{|u}$ and ... and $X_p = \frac{\partial}{\partial x^{p+1}}_{|u}$.

Thus the vector space of all A^4 (in question) is of dimension not more than 3.

On the other hand, the collection of $\mathcal{VB}_{m,n}$-gauge natural operators $E^i : \Gamma^l(T) \times \Gamma^l(\wedge^p T^*) \rightsquigarrow \Gamma^l(\wedge^p T^*)$ for $i = 1, 2, 3$ given by $E^1(X, \omega) = \mathcal{L}_X\omega$, $E^2(X, \omega) = i_X d\omega$ and $E^3(X, \omega) = \mathcal{L}_X d i_L \omega$ is $\mathbf{R}$-linearly independent. Indeed, it follows easily from

$$
\begin{aligned}
&E^1\Big(\frac{\partial}{\partial x^1}, y^1 dx^1 \wedge \tilde{\omega}_o\Big)(e_1) = 0\,,\\
&E^2\Big(\frac{\partial}{\partial x^1}, y^1 dx^1 \wedge \tilde{\omega}_o\Big)(e_1) = -d_{e_1}y^1 \wedge \tilde{\omega}_o(e_1)\,,\\
&E^3\Big(\frac{\partial}{\partial x^1}, y^1 dx^1 \wedge \tilde{\omega}_o\Big)(e_1) = 0\,,\\
&E^1\Big(\frac{\partial}{\partial x^1}, x^1 dy^1 \wedge \tilde{\omega}_o\Big)(e_1) = d_{e_1}y^1 \wedge \tilde{\omega}_o(e_1)\,,\\
&E^2\Big(\frac{\partial}{\partial x^1}, x^1 dy^1 \wedge \tilde{\omega}_o\Big)(e_1) = d_{e_1}y^1 \wedge \tilde{\omega}_o(e_1)\,,\\
&E^3\Big(\frac{\partial}{\partial x^1}, x^1 dy^1 \wedge \tilde{\omega}_o\Big)(e_1) = d_{e_1}y^1 \wedge \tilde{\omega}_o(e_1)\,,\\
&E^1\Big(\frac{\partial}{\partial x^1}, (x^1)^2 dy^1 \wedge \tilde{\omega}_o\Big)(e_1) = 0\,,\\
&E^2\Big(\frac{\partial}{\partial x^1}, (x^1)^2 dy^1 \wedge \tilde{\omega}_o\Big)(e_1) = 0\,,\\
&E^3\Big(\frac{\partial}{\partial x^1}, (x^1)^2 dy^1 \wedge \tilde{\omega}_o\Big)(e_1) = 2d_{e_1}x^1 \wedge \tilde{\omega}_o(e_1)\,,
\end{aligned}
$$

where $\tilde{\omega}_o := dx^2 \wedge \ldots \wedge dx^p$ (if $p = 1$, $\omega_o := 1$) and $e_1 = (1, 0 \ldots, 0) \in (\mathbf{R}^{m,n})_0$.

Now, the lemma is complete because of the dimension argument. □

Lemma 2.15. *Let m, n, p be positive integers. Any $\mathcal{VB}_{m,n}$-gauge-natural bilinear operator*

$$A^5 : \Gamma^l\Big(\bigwedge^p T^*\Big) \times \Gamma^l(T) \rightsquigarrow \Gamma^l(T)$$

is 0.

Proof. It is sufficient to apply Lemma 2.13 for $A^3(X, \omega) := A^5(\omega, X)$. □

Lemma 2.16. *Let m and $n \geq 1$ and $p \geq 1$ be natural numbers such that $m \geq p + 1$. Any $\mathcal{VB}_{m,n}$-gauge-natural bilinear operator*

$$A^6 : \Gamma^l\Big(\bigwedge^p T^*\Big) \times \Gamma^l(T) \rightsquigarrow \Gamma^l\Big(\bigwedge^p T^*\Big)$$

is of the form

$$A^6(\omega, X) = \mu_1 \mathcal{L}_X\omega + \mu_2 i_X d\omega + \mu_3 \mathcal{L}_X d i_L \omega$$

for the (uniquely determined by A^6) real numbers μ_1, μ_2, μ_3.

Proof. It is sufficient to apply Lemma 2.14 for $A^4(X, \omega) := A^6(\omega, X)$. □

Lemma 2.17. *Any $\mathcal{VB}_{m,n}$-gauge-natural bilinear operator*

$$A^7 : \Gamma^l\Big(\bigwedge^p T^*\Big) \times \Gamma^l\Big(\bigwedge^p T^*\Big) \rightsquigarrow \Gamma^l(T)$$

is 0.

Proof. Using the invariance of A^7 with respect to the fiber homotheties we get $A^7(t\omega, t\omega_1) = A^7(\omega, \omega_1)$ for any linear p-forms ω and ω_1 on a $\mathcal{VB}_{m,n}$-object E and any $t > 0$. Then, by bi-linearity of A^7, $t^2 A^7(\omega, \omega_1) = A^7(\omega, \omega_1)$, i.e. $A^7 = 0$. □

Lemma 2.18. *Any $\mathcal{VB}_{m,n}$-gauge-natural bilinear operator*

$$A^8 : \Gamma^l\Big(\bigwedge^p T^*\Big) \times \Gamma^l\Big(\bigwedge^p T^*\Big) \rightsquigarrow \Gamma^l\Big(\bigwedge^p T^*\Big)$$

is 0.

Proof. Using the invariance of A^8 with respect to the fiber homotheties we get $A^8(t\omega, t\omega_1) = tA^8(\omega, \omega_1)$ for any linear p-forms ω and ω_1 on a $\mathcal{VB}_{m,n}$-object E and any $t > 0$. Then, by bi-linearity of A^8, $t^2 A^8(\omega, \omega_1) = tA^8(\omega, \omega_1)$, i.e. $A^8 = 0$. □

3. The gauge-natural bilinear brackets on couples of linear vector fields and linear p-forms with the Jacobi identity in Leibniz form. A $\mathcal{VB}_{m,n}$-gauge-natural $\mathbf{R}$-bilinear operator A satisfies the Jacobi identity in Leibniz form if

$$A(\nu^1, A(\nu^2, \nu^3)) = A(A(\nu^1, \nu^2), \nu^3) + A(\nu^2, A(\nu^1, \nu^3)) \tag{5}$$

for any $\nu^i \in \Gamma^l_E(TE \oplus \bigwedge^p T^*E)$ for $i = 1, 2, 3$.

We are going to prove Theorem 1.2, i.e., the following theorem.

Theorem 3.1. *Let m, n, p be positive integers such that $m \geq p + 2$. Any $\mathcal{VB}_{m,n}$-gauge-natural $\mathbf{R}$-bilinear operator A of the form (4) satisfies the Jacobi identity in Leibniz form if and only if the 7-tuple $(a, b_1, b_2, b_3, b_4, b_5, b_6)$ is from the following list of 7-tuples:*

$$\begin{aligned}
&(c,0,0,0,0,c,0)\,,\ (c,0,0,0,0,c,-c)\,,\\
&(c,c,0,0,0,-c,0)\,,\ (c,c,-c,0,0,-c,c)\,,\\
&(c,0,0,0,0,0,0)\,,\ (c,c,0,0,0,0,0)\,,\\
&(c,c,0,0,0,0,-c)\,,\ (c,c,-c,0,0,0,0)\,,\\
&(c,c,-c,0,c-\lambda,0,\lambda)\,,\ (0,0,0,\lambda,\mu,-\lambda,-\mu)\,,
\end{aligned} \tag{6}$$

where c, λ, μ are arbitrary real numbers with $c \neq 0$.

The above Theorem 3.1 for $p = 1$ is proved in [2]. So, to prove Theorem 3.1 it is sufficient to prove the following two propositions.

Proposition 3.2. *Let m, n, p be positive integers such that $m \geq p + 2$ and $p \geq 2$. Let $(a, b_1, \ldots, b_6)$ be a 7-tuple such that the $\mathcal{VB}_{m,n}$-gauge-natural bilinear operator A given by* (4) *satisfies the Jacobi identity in Leibniz form. Then the $\mathcal{VB}_{3,n}$-gauge-natural bilinear operator*

$$A^o : \Gamma^l(T \oplus T^*) \times \Gamma^l(T \oplus T^*) \rightsquigarrow \Gamma^l(T \oplus T^*)$$

given by

$$\begin{aligned} A^o(X^1 \oplus \omega^1, X^2 \oplus \omega^2) = a[X^1, X^2] \oplus \{b_1 \mathcal{L}_{X^1}\omega^2 + b_2 \mathcal{L}_{X^2}\omega^1 \\ + b_3 di_{X^1}\omega^2 + b_4 di_{X^2}\omega^1 + b_5 \mathcal{L}_{X^1} di_L \omega^2 + b_6 \mathcal{L}_{X^2} di_L \omega^1\} \end{aligned} \tag{7}$$

satisfies the Jacobi identity in Leibniz form, too.

Proposition 3.3. *Let $(a, b_1, \ldots, b_6)$ be from the list* (6)*. Then the $\mathcal{VB}_{m,n}$-gauge-natural bilinear operator A given by* (4) *satisfies the Jacobi identity in Leibniz form.*

The proofs of the above two propositions will occupy the rest of the paper.

From now on, let $\mathbf{R}^{m,n}$ be the trivial vector bundle over $\mathbf{R}^m$ with the standard fibre $\mathbf{R}^n$ and let $x^1, \ldots, x^m, y^1, \ldots, y^n$ be the usual coordinates on $\mathbf{R}^{m,n}$.

We can write $\mathbf{R}^{m,n} = \mathbf{R}^{3,n} \times \mathbf{R}^{m-3}$. Then $x^1, x^2, x^3, y^1, \ldots, y^n$ are the usual coordinates on $\mathbf{R}^{3,n}$ and $x^4, \ldots, x^m$ are the usual coordinates on $\mathbf{R}^{m-3}$.

Given a linear section $\nu = X \oplus \omega$ of $T\mathbf{R}^{3,n} \oplus T^*\mathbf{R}^{3,n} \to \mathbf{R}^{3,n}$ we have a linear section

$$\nu^{\#} := X^{\#} \oplus \omega^{\#}$$

of $T\mathbf{R}^{m,n} \oplus \bigwedge^p T^*\mathbf{R}^{m,n} \to \mathbf{R}^{m,n}$, where $X^{\#} := X \times 0$ and $\omega^{\#} := \omega \wedge \omega_o$, where 0 is the zero vector field on $\mathbf{R}^{m-3}$ and $\omega_o := dx^4 \wedge \ldots \wedge dx^{p+2}$ is the $(p-1)$-form on $\mathbf{R}^{m-3}$ (and where we do not indicate the pullbacks with respect to respective projections).

Lemma 3.4. *Let A and A^o be as in Proposition 3.2. We have*

$$A((\nu^1)^{\#}, (\nu^2)^{\#}) = (A^o(\nu^1, \nu^2))^{\#}$$

for any linear sections $\nu^1 = X^1 \oplus \omega^1$ and $\nu^2 = X^2 \oplus \omega^2$ of $T\mathbf{R}^{3,n} \oplus T^*\mathbf{R}^{3,n} \to \mathbf{R}^{3,n}$.

Proof. It follows immediately from the formulas:

$$\begin{aligned} [(X^1)^{\#}, (X^2)^{\#}] &= [X^1, X^2]^{\#}, \\ \mathcal{L}_{(X^1)^{\#}}(\omega^2)^{\#} &= (\mathcal{L}_{X^1}\omega^2)^{\#}, \\ di_{(X^1)^{\#}}(\omega^2)^{\#} &= (di_{X^1}\omega^2)^{\#}, \\ \mathcal{L}_{(X^1)^{\#}} di_L(\omega^2)^{\#} &= (\mathcal{L}_{X^1} di_L \omega^2)^{\#}. \end{aligned}$$

These formulas are easy to verify. □

Now, we are in position to prove Proposition 3.2.

Proof. By Lemma 3.4, for any linear sections ν^1 and ν^2 and ν^3 of $T\mathbf{R}^{3,n} \oplus T^*\mathbf{R}^{3,n} \to \mathbf{R}^{3,n}$ we have:

$$(A^o(\nu^1, A^o(\nu^2,\nu^3)))^\# = A((\nu^1)^\#, A((\nu^2)^\#,(\nu^3)^\#)),$$
$$(A^o(A^o(\nu^1,\nu^2),\nu^3))^\# = A(A((\nu^1)^\#,(\nu^2)^\#),(\nu^3)^\#),$$
$$(A^o(\nu^2, A^o(\nu^1,\nu^3)))^\# = A((\nu^2)^\#, A((\nu^1)^\#,(\nu^3)^\#)).$$

Then using the Jacobi identity in Leibniz form (5) of A we get

$$(A^o(\nu^1, A^o(\nu^2,\nu^3)))^\# = (A^o(A^o(\nu^1,\nu^2),\nu^3))^\# + (A^o(\nu^2, A^o(\nu^1,\nu^3)))^\#.$$

Hence (applying $i_{(\frac{\partial}{\partial x^4})^\#}\dots i_{(\frac{\partial}{\partial x^{p+2}})^\#}$ to both sides of the last equality) we get

$$A^o(\nu^1, A^o(\nu^2,\nu^3)) = A^o(A^o(\nu^1,\nu^2),\nu^3) + A^o(\nu^2, A^o(\nu^1,\nu^3)).$$

Then, since A^o is $\mathcal{VB}_{3,n}$-invariant, A^o satisfies the Jacobi identity in Leibniz form.

The proof of Proposition 3.2 is complete. □

In the proof of Proposition 3.3 we will use the following well-known formulas:

$$\mathcal{L}_X\mathcal{L}_Y\omega - \mathcal{L}_Y\mathcal{L}_X\omega = \mathcal{L}_{[X,Y]}\omega, \tag{8}$$
$$i_X\mathcal{L}_Y\omega - \mathcal{L}_Y i_X\omega = i_{[X,Y]}\omega, \tag{9}$$
$$i_X d\omega + d i_X\omega = \mathcal{L}_X\omega, \tag{10}$$
$$d\mathcal{L}_X\omega = \mathcal{L}_X d\omega, \tag{11}$$

where X, Y are vector fields and ω is a p-form on a manifold M.

We will also use some formulas from the following lemma.

Lemma 3.5. *For any linear vector field X and any linear p-form ω on a vector bundle E, we have:*

$$i_L\mathcal{L}_X\omega = \mathcal{L}_X i_L\omega, \tag{12}$$
$$d i_L\mathcal{L}_X\omega = \mathcal{L}_X d i_L\omega = d i_L\mathcal{L}_X d i_L\omega, \tag{13}$$
$$d i_L d i_X\omega = d i_X\omega, \tag{14}$$

where L is the Euler vector field on E.

Proof. By the well-known formula (9) and the condition $[L,X]=0$ (as X is linear), we get (12) (also for not necessarily linear ω). Now, using the well-known formula (11), we get

$$d i_L\mathcal{L}_X\omega = d\mathcal{L}_X i_L\omega = \mathcal{L}_X d i_L\omega.$$

Then, replacing ω by $d i_L\omega$, we get

$$d i_L\mathcal{L}_X d i_L\omega = \mathcal{L}_X d i_L d i_L\omega.$$

Further, using the well-known formula (10) (for $X = L$) and the condition $\mathcal{L}_L\omega = \omega$ (as ω is linear) and the obvious formula $i_L i_L = 0$, we get

$$di_L di_L\omega = di_L\mathcal{L}_L\omega = di_L\omega\,.$$

Then

$$di_L\mathcal{L}_X di_L\omega = \mathcal{L}_X di_L di_L\omega = \mathcal{L}_X di_L\omega\,.$$

Further, by the formula (10) (for $X = L$) and the well-known formula $dd = 0$ and the formula (11) (for $X = L$) and the condition (12), we get

$$di_L di_X\omega = \mathcal{L}_L di_X\omega = di_X\mathcal{L}_L\omega = di_X\omega$$

as ω is linear.

The proof of Lemma 3.5 is complete. □

We will also apply the following lemma.

Lemma 3.6. *Let $(a, b_1, \dots, b_6)$ be a 7-tuple of real numbers. If A given by (4) satisfies the system consisting of conditions*

$$(b_2, b_1) = (0,0) \text{ or } (b_2, b_1) = (0,a) \text{ or } (b_2, b_1) = (-a,a)\,, \tag{15}$$

$$\begin{aligned} &b_1\mathcal{L}_{X^1}\{b_3 di_{X^2}\omega^3 + b_5\mathcal{L}_{X^2}di_L\omega^3\}\\ &\quad + b_3 di_{X^1}\{b_1\mathcal{L}_{X^2}\omega^3 + b_3 di_{X^2}\omega^3 + b_5\mathcal{L}_{X^2}di_L\omega^3\}\\ &\quad + b_5\mathcal{L}_{X^1}di_L\{b_1\mathcal{L}_{X^2}\omega^3 + b_3 di_{X^2}\omega^3 + b_5\mathcal{L}_{X^2}di_L\omega^3\}\\ &= b_3 di_{a[X^1,X^2]}\omega^3 + b_5\mathcal{L}_{a[X^1,X^2]}di_L\omega^3\\ &\quad + b_1\mathcal{L}_{X^2}\{b_3 di_{X^1}\omega^3 + b_5\mathcal{L}_{X^1}di_L\omega^3\}\\ &\quad + b_3 di_{X^2}\{b_1\mathcal{L}_{X^1}\omega^3 + b_3 di_{X^1}\omega^3 + b_5\mathcal{L}_{X^1}di_L\omega^3\}\\ &\quad + b_5\mathcal{L}_{X^2}di_L\{b_1\mathcal{L}_{X^1}\omega^3 + b_3 di_{X^1}\omega^3 + b_5\mathcal{L}_{X^1}di_L\omega^3\}\,, \end{aligned} \tag{16}$$

$$\begin{aligned} &b_1\mathcal{L}_{X^1}\{b_4 di_{X^3}\omega^2 + b_6\mathcal{L}_{X^3}di_L\omega^2\}\\ &\quad + b_3 di_{X^1}\{b_2\mathcal{L}_{X^3}\omega^2 + b_4 di_{X^3}\omega^2 + b_6\mathcal{L}_{X^3}di_L\omega^2\}\\ &\quad + b_5\mathcal{L}_{X^1}di_L\{b_2\mathcal{L}_{X^3}\omega^2 + b_4 di_{X^3}\omega^2 + b_6\mathcal{L}_{X^3}di_L\omega^2\}\\ &= b_2\mathcal{L}_{X^3}\{b_3 di_{X^1}\omega^2 + b_5\mathcal{L}_{X^1}di_L\omega^2\}\\ &\quad + b_4 di_{X^3}\{b_1\mathcal{L}_{X^1}\omega^2 + b_3 di_{X^1}\omega^2 + b_5\mathcal{L}_{X^1}di_L\omega^2\}\\ &\quad + b_6\mathcal{L}_{X^3}di_L\{b_1\mathcal{L}_{X^1}\omega^2 + b_3 di_{X^1}\omega^2 + b_5\mathcal{L}_{X^1}di_L\omega^2\}\\ &\quad + b_4 di_{a[X^1,X^3]}\omega^2 + b_6\mathcal{L}_{a[X^1,X^3]}di_L\omega^2\,, \end{aligned} \tag{17}$$

$$
\begin{aligned}
&b_4 di_{a[X^2,X^3]}\omega^1 + b_6\mathcal{L}_{a[X^2,X^3]} di_L\omega^1\\
&= b_2\mathcal{L}_{X^3}\{b_4 di_{X^2}\omega^1 + b_6\mathcal{L}_{X^2}di_L\omega^1\}\\
&\quad + b_4 di_{X^3}\{b_2\mathcal{L}_{X^2}\omega^1 + b_4 di_{X^2}\omega^1 + b_6\mathcal{L}_{X^2}di_L\omega^1\}\\
&\quad + b_6\mathcal{L}_{X^3}di_L\{b_2\mathcal{L}_{X^2}\omega^1 + b_4 di_{X^2}\omega^1 + b_6\mathcal{L}_{X^2}di_L\omega^1\}\\
&\quad + b_1\mathcal{L}_{X^2}\{b_4 di_{X^3}\omega^1 + b_6\mathcal{L}_{X^3}di_L\omega^1\}\\
&\quad + b_3 di_{X^2}\{b_2\mathcal{L}_{X^3}\omega^1 + b_4 di_{X^3}\omega^1 + b_6\mathcal{L}_{X^3}di_L\omega^1\}\\
&\quad + b_5\mathcal{L}_{X^2}di_L\{b_2\mathcal{L}_{X^3}\omega^1 + b_4 di_{X^3}\omega^1 + b_6\mathcal{L}_{X^3}di_L\omega^1\}
\end{aligned}
\tag{18}
$$

for all linear vector fields X^1, X^2, X^3 *and all linear p-forms* $\omega^1, \omega^2, \omega^3$ *on* E*, then* A *satisfies the Jacobi identity in Leibniz form.*

Proof. Since b_1, b_2 and a satisfy (15), then using formula (8), we get

$$
b_1^2\mathcal{L}_{X^1}\mathcal{L}_{X^2}\omega^3 = b_1 a\mathcal{L}_{[X^1,X^2]}\omega^3 + b_1^2\mathcal{L}_{X^2}\mathcal{L}_{X^1}\omega^3 \tag{19}
$$

for all linear X^1, X^2, ω^3, and

$$
b_1 b_2\mathcal{L}_{X^1}\mathcal{L}_{X^3}\omega^2 = b_2 b_1\mathcal{L}_{X^3}\mathcal{L}_{X^1}\omega^2 + b_2 a\mathcal{L}_{[X^1,X^3]}\omega^2 \tag{20}
$$

for all linear X^1, X^3, ω^2, and

$$
b_2 a\mathcal{L}_{[X^2,X^3]}\omega^1 = b_2^2\mathcal{L}_{X^3}\mathcal{L}_{X^2}\omega^1 + b_1 b_2\mathcal{L}_{X^2}\mathcal{L}_{X^3}\omega^1 \tag{21}
$$

for all linear X^2, X^3, ω^1.

Then applying (16) and (17) and (18) we get

$$
\begin{aligned}
&b_1\mathcal{L}_{X^1}\{b_1\mathcal{L}_{X^2}\omega^3 + b_3 di_{X^2}\omega^3 + b_5\mathcal{L}_{X^2}di_L\omega^3\}\\
&\quad + b_3 di_{X^1}\{b_1\mathcal{L}_{X^2}\omega^3 + b_3 di_{X^2}\omega^3 + b_5\mathcal{L}_{X^2}di_L\omega^3\}\\
&\quad + b_5\mathcal{L}_{X^1}di_L\{b_1\mathcal{L}_{X^2}\omega^3 + b_3 di_{X^2}\omega^3 + b_5\mathcal{L}_{X^2}di_L\omega^3\}\\
&= b_1\mathcal{L}_{a[X^1,X^2]}\omega^3 + b_3 di_{a[X^1,X^2]}\omega^3 + b_5\mathcal{L}_{a[X^1,X^2]}di_L\omega^3\\
&\quad + b_1\mathcal{L}_{X^2}\{b_1\mathcal{L}_{X^1}\omega^3 + b_3 di_{X^1}\omega^3 + b_5\mathcal{L}_{X^1}di_L\omega^3\}\\
&\quad + b_3 di_{X^2}\{b_1\mathcal{L}_{X^1}\omega^3 + b_3 di_{X^1}\omega^3 + b_5\mathcal{L}_{X^1}di_L\omega^3\}\\
&\quad + b_5\mathcal{L}_{X^2}di_L\{b_1\mathcal{L}_{X^1}\omega^3 + b_3 di_{X^1}\omega^3 + b_5\mathcal{L}_{X^1}di_L\omega^3\},
\end{aligned}
\tag{22}
$$

$$
\begin{aligned}
&b_1\mathcal{L}_{X^1}\{b_2\mathcal{L}_{X^3}\omega^2 + b_4 di_{X^3}\omega^2 + b_6\mathcal{L}_{X^3}di_L\omega^2\}\\
&\quad + b_3 di_{X^1}\{b_2\mathcal{L}_{X^3}\omega^2 + b_4 di_{X^3}\omega^2 + b_6\mathcal{L}_{X^3}di_L\omega^2\}\\
&\quad + b_5\mathcal{L}_{X^1}di_L\{b_2\mathcal{L}_{X^3}\omega^2 + b_4 di_{X^3}\omega^2 + b_6\mathcal{L}_{X^3}di_L\omega^2\}\\
&= b_2\mathcal{L}_{X^3}\{b_1\mathcal{L}_{X^1}\omega^2 + b_3 di_{X^1}\omega^2 + b_5\mathcal{L}_{X^1}di_L\omega^2\}\\
&\quad + b_4 di_{X^3}\{b_1\mathcal{L}_{X^1}\omega^2 + b_3 di_{X^1}\omega^2 + b_5\mathcal{L}_{X^1}di_L\omega^2\}\\
&\quad + b_6\mathcal{L}_{X^3}di_L\{b_1\mathcal{L}_{X^1}\omega^2 + b_3 di_{X^1}\omega^2 + b_5\mathcal{L}_{X^1}di_L\omega^2\}\\
&\quad + b_2\mathcal{L}_{a[X^1,X^3]}\omega^2 + b_4 di_{a[X^1,X^3]}\omega^2 + b_6\mathcal{L}_{a[X^1,X^3]}di_L\omega^2,
\end{aligned}
\tag{23}
$$

$$
\begin{aligned}
& b_2\mathcal{L}_{a[X^2,X^3]}\omega^1 + b_4 di_{a[X^2,X^3]}\omega^1 + b_6\mathcal{L}_{a[X^2,X^3]}di_L\omega^1 \\
&= b_2\mathcal{L}_{X^3}\{b_2\mathcal{L}_{X^2}\omega^1 + b_4 di_{X^2}\omega^1 + b_6\mathcal{L}_{X^2}di_L\omega^1\} \\
&\quad + b_4 di_{X^3}\{b_2\mathcal{L}_{X^2}\omega^1 + b_4 di_{X^2}\omega^1 + b_6\mathcal{L}_{X^2}di_L\omega^1\} \\
&\quad + b_6\mathcal{L}_{X^3}di_L\{b_2\mathcal{L}_{X^2}\omega^1 + b_4 di_{X^2}\omega^1 + b_6\mathcal{L}_{X^2}di_L\omega^1\} \\
&\quad + b_1\mathcal{L}_{X^2}\{b_2\mathcal{L}_{X^3}\omega^1 + b_4 di_{X^3}\omega^1 + b_6\mathcal{L}_{X^3}di_L\omega^1\} \\
&\quad + b_3 di_{X^2}\{b_2\mathcal{L}_{X^3}\omega^1 + b_4 di_{X^3}\omega^1 + b_6\mathcal{L}_{X^3}di_L\omega^1\} \\
&\quad + b_5\mathcal{L}_{X^2}di_L\{b_2\mathcal{L}_{X^3}\omega^1 + b_4 di_{X^3}\omega^1 + b_6\mathcal{L}_{X^3}di_L\omega^1\}
\end{aligned} \tag{24}
$$

for any linear vector fields X^1, X^2, X^3 and any linear p-forms $\omega^1, \omega^2, \omega^3$ on E.

Adding equalities (22) and (23) and (24) we get

$$\Omega = \Theta + \mathcal{T}\,, \tag{25}$$

where

$$
\begin{aligned}
\Omega = {} & b_1\mathcal{L}_{X^1}\{b_1\mathcal{L}_{X^2}\omega^3 + b_2\mathcal{L}_{X^3}\omega^2 + b_3 di_{X^2}\omega^3 + b_4 di_{X^3}\omega^2 \\
& + b_5\mathcal{L}_{X^2}di_L\omega^3 + b_6\mathcal{L}_{X^3}di_L\omega^2\} + b_2\mathcal{L}_{a[X^2,X^3]}\omega^1 + b_3 di_{X^1}\{b_1\mathcal{L}_{X^2}\omega^3 \\
& + b_2\mathcal{L}_{X^3}\omega^2 + b_3 di_{X^2}\omega^3 + b_4 di_{X^3}\omega^2 + b_5\mathcal{L}_{X^2}di_L\omega^3 + b_6\mathcal{L}_{X^3}di_L\omega^2\} \\
& + b_4 di_{a[X^2,X^3]}\omega^1 + b_5\mathcal{L}_{X^1}di_L\{b_1\mathcal{L}_{X^2}\omega^3 + b_2\mathcal{L}_{X^3}\omega^2 + b_3 di_{X^2}\omega^3 \\
& + b_4 di_{X^3}\omega^2 + b_5\mathcal{L}_{X^2}di_L\omega^3 + b_6\mathcal{L}_{X^3}di_L\omega^2\} + b_6\mathcal{L}_{a[X^2,X^3]}di_L\omega^1\,,
\end{aligned}
$$

$$
\begin{aligned}
\Theta = {} & b_1\mathcal{L}_{a[X^1,X^2]}\omega^3 + b_2\mathcal{L}_{X^3}\{b_1\mathcal{L}_{X^1}\omega^2 + b_2\mathcal{L}_{X^2}\omega^1 + b_3 di_{X^1}\omega^2 \\
& + b_4 di_{X^2}\omega^1 + b_5\mathcal{L}_{X^1}di_L\omega^2 + b_6\mathcal{L}_{X^2}di_L\omega^1\} + \ b_3 di_{a[X^1,X^2]}\omega^3 \\
& + b_4 di_{X^3}\{b_1\mathcal{L}_{X^1}\omega^2 + b_2\mathcal{L}_{X^2}\omega^1 + b_3 di_{X^1}\omega^2 + b_4 di_{X^2}\omega^1 + b_5\mathcal{L}_{X^1}di_L\omega^2 \\
& + b_6\mathcal{L}_{X^2}di_L\omega^1\} + b_5\mathcal{L}_{a[X^1,X^2]}di_L\omega^3 + b_6\mathcal{L}_{X^3}di_L\{b_1\mathcal{L}_{X^1}\omega^2 + b_2\mathcal{L}_{X^2}\omega^1 \\
& + b_3 di_{X^1}\omega^2 + b_4 di_{X^2}\omega^1 + b_5\mathcal{L}_{X^1}di_L\omega^2 + b_6\mathcal{L}_{X^2}di_L\omega^1\}\,,
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{T} = {} & b_1\mathcal{L}_{X^2}\{b_1\mathcal{L}_{X^1}\omega^3 + b_2\mathcal{L}_{X^3}\omega^1 + b_3 di_{X^1}\omega^3 + b_4 di_{X^3}\omega^1 \\
& + b_5\mathcal{L}_{X^1}di_L\omega^3 + b_6\mathcal{L}_{X^3}di_L\omega^1\} + b_2\mathcal{L}_{a[X^1,X^3]}\omega^2 + b_3 di_{X^2}\{b_1\mathcal{L}_{X^1}\omega^3 \\
& + b_2\mathcal{L}_{X^3}\omega^1 + b_3 di_{X^1}\omega^3 + b_4 di_{X^3}\omega^1 + b_5\mathcal{L}_{X^1}di_L\omega^3 + b_6\mathcal{L}_{X^3}di_L\omega^1\} \\
& + b_4 di_{a[X^1,X^3]}\omega^2 + b_5\mathcal{L}_{X^2}di_L\{b_1\mathcal{L}_{X^1}\omega^3 + b_2\mathcal{L}_{X^3}\omega^1 + b_3 di_{X^1}\omega^3 \\
& + b_4 di_{X^3}d\omega^1 + b_5\mathcal{L}_{X^1}di_L\omega^3 + b_6\mathcal{L}_{X^3}di_L\omega^1\} + b_6\mathcal{L}_{a[X^1,X^3]}di_L\omega^2\,.
\end{aligned}
$$

On the other hand, we can see that

$$
\begin{aligned}
A(X^1, A(X^2\oplus\omega^2, X^3\oplus\omega^3)) &= a^2[X^1,[X^2,X^3]]\oplus\Omega\,, \\
A(A(X^1\oplus\omega^1, X^2\oplus\omega^2), X^3\oplus\omega^3) &= a^2[[X^1,X^2],X^3]\oplus\Theta\,, \\
A(X^2\oplus\omega^2, A(X^1\oplus\omega^1, X^3\oplus\omega^3)) &= a^2[X^2,[X^1,X^3]]\oplus\mathcal{T}\,.
\end{aligned}
$$

Then A satisfies the Jacobi identity in Leibniz form. The proof of Lemma 3.6 is complete. □

Now, we are in position to prove Proposition 3.3.

Proof. Let $(a, b_1, b_2, b_3, b_4, b_5, b_6)$ be a arbitrary 7-tuple from the list (6). Let A be given by (4). We are going to show that A satisfies the Jacobi identity in Leibniz form. By Lemma 3.6, it is sufficient to show that $(a, b_1, b_2, b_3, b_4, b_5, b_6)$ satisfies conditions (15)–(18) for all linear vector fields X^1, X^2, X^3 and all linear p-forms $\omega^1, \omega^2, \omega^2$ on E. But one can easily directly observe that such 7-tuple satisfies (15). So, it remains to show that it satisfies (16)–(18).

We consider several cases.

Case 1. $(a, b_1, b_2, b_3, b_4, b_5, b_6) = (a, 0, 0, 0, 0, 0, 0)$ and $a = c \neq 0$.
The equalities (16)–(18) hold. They are $0 = 0$.

Case 2. $(a, b_1, b_2, b_3, b_4, b_5, b_6) = (a, a, 0, 0, 0, 0, 0)$ and $a = c \neq 0$.
The equalities (16)–(18) hold. They are $0 = 0$.

Case 3. $(a, b_1, b_2, b_3, b_4, b_5, b_6) = (a, 0, 0, 0, 0, a, 0)$ and $a = c \neq 0$.
The equalities (17) and (18) hold as they are $0 = 0$. Further, using (13), condition (16) is

$$a^2 \mathcal{L}_{X^1} \mathcal{L}_{X^2} d i_L \omega^3 = a^2 \mathcal{L}_{[X^1, X^2]} d i_L \omega^3 + a^2 \mathcal{L}_{X^2} \mathcal{L}_{X^1} d i_L \omega^3 .$$

It holds because of the well-known formula (8).

Case 4. $(a, b_1, b_2, b_3, b_4, b_5, b_6) = (a, a, -a, 0, 0, 0, 0)$ and $a = c \neq 0$.
The equalities (16)–(18) hold as they are $0 = 0$.

Case 5. $(a, b_1, b_2, b_3, b_4, b_5, b_6) = (a, 0, 0, 0, 0, a, -a)$ and $a = c \neq 0$.
Using (13), we can see that conditions (16)–(18) are

$$\begin{aligned} a^2 \mathcal{L}_{X^1} \mathcal{L}_{X^2} d i_L \omega^3 &= a^2 \mathcal{L}_{[X^1, X^2]} d i_L \omega^3 + a^2 \mathcal{L}_{X^2} \mathcal{L}_{X^1} d i_L \omega^3 , \\ -a^2 \mathcal{L}_{X^1} \mathcal{L}_{X^3} d i_L \omega^2 &= -a^2 \mathcal{L}_{X^3} \mathcal{L}_{X^1} d i_L \omega^2 - a^2 \mathcal{L}_{[X^1, X^3]} d i_L \omega^2 , \\ -a^2 \mathcal{L}_{[X^2, X^3]} d i_L \omega^1 &= a^2 \mathcal{L}_{X^3} \mathcal{L}_{X^2} d i_L \omega^1 - a^2 \mathcal{L}_{X^2} \mathcal{L}_{X^3} d i_L \omega^1 . \end{aligned}$$

They hold because of the well-known formula (8).

Case 6. $(a, b_1, b_2, b_3, b_4, b_5, b_6) = (a, a, 0, 0, 0, 0, -a)$ and $a = c \neq 0$.
The equality (16) holds as it is $0 = 0$. Further, using (13), conditions (17) and (18) are

$$\begin{aligned} -a^2 \mathcal{L}_{X^1} \mathcal{L}_{X^3} d i_L \omega^2 &= -a^2 \mathcal{L}_{X^3} \mathcal{L}_{X^1} d i_L \omega^2 - a^2 \mathcal{L}_{[X^1, X^3]} d i_L \omega^2 , \\ -a^2 \mathcal{L}_{[X^2, X^3]} d i_L \omega^1 &= a^2 \mathcal{L}_{X^3} \mathcal{L}_{X^2} d i_L \omega^1 - a^2 \mathcal{L}_{X^2} \mathcal{L}_{X^3} d i_L \omega^1 . \end{aligned}$$

They hold because of the well-known formula (8).

Case 7. $(a, b_1, b_2, b_3, b_4, b_5, b_6) = (a, a, 0, 0, 0, -a, 0)$ and $a = c \neq 0$.

The conditions (17) and (18) are satisfied as they are $0 = 0$. Further, using (13), equality (16) is

$$\begin{aligned}&- a^2\mathcal{L}_{X^1}\mathcal{L}_{X^2}di_L\omega^3 - a^2\mathcal{L}_{X^1}\mathcal{L}_{X^2}di_L\omega^3 + a^2\mathcal{L}_{X^1}\mathcal{L}_{X^2}di_L\omega^3\\ &\quad = -a^2\mathcal{L}_{[X^1,X^2]}di_L\omega^3 - a^2\mathcal{L}_{X^2}\mathcal{L}_{X^1}di_L\omega^3 - a^2\mathcal{L}_{X^2}\mathcal{L}_{X^1}di_L\omega^3\\ &\qquad + a^2\mathcal{L}_{X^2}\mathcal{L}_{X^1}di_L\omega^3\end{aligned}$$

or (after reduction)

$$-a^2\mathcal{L}_{X^1}\mathcal{L}_{X^2}di_L\omega^3 = -a^2\mathcal{L}_{[X^1,X^2]}di_L\omega^3 - a^2\mathcal{L}_{X^2}\mathcal{L}_{X^1}di_L\omega^3 .$$

It holds because of the well-known formula (8).

Case 8. $(a, b_1, b_2, b_3, b_4, b_5, b_6) = (a, a, -a, 0, 0, -a, a)$ and $a = c \neq 0$.

Using (13), equality (16) is

$$\begin{aligned}&- a^2\mathcal{L}_{X^1}\mathcal{L}_{X^2}di_L\omega^3 - a^2\mathcal{L}_{X^1}\mathcal{L}_{X^2}di_L\omega^3 + a^2\mathcal{L}_{X^1}\mathcal{L}_{X^2}di_L\omega^3\\ &\quad = -a^2\mathcal{L}_{[X^1,X^2]}di_L\omega^3 - a^2\mathcal{L}_{X^2}\mathcal{L}_{X^1}di_L\omega^3 - a^2\mathcal{L}_{X^2}\mathcal{L}_{X^1}di_L\omega^3\\ &\qquad + a^2\mathcal{L}_{X^2}\mathcal{L}_{X^1}di_L\omega^3\end{aligned}$$

or (after reduction)

$$-a^2\mathcal{L}_{X^1}\mathcal{L}_{X^2}di_L\omega^3 = -a^2\mathcal{L}_{[X^1,X^2]}di_L\omega^3 - a^2\mathcal{L}_{X^2}\mathcal{L}_{X^1}di_L\omega^3 .$$

Similarly, (17) is

$$\begin{aligned}&a^2\mathcal{L}_{X^1}\mathcal{L}_{X^3}di_L\omega^2 + a^2\mathcal{L}_{X^1}\mathcal{L}_{X^3}di_L\omega^2 - a^2\mathcal{L}_{X^1}\mathcal{L}_{X^3}di_L\omega^2\\ &\quad = a^2\mathcal{L}_{X^3}\mathcal{L}_{X^1}di_L\omega^2 + a^2\mathcal{L}_{X^3}\mathcal{L}_{X^1}di_L\omega^2 - a^2\mathcal{L}_{X^3}\mathcal{L}_{X^1}di_L\omega^2\\ &\qquad + a^2\mathcal{L}_{[X^1,X^3]}di_L\omega^2\end{aligned}$$

or (after reduction)

$$a^2\mathcal{L}_{X^1}\mathcal{L}_{X^3}di_L\omega^2 = a^2\mathcal{L}_{X^3}\mathcal{L}_{X^1}di_L\omega^2 + a^2\mathcal{L}_{[X^1,X^3]}di_L\omega^2 .$$

Similarly, (18) is

$$\begin{aligned}a^2\mathcal{L}_{[X^2,X^3]}di_L\omega^1 = &- a^2\mathcal{L}_{X^3}\mathcal{L}_{X^2}di_L\omega^1 - a^2\mathcal{L}_{X^3}\mathcal{L}_{X^2}di_L\omega^1\\ &+ a^2\mathcal{L}_{X^3}\mathcal{L}_{X^2}di_L\omega^1 + a^2\mathcal{L}_{X^2}\mathcal{L}_{X^3}di_L\omega^1\\ &+ a^2\mathcal{L}_{X^2}\mathcal{L}_{X^3}di_L\omega^1 - a^2\mathcal{L}_{X^2}\mathcal{L}_{X^3}di_L\omega^1\end{aligned}$$

or (after reduction)

$$a^2\mathcal{L}_{[X^2,X^3]}di_L\omega^1 = -a^2\mathcal{L}_{X^3}\mathcal{L}_{X^2}di_L\omega^1 + a^2\mathcal{L}_{X^2}\mathcal{L}_{X^3}di_L\omega^1 .$$

So, (16)–(18) hold because of the well-known formula (8).

Case 9. $(a, b_1, b_2, b_3, b_4, b_5, b_6) = (a, a, -a, 0, a - \lambda, 0, \lambda)$ and $a = c \neq 0$.

Condition (16) holds as it is $0 = 0$. Using (13), condition (17) is

$$a(a-\lambda)\mathcal{L}_{X^1} di_{X^3}\omega^2 + a\lambda\mathcal{L}_{X^1}\mathcal{L}_{X^3} di_L\omega^2 = (a-\lambda) a di_{X^3}\mathcal{L}_{X^1}\omega^2 + \lambda a\mathcal{L}_{X^3}\mathcal{L}_{X^1} di_L\omega^2 + (a-\lambda) a di_{[X^1,X^3]}\omega^2 + \lambda a\mathcal{L}_{[X^1,X^3]} di_L\omega^2 \,.$$

Then, using formulas (8) and (11), condition (17) is

$$a(a-\lambda) d\mathcal{L}_{X^1} i_{X^3}\omega^2 = (a-\lambda) a di_{X^3}\mathcal{L}_{X^1}\omega^2 + (a-\lambda) a di_{[X^1,X^3]}\omega^2 \,.$$

Then (17) holds because of the well-known formula (9).

Using (13) and the well-known-formula (11), we can see that (18) is

$$\begin{aligned}
&(a-\lambda) a di_{[X^2,X^3]}\omega^1 + \lambda a\mathcal{L}_{[X^2,X^3]} di_L\omega^1 \\
&\quad = -(a-\lambda) a d\mathcal{L}_{X^3} i_{X^2}\omega^1 - a\lambda\mathcal{L}_{X^3}\mathcal{L}_{X^2} di_L\omega^1 \\
&\qquad - (a-\lambda) a di_{X^3}\mathcal{L}_{X^2}\omega^1 + (a-\lambda)^2 di_{X^3} di_{X^2}\omega^1 \\
&\qquad + (a-\lambda)\lambda di_{X^3}\mathcal{L}_{X^2} di_L\omega^1 - \lambda a\mathcal{L}_{X^3}\mathcal{L}_{X^2} di_L\omega^1 \\
&\qquad + \lambda(a-\lambda)\mathcal{L}_{X^3} di_L di_{X^2}\omega^1 + \lambda^2\mathcal{L}_{X^3}\mathcal{L}_{X^2} di_L\omega^1 \\
&\qquad + a(a-\lambda) d\mathcal{L}_{X^2} i_{X^3}\omega^1 + a\lambda\mathcal{L}_{X^2}\mathcal{L}_{X^3} di_L\omega^1 \,.
\end{aligned} \tag{26}$$

So, to prove that (18) holds, it remains to show that the coefficients on λ^k of both sides of (26) are equal (for $k = 0, 1, 2$).

Comparing the coefficients on λ^0 in (26), we have

$$\begin{aligned}
a^2 di_{[X^2,X^3]}\omega^1 = &- a^2 d\mathcal{L}_{X^3} i_{X^2}\omega^1 - a^2 di_{X^3}\mathcal{L}_{X^2}\omega^1 \\
&+ a^2 di_{X^3} di_{X^2}\omega^1 + a^2 d\mathcal{L}_{X^2} i_{X^3}\omega^1 \,.
\end{aligned}$$

This condition holds because

$$\begin{aligned}
di_{[X^2,X^3]} &= d(\mathcal{L}_{X^2} i_{X^3} - i_{X^3}\mathcal{L}_{X^2}) = d\mathcal{L}_{X^2} i_{X^3} - di_{X^3}\mathcal{L}_{X^2} \\
&= (d\mathcal{L}_{X^2} i_{X^3} - di_{X^3}\mathcal{L}_{X^2}) + (di_{X^3} di_{X^2} - d\mathcal{L}_{X^3} i_{X^2}) \\
&= -d\mathcal{L}_{X^3} i_{X^2} - di_{X^3}\mathcal{L}_{X^2} + di_{X^3} di_{X^2} + d\mathcal{L}_{X^2} i_{X^3}
\end{aligned}$$

as $di_{X^3} di_{X^2} = d(di_{X^3} + i_{X^3} d) i_{X^2} = d\mathcal{L}_{X^3} i_{X^2}$.

Comparing the coefficients on λ in (26) and using the well-known formula (11), we have

$$\begin{aligned}
&- a di_{[X^2,X^3]}\omega^1 + a\mathcal{L}_{[X^2,X^3]} di_L\omega^1 \\
&\quad = a\mathcal{L}_{X^3} di_{X^2}\omega^1 - a\mathcal{L}_{X^3}\mathcal{L}_{X^2} di_L\omega^1 + a di_{X^3}\mathcal{L}_{X^2}\omega^1 \\
&\qquad - 2a di_{X^3} di_{X^2}\omega^1 + a di_{X^3}\mathcal{L}_{X^2} di_L\omega^1 - a\mathcal{L}_{X^3}\mathcal{L}_{X^2} di_L\omega^1 \\
&\qquad + a\mathcal{L}_{X^3} di_L di_{X^2}\omega^1 - a d\mathcal{L}_{X^2} i_{X^3}\omega^1 + a\mathcal{L}_{X^2}\mathcal{L}_{X^3} di_L\omega^1 \,.
\end{aligned} \tag{27}$$

Then using the well-known formulas (8) and (9), we can equivalently reduce (27) to

$$\begin{aligned}0 = a\mathcal{L}_{X^3} di_{X^2}\omega^1 - 2a di_{X^3} di_{X^2}\omega^1 + a di_{X^3}\mathcal{L}_{X^2} di_L\omega^1 \\ - a\mathcal{L}_{X^3}\mathcal{L}_{X^2} di_L\omega^1 + a\mathcal{L}_{X^3} di_L di_{X^2}\omega^1 .\end{aligned} \tag{28}$$

By (14), $di_L di_{X^2}\omega^1 = di_{X^2}\omega^1$. Then $\mathcal{L}_{X^3} di_L di_{X^2}\omega^1 = \mathcal{L}_{X^3} di_{X^2}\omega^1$. Moreover, by formulas (10) and $d^2 = 0$ and (11), we have

$$di_{X^3} di_{X^2}\omega^1 = (di_{X^3} + i_{X^3} d) di_{X^2}\omega^1 = \mathcal{L}_{X^3} di_{X^2}\omega^1 . \tag{29}$$

Also $di_{X^3}\mathcal{L}_{X^2} di_L\omega^1 = (di_{X^3} + i_{X^3} d) d\mathcal{L}_{X^2} i_L\omega^1 = \mathcal{L}_{X^3}\mathcal{L}_{X^2} di_L\omega^1$, i.e.,

$$di_{X^3}\mathcal{L}_{X^2} di_L\omega^1 = \mathcal{L}_{X^3}\mathcal{L}_{X^2} di_L\omega^1 . \tag{30}$$

So, our equality (28) can be equivalently transformed into

$$\begin{aligned}0 = a\mathcal{L}_{X^3} di_{X^2}\omega^1 - 2a\mathcal{L}_{X^3} di_{X^2}\omega^1 + a\mathcal{L}_{X^3}\mathcal{L}_{X^2} di_L\omega^1 \\ - a\mathcal{L}_{X^3}\mathcal{L}_{X^2} di_L\omega^1 + a\mathcal{L}_{X^3} di_{X^2}\omega^1 ,\end{aligned}$$

i.e., into $0 = 0$. So, (27) holds.

Comparing the coefficients on λ^2 in (26), we get

$$0 = di_{X^3} di_{X^2}\omega^1 - di_{X^3}\mathcal{L}_{X^2} di_L\omega^1 - \mathcal{L}_{X^3} di_L di_{X^2}\omega^1 + \mathcal{L}_{X^3}\mathcal{L}_{X^2} di_L\omega^1 .$$

This condition is satisfied because by (29), (13) and (30) it can be rewritten as

$$0 = \mathcal{L}_{X^3} di_{X^2}\omega^1 - \mathcal{L}_{X^3}\mathcal{L}_{X^2} di_L\omega^1 - \mathcal{L}_{X^3} di_{X^2}\omega^1 + \mathcal{L}_{X^3}\mathcal{L}_{X^2} di_L\omega^1 .$$

Case 10. $(a, b_1, b_2, b_3, b_4, b_5, b_6) = (0, 0, 0, \lambda, \mu, -\lambda, -\mu)$.

Condition (16) is

$$\begin{aligned}&\lambda^2 di_{X^1} di_{X^2}\omega^3 - \lambda^2 di_{X^1}\mathcal{L}_{X^2} di_L\omega^3 - \lambda^2\mathcal{L}_{X^1} di_L di_{X^2}\omega^3 + \lambda^2\mathcal{L}_{X^1} di_L\mathcal{L}_{X^2} di_L\omega^3 \\ &= \lambda^2 di_{X^2} di_{X^1}\omega^3 - \lambda^2 di_{X^2}\mathcal{L}_{X^1} di_L\omega^3 - \lambda^2\mathcal{L}_{X^2} di_L di_{X^1}\omega^3 \\ &\quad + \lambda^2\mathcal{L}_{X^2} di_L\mathcal{L}_{X^1} di_L\omega^3 .\end{aligned}$$

This condition holds because by (29), (30), (13) and (14) it can be transformed into

$$\begin{aligned}&\lambda^2\mathcal{L}_{X^1} di_{X^2}\omega^3 - \lambda^2\mathcal{L}_{X^1}\mathcal{L}_{X^2} di_L\omega^3 - \lambda^2\mathcal{L}_{X^1} di_{X^2}\omega^3 + \lambda^2\mathcal{L}_{X^1}\mathcal{L}_{X^2} di_L\omega^3 \\ &= \lambda^2\mathcal{L}_{X^2} di_{X^1}\omega^3 - \lambda^2\mathcal{L}_{X^2}\mathcal{L}_{X^1} di_L\omega^3 - \lambda^2\mathcal{L}_{X^2} di_{X^1}\omega^3 \\ &\quad + \lambda^2\mathcal{L}_{X^2}\mathcal{L}_{X^1} di_L\omega^3 ,\end{aligned}$$

i.e., into $0 = 0$.

Condition (17) is

$$\begin{aligned}&\lambda\mu di_{X^1} di_{X^3}\omega^2 - \lambda\mu di_{X^1}\mathcal{L}_{X^3} di_L\omega^2 - \lambda\mu\mathcal{L}_{X^1} di_L di_{X^3}\omega^2 + \lambda\mu\mathcal{L}_{X^1} d_L\mathcal{L}_{X^3} di_L\omega^2 \\ &= \mu\lambda di_{X^3} di_{X^1}\omega^2 - \mu\lambda di_{X^3}\mathcal{L}_{X^1} di_L\omega^2 - \mu\lambda\mathcal{L}_{X^3} di_L di_{X^1}\omega^2 \\ &\quad + \mu\lambda\mathcal{L}_{X^3} d_L\mathcal{L}_{X^1} di_L\omega^2 .\end{aligned}$$

This condition holds because by (29), (30), (13) and (14) it can be transformed into

$$\lambda\mu\mathcal{L}_{X^1}di_{X^3}\omega^2-\lambda\mu\mathcal{L}_{X^1}\mathcal{L}_{X^3}di_L\omega^2-\lambda\mu\mathcal{L}_{X^1}di_{X^3}\omega^2++\lambda\mu\mathcal{L}_{X^1}\mathcal{L}_{X^3}di_L\omega^2$$
$$=\mu\lambda\mathcal{L}_{X^3}di_{X^1}\omega^2-\mu\lambda\mathcal{L}_{X^3}\mathcal{L}_{X^1}di_L\omega^2-\mu\lambda\mathcal{L}_{X^3}di_{X^1}\omega^2$$
$$+\mu\lambda\mathcal{L}_{X^3}\mathcal{L}_{X^1}di_L\omega^2\,,$$

i.e., into $0=0$.

Condition (18) is

$$0=\mu^2di_{X^3}di_{X^2}\omega^1-\mu^2di_{X^3}\mathcal{L}_{X^2}di_L\omega^1-\mu^2\mathcal{L}_{X^3}di_Ldi_{X^2}\omega^1$$
$$+\mu^2\mathcal{L}_{X^3}di_L\mathcal{L}_{X^2}\omega^1+\lambda\mu di_{X^2}di_{X^3}\omega^1-\lambda\mu di_{X^2}\mathcal{L}_{X^3}di_L\omega^1$$
$$-\lambda\mu\mathcal{L}_{X^2}di_Ldi_{X^3}\omega^1+\lambda\mu\mathcal{L}_{X^2}di_L\mathcal{L}_{X^3}\omega^1\,.$$

This condition is satisfied because by (29), (30), (13) and (14) it can be transformed into

$$0=\mu^2\mathcal{L}_{X^3}di_{X^2}\omega^1-\mu^2\mathcal{L}_{X^3}\mathcal{L}_{X^2}di_L\omega^1-\mu^2\mathcal{L}_{X^3}di_{X^2}\omega^1+$$
$$+\mu^2\mathcal{L}_{X^3}\mathcal{L}_{X^2}di_L\omega^1+\lambda\mu\mathcal{L}_{X^2}di_{X^3}\omega^1-\lambda\mu\mathcal{L}_{X^2}\mathcal{L}_{X^3}di_L\omega^1$$
$$-\lambda\mu\mathcal{L}_{X^2}di_{X^3}\omega^1+\lambda\mu\mathcal{L}_{X^2}\mathcal{L}_{X^3}di_L\omega^1\,,$$

i.e., into $0=0$.

The proof of Proposition 3.3 is complete. □

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