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Upper and lower bounds for an integral transform of positive operators in Hilbert spaces with applications

Abstract. For a continuous and positive function $w(\lambda)$, $\lambda > 0$ and a positive measure μ on $(0, \infty)$ we consider the following *integral transform*

$$\mathcal{D}(w, \mu)(T) := \int_0^\infty w(\lambda)(\lambda + T)^{-1} d\mu(\lambda),$$

where the integral is assumed to exist for any positive operator T on a complex Hilbert space H. In this paper we obtain several upper and lower bounds for the difference $\mathcal{D}(w, \mu)(A) - \mathcal{D}(w, \mu)(B)$ under certain assumptions for the operators A and B. Some natural applications for operator monotone and operator convex functions are also given.

1. Introduction. Consider a complex Hilbert space $(H, \langle \cdot, \cdot \rangle)$. An operator T on H is said to be positive (denoted by $T \geq 0$) if $\langle Tx, x \rangle \geq 0$ for all $x \in H$ and also an operator T on H is said to be *strictly positive* (denoted by $T > 0$) if T is positive and invertible. A real valued continuous function f on $(0, \infty)$ is said to be operator monotone if $f(A) \geq f(B)$ holds for any $A \geq B > 0$.

We have the following representation of operator monotone functions [10], see for instance [1, p. 144–145]:

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Theorem 1. *A function* $f:[0,\infty)\to\mathbb{R}$ *is operator monotone in* $[0,\infty)$ *if and only if it has the representation*

$$f(t)=f(0)+bt+\int_0^\infty \frac{t\lambda}{t+\lambda}d\mu(\lambda), \tag{1.1}$$

where $b\geq 0$ *and* μ *is a positive measure on* $[0,\infty)$ *such that*

$$\int_0^\infty \frac{\lambda}{1+\lambda}d\mu(\lambda)<\infty. \tag{1.2}$$

A real valued continuous function f on an interval I is said to be *operator convex (operator concave)* on I if

$$f((1-\lambda)A+\lambda B)\leq(\geq)(1-\lambda)f(A)+\lambda f(B) \tag{OC}$$

in the operator order, for all $\lambda\in[0,1]$ and for every selfadjoint operators A and B on a Hilbert space H whose spectra are contained in I. Notice that a function f is operator concave if $-f$ is operator convex.

We have the following representation of operator convex functions [1, p. 147]:

Theorem 2. *A function* $f:[0,\infty)\to\mathbb{R}$ *is operator convex in* $[0,\infty)$ *if and only if it has the representation*

$$f(t)=f(0)+f'_+(0)t+ct^2+\int_0^\infty \frac{t^2\lambda}{t+\lambda}d\mu(\lambda), \tag{1.3}$$

where $c\geq 0$ *and* μ *is a positive measure on* $[0,\infty)$ *such that* (1.2) *holds.*

We have the following integral representation for the power function when $t>0$, $r\in(0,1)$, see for instance [1, p. 145]:

$$t^{r-1}=\frac{\sin(r\pi)}{\pi}\int_0^\infty \frac{\lambda^{r-1}}{\lambda+t}d\lambda.$$

Motivated by these representations, we introduce, for a continuous and positive function $w(\lambda)$, $\lambda>0$, the following *integral transform*:

$$\mathcal{D}(w,\mu)(t):=\int_0^\infty \frac{w(\lambda)}{\lambda+t}d\mu(\lambda),\ t>0, \tag{1.4}$$

where μ is a positive measure on $(0,\infty)$ and the integral (1.4) exists for all $t>0$.

If μ is the usual Lebesgue measure, we put

$$\mathcal{D}(w)(t):=\int_0^\infty \frac{w(\lambda)}{\lambda+t}d\lambda,\ t>0. \tag{1.5}$$

Now, assume that $T>0$, then by the continuous functional calculus for selfadjoint operators, we can define the positive operator

$$\mathcal{D}(w,\mu)(T):=\int_0^\infty w(\lambda)(\lambda+T)^{-1}d\mu(\lambda), \tag{1.6}$$

where w and μ are as above. Also, when μ is the usual Lebesgue measure, then

$$\mathcal{D}(w)(T) := \int_0^\infty w(\lambda)(\lambda+T)^{-1} d\lambda, \tag{1.7}$$

for $T > 0$.

If we take μ to be the usual Lebesgue measure and the kernel $w_r(\lambda) = \lambda^{r-1}$, $r \in (0,1]$, then

$$t^{r-1} = \frac{\sin(r\pi)}{\pi}\mathcal{D}(w_r)(t), \ t > 0. \tag{1.8}$$

We define the *upper incomplete Gamma function* as in [8]:

$$\Gamma(a,z) := \int_z^\infty t^{a-1}e^{-t}dt,$$

which for $z = 0$ gives the *Gamma function*

$$\Gamma(a) := \int_0^\infty t^{a-1}e^{-t}dt \quad \text{for} \quad \operatorname{Re} a > 0.$$

We have the integral representation (see [9]):

$$\Gamma(a,z) = \frac{z^a e^{-z}}{\Gamma(1-a)}\int_0^\infty \frac{t^{-a}e^{-t}}{z+t}dt \tag{1.9}$$

for $\operatorname{Re} a < 1$ and $|\operatorname{ph} z| < \pi$.

Now, we consider the weight $w_{\cdot^{-a}e^{-\cdot}}(\lambda) := \lambda^{-a}e^{-\lambda}$ for $\lambda > 0$. Then by (1.9) we have

$$\mathcal{D}(w_{\cdot^{-a}e^{-\cdot}})(t) = \int_0^\infty \frac{\lambda^{-a}e^{-\lambda}}{t+\lambda}d\lambda = \Gamma(1-a)t^{-a}e^t\Gamma(a,t) \tag{1.10}$$

for $a < 1$ and $t > 0$.

For $a = 0$ in (1.10) we get

$$\mathcal{D}(w_{e^{-\cdot}})(t) = \int_0^\infty \frac{e^{-\lambda}}{t+\lambda}d\lambda = \Gamma(1)e^t\Gamma(0,t) = e^t E_1(t) \tag{1.11}$$

for $t > 0$, where

$$E_1(t) := \int_t^\infty \frac{e^{-u}}{u}du. \tag{1.12}$$

Let $a = 1-n$, with a natural number n, then by (1.10) we have

$$\begin{aligned}\mathcal{D}(w_{\cdot^{n-1}e^{-\cdot}})(t) &= \int_0^\infty \frac{\lambda^{n-1}e^{-\lambda}}{t+\lambda}d\lambda = \Gamma(n)t^{n-1}e^t\Gamma(1-n,t) \\ &= (n-1)!t^{n-1}e^t\Gamma(1-n,t).\end{aligned} \tag{1.13}$$

If we define the generalized exponential integral (see [6]) by

$$E_p(z) := z^{p-1}\Gamma(1-p,z) = z^{p-1}\int_z^\infty \frac{e^{-t}}{t^p}dt,$$

then

$$t^{n-1}\Gamma(1-n,t)=E_n\left(t\right)$$

for $n\geq 1$ and $t>0$.

Using the identity [6, Eq 8.19.7], for $n\geq 2$:

$$E_n\left(z\right)=\frac{\left(-z\right)^{n-1}}{(n-1)!}E_1\left(z\right)+\frac{e^{-z}}{(n-1)!}\sum_{k=0}^{n-2}(n-k-2)!\left(-z\right)^k,$$

we get

$$\begin{aligned}
&\mathcal{D}\left(w_{\cdot^{n-1}e^{-\cdot}}\right)(t)\\
&=(n-1)!e^tE_n\left(t\right)\\
&=(n-1)!e^t\left[\frac{\left(-t\right)^{n-1}}{(n-1)!}E_1\left(t\right)+\frac{e^{-t}}{(n-1)!}\sum_{k=0}^{n-2}(n-k-2)!\left(-t\right)^k\right] \quad (1.14)\\
&=\sum_{k=0}^{n-2}\left(-1\right)^k(n-k-2)!t^k+\left(-1\right)^{n-1}t^{n-1}e^tE_1\left(t\right)
\end{aligned}$$

for $n\geq 2$ and $t>0$.

If $T>0$, then we have

$$\begin{aligned}
\mathcal{D}\left(w_{\cdot^{-a}e^{-\cdot}}\right)(T)&=\int_0^\infty\lambda^{-a}e^{-\lambda}\left(t+\lambda\right)^{-1}d\lambda \quad (1.15)\\
&=\Gamma(1-a)T^{-a}\exp\left(T\right)\Gamma(a,T)
\end{aligned}$$

for $a<1$.

In particular, for $a=0$,

$$\mathcal{D}\left(w_{e^{-\cdot}}\right)(T)=\int_0^\infty e^{-\lambda}\left(T+\lambda\right)^{-1}d\lambda=\exp\left(T\right)E_1\left(T\right) \quad (1.16)$$

and, for $n\geq 2$,

$$\begin{aligned}
&\mathcal{D}\left(w_{\cdot^{n-1}e^{-\cdot}}\right)(t)\\
&=\int_0^\infty\lambda^{n-1}e^{-\lambda}\left(T+\lambda\right)^{-1}d\lambda \quad (1.17)\\
&=\sum_{k=0}^{n-2}\left(-1\right)^k(n-k-2)!T^k+\left(-1\right)^{n-1}T^{n-1}\exp\left(T\right)E_1\left(T\right),
\end{aligned}$$

where $T>0$.

For $n=2$, we also get

$$\mathcal{D}\left(w_{\cdot e^{-\cdot}}\right)(T)=\int_0^\infty\lambda e^{-\lambda}\left(T+\lambda\right)^{-1}d\lambda=1-T\exp\left(T\right)E_1\left(T\right) \quad (1.18)$$

for $T>0$.

We consider the weight $w_{(\cdot+a)^{-1}}(\lambda) := \frac{1}{\lambda+a}$ for $\lambda > 0$ and $a > 0$. Then, by simple calculations, we get

$$\mathcal{D}\left(w_{(\cdot+a)^{-1}}\right)(t) := \int_0^\infty \frac{1}{(\lambda+t)(\lambda+a)} d\lambda = \frac{\ln t - \ln a}{t-a} \tag{1.19}$$

for all $a > 0$ and $t > 0$ with $t \neq a$.

From this, we get

$$\ln t = \ln a + (t-a)\,\mathcal{D}\left(w_{(\cdot+a)^{-1}}\right)(t)$$

for all $t,\, a > 0$.

If $T > 0$, then

$$\begin{aligned} \ln T &= \ln a + (T-a)\,\mathcal{D}\left(w_{(\cdot+a)^{-1}}\right)(t) \\ &= \ln a + (T-a)\int_0^\infty \frac{1}{(\lambda+a)}(\lambda+T)^{-1}\, d\lambda. \end{aligned} \tag{1.20}$$

Let $a > 0$. Assume that either $0 < T < a$ or $T > a$, then by (1.20) we get

$$(\ln T - \ln a)(T-a)^{-1} = \int_0^\infty \frac{1}{(\lambda+a)}(\lambda+T)^{-1}\, d\lambda. \tag{1.21}$$

We can also consider the weight $w_{(\cdot^2+a^2)^{-1}}(\lambda) := \frac{1}{\lambda^2+a^2}$ for $\lambda > 0$ and $a > 0$. Then, by simple calculations, we get

$$\begin{aligned} \mathcal{D}\left(w_{(\cdot^2+a^2)^{-1}}\right)(t) &:= \int_0^\infty \frac{1}{(\lambda+t)(\lambda^2+a^2)} d\lambda \\ &= \frac{\pi t}{2a(t^2+a^2)} - \frac{\ln t - \ln a}{t^2+a^2} \end{aligned}$$

for $t > 0$ and $a > 0$.

For $a = 1$ we also have

$$\mathcal{D}\left(w_{(\cdot^2+1)^{-1}}\right)(t) := \int_0^\infty \frac{1}{(\lambda+t)(\lambda^2+1)} d\lambda = \frac{\pi t}{2(t^2+1)} - \frac{\ln t}{t^2+1}$$

for $t > 0$.

If $T > 0$ and $a > 0$, then

$$\begin{aligned} &\frac{\pi}{2a} T\left(T^2+a^2\right)^{-1} - (\ln T - \ln a)\left(T^2+a^2\right)^{-1} \\ &\quad = \int_0^\infty \frac{1}{(\lambda^2+a^2)}(\lambda+T)^{-1}\, d\lambda \end{aligned} \tag{1.22}$$

and, in particular,

$$\frac{\pi}{2} T\left(T^2+1\right)^{-1} - \left(T^2+1\right)^{-1}\ln T = \int_0^\infty \frac{1}{(\lambda^2+1)}(\lambda+T)^{-1}\, d\lambda. \tag{1.23}$$

Assume that $0 < A < B$. We say that these operators are *separated* if there exist $0 < \beta < \gamma$ such that $0 < A \le \beta < \gamma \le B$.

For a positive operator $T>0$, we have the operator inequalities

$$\left\|T^{-1}\right\|^{-1}\le T\le\|T\|.$$

Therefore, if A, $B>0$ with $\|A\|\left\|B^{-1}\right\|<1$, then

$$0<\left\|A^{-1}\right\|^{-1}\le A\le\|A\|<\left\|B^{-1}\right\|^{-1}\le B\le\|B\|.$$

The class of two separated positive operators play an important role in establishing various refinements and reverses of operator Young inequalities as pointed out in numerous recent papers from which we only mention [3], [13] and the references therein.

2. Main results. In the following, whenever we write $\mathcal{D}(w,\mu)$ we mean that the integral from (2.3) exists and is finite for all $t>0$.

Lemma 1. *For all A, $B>0$ we have the representation*

$$\begin{aligned}&\mathcal{D}(w,\mu)(B)-\mathcal{D}(w,\mu)(A)\\ &=\int_0^\infty\left(\int_0^1(\lambda+sB+(1-s)A)^{-1}(A-B)(\lambda+sB+(1-s)A)^{-1}ds\right)\\ &\quad\times w(\lambda)\,d\mu(\lambda).\end{aligned}\tag{2.1}$$

Proof. Observe that, for all A, $B>0$,

$$\begin{aligned}&\mathcal{D}(w,\mu)(B)-\mathcal{D}(w,\mu)(A)\\ &=\int_0^\infty w(\lambda)\left[(\lambda+B)^{-1}-(\lambda+A)^{-1}\right]d\mu(\lambda).\end{aligned}\tag{2.2}$$

Let T, $S>0$. The function $f(t)=-t^{-1}$ is operator monotone on $(0,\infty)$, operator Gâteaux differentiable and the Gâteaux derivative is given by

$$\nabla f_T(S):=\lim_{t\to0}\left[\frac{f(T+tS)-f(T)}{t}\right]=T^{-1}ST^{-1}$$

for T, $S>0$.

Consider the continuous function f defined on an interval I for which the corresponding operator function is Gâteaux differentiable on the segment $[C,D]:=\{(1-t)C+tD,\ t\in[0,1]\}$ for selfadjoint operators C, D with spectra in I. We consider the auxiliary function defined on $[0,1]$ by

$$f_{C,D}(t):=f((1-t)C+tD),\ t\in[0,1].$$

Then, by the properties of the Bochner integral, we have

$$f(D)-f(C)=\int_0^1\frac{d}{dt}(f_{C,D}(t))\,dt=\int_0^1\nabla f_{(1-t)C+tD}(D-C)\,dt.$$

If we write this equality for the function $f(t)=-t^{-1}$ and C, $D>0$, then we get the representation

$$C^{-1}-D^{-1}=\int_0^1((1-t)C+tD)^{-1}(D-C)((1-t)C+tD)^{-1}dt\tag{2.3}$$

Now, if we take $C = \lambda + B$, $D = \lambda + A$ in (2.3), then

$$\begin{aligned}
&(\lambda + B)^{-1} - (\lambda + A)^{-1} \\
&= \int_0^1 ((1-t)(\lambda + B) + t(\lambda + A))^{-1} (A - B) \\
&\qquad \times ((1-t)(\lambda + B) + t(\lambda + A))^{-1} dt \\
&= \int_0^1 (\lambda + (1-t) B + tA)^{-1} (A - B) (\lambda + (1-t) B + tA)^{-1} dt
\end{aligned}$$

and by (2.2) we derive

$$\begin{aligned}
&\mathcal{D}(w, \mu)(B) - \mathcal{D}(w, \mu)(A) \\
&= \int_0^\infty \left(\int_0^1 (\lambda + (1-t) B + tA)^{-1} (A - B) (\lambda + (1-t) B + tA)^{-1} dt \right) \\
&\quad \times w(\lambda)\, d\mu(\lambda),
\end{aligned}$$

which, by the change of variable $t = 1 - s$, gives (2.1). □

We have the following double inequality for two positive separated operators:

Theorem 3. *If the positive operators satisfy the separation condition*

$$0 < \alpha \le A \le \beta < \gamma \le B \le \delta \tag{2.4}$$

for some positive constants α, β, γ, δ, *then*

$$\begin{aligned}
0 &\le \frac{\gamma - \beta}{\delta - \beta} [\mathcal{D}(w, \mu)(\beta) - \mathcal{D}(w, \mu)(\delta)] \\
&\le \mathcal{D}(w, \mu)(A) - \mathcal{D}(w, \mu)(B) \\
&\le \frac{\delta - \alpha}{\gamma - \alpha} [\mathcal{D}(w, \mu)(\alpha) - \mathcal{D}(w, \mu)(\gamma)] .
\end{aligned} \tag{2.5}$$

Proof. From (2.4) we have

$$0 < \gamma - \beta \le B - A \le \delta - \alpha,$$

which implies that

$$\begin{aligned}
0 &\le (\gamma - \beta) ((1-s) A + sB + \lambda)^{-2} \\
&\le ((1-s) A + sB + \lambda)^{-1} (B - A) ((1-s) A + sB + \lambda)^{-1} \\
&\le (\delta - \alpha) ((1-s) A + sB + \lambda)^{-2}
\end{aligned}$$

for all $s \in [0, 1]$ and $\lambda \ge 0$.

By integration over $s\in[0,1]$ we deduce that

$$\begin{aligned}0&\leq(\gamma-\beta)\int_0^1((1-s)A+sB+\lambda)^{-2}ds\\&\leq\int_0^1((1-s)A+sB+\lambda)^{-1}(B-A)((1-s)A+sB+\lambda)^{-1}ds\\&\leq(\delta-\alpha)\int_0^1((1-s)A+sB+\lambda)^{-2}ds\end{aligned}$$

for all $\lambda\geq 0$.

If we multiply this inequality by $w(\lambda)\geq 0$ and integrate over the measure $\mu(\lambda)$, we get

$$\begin{aligned}0&\leq(\gamma-\beta)\int_0^\infty\left(\int_0^1((1-s)A+sB+\lambda)^{-2}ds\right)w(\lambda)\,d\mu(\lambda)\\&\leq\int_0^\infty\left(\int_0^1((1-s)A+sB+\lambda)^{-1}(B-A)((1-s)A+sB+\lambda)^{-1}ds\right)\\&\qquad\times w(\lambda)\,d\mu(\lambda)\\&\leq(\delta-\alpha)\int_0^\infty\left(\int_0^1((1-s)A+sB+\lambda)^{-2}ds\right)w(\lambda)\,d\mu(\lambda),\end{aligned}$$

and, by (2.1) we derive the inequality of interest

$$\begin{aligned}0&\leq(\gamma-\beta)\int_0^\infty\left(\int_0^1((1-s)A+sB+\lambda)^{-2}ds\right)w(\lambda)\,d\mu(\lambda)\\&\leq\mathcal{D}(w,\mu)(A)-\mathcal{D}(w,\mu)(B)\\&\leq(\delta-\alpha)\int_0^\infty\left(\int_0^1((1-s)A+sB+\lambda)^{-2}ds\right)w(\lambda)\,d\mu(\lambda).\end{aligned}\tag{2.6}$$

From (2.4) we derive that

$$(1-s)A+sB+\lambda\leq(1-s)\beta+s\delta+\lambda,$$

which implies that

$$((1-s)A+sB+\lambda)^{-1}\geq((1-s)\beta+s\delta+\lambda)^{-1}$$

and

$$((1-s)A+sB+\lambda)^{-2}\geq((1-s)\beta+s\delta+\lambda)^{-2}$$

for all $s\in[0,1]$ and $\lambda\geq 0$.

Also

$$(1-s)A+sB+\lambda\geq(1-s)\alpha+s\gamma+\lambda,$$

which implies that

$$((1-s)A+sB+\lambda)^{-1}\leq((1-s)\alpha+s\gamma+\lambda)^{-1}$$

and

$$((1-s)A+sB+\lambda)^{-2}\leq((1-s)\alpha+s\gamma+\lambda)^{-2}$$

for all $s \in [0,1]$ and $\lambda \geq 0$.

Therefore

$$\begin{aligned}(2.7)\qquad &(\gamma-\beta)\int_0^\infty\left(\int_0^1((1-s)\beta+s\delta+\lambda)^{-2}ds\right)w(\lambda)\,d\mu(\lambda)\\ &\leq(\gamma-\beta)\int_0^\infty\left(\int_0^1((1-s)A+sB+\lambda)^{-2}ds\right)w(\lambda)\,d\mu(\lambda)\end{aligned}$$

and

$$\begin{aligned}(2.8)\qquad &(\delta-\alpha)\int_0^\infty\left(\int_0^1((1-s)A+sB+\lambda)^{-2}ds\right)w(\lambda)\,d\mu(\lambda)\\ &\leq(\delta-\alpha)\int_0^\infty\left(\int_0^1((1-s)\alpha+s\gamma+\lambda)^{-2}ds\right)w(\lambda)\,d\mu(\lambda).\end{aligned}$$

Since

$$\begin{aligned}&(\gamma-\beta)\int_0^\infty\left(\int_0^1((1-s)\beta+s\delta+\lambda)^{-2}ds\right)w(\lambda)\,d\mu(\lambda)\\ &=\frac{\gamma-\beta}{\delta-\beta}\int_0^\infty\left(\int_0^1((1-s)\beta+s\delta+\lambda)^{-1}(\delta-\beta)((1-s)\beta+s\delta+\lambda)^{-1}ds\right)\\ &\times w(\lambda)\,d\mu(\lambda)\\ &=\frac{\gamma-\beta}{\delta-\beta}[\mathcal{D}(w,\mu)(\beta)-\mathcal{D}(w,\mu)(\delta)]\quad\text{(by (2.1) for } A=\beta I \text{ and } B=\delta I)\end{aligned}$$

and

$$\begin{aligned}&(\delta-\alpha)\int_0^\infty\left(\int_0^1((1-s)\alpha+s\gamma+\lambda)^{-2}ds\right)w(\lambda)\,d\mu(\lambda)\\ &=\frac{\delta-\alpha}{\gamma-\alpha}\int_0^\infty\left(\int_0^1((1-s)\alpha+s\gamma+\lambda)^{-1}(\gamma-\alpha)((1-s)\alpha+s\gamma+\lambda)^{-1}ds\right)\\ &\times w(\lambda)\,d\mu(\lambda)\\ &=\frac{\delta-\alpha}{\gamma-\alpha}[\mathcal{D}(w,\mu)(\alpha)-\mathcal{D}(w,\mu)(\gamma)]\quad\text{(by (2.1) for } A=\alpha I \text{ and } B=\gamma I),\end{aligned}$$

then (2.7) and (2.8) become

$$\begin{aligned}(2.9)\qquad &\frac{\gamma-\beta}{\delta-\beta}[\mathcal{D}(w,\mu)(\beta)-\mathcal{D}(w,\mu)(\delta)]\\ &\leq(\gamma-\beta)\int_0^\infty\left(\int_0^1((1-s)A+sB+\lambda)^{-2}ds\right)w(\lambda)\,d\mu(\lambda)\end{aligned}$$

and

$$\begin{aligned}(2.10)\qquad &(\delta-\alpha)\int_0^\infty\left(\int_0^1((1-s)A+sB+\lambda)^{-2}ds\right)w(\lambda)\,d\mu(\lambda)\\ &\leq\frac{\delta-\alpha}{\gamma-\alpha}[\mathcal{D}(w,\mu)(\alpha)-\mathcal{D}(w,\mu)(\gamma)].\end{aligned}$$

Finally, on making use of (2.6), (2.9) and (2.10), we derive (2.5). □

Corollary 1. *If* A, $B>0$ *with* $\|A\|\left\|B^{-1}\right\|<1$, *then*

$$\begin{aligned}0&\le\frac{1-\|A\|\left\|B^{-1}\right\|}{(\|B\|-\|A\|)\left\|B^{-1}\right\|}\left[\mathcal{D}(w,\mu)(\|A\|)-\mathcal{D}(w,\mu)(\|B\|)\right]\\&\le\mathcal{D}(w,\mu)(A)-\mathcal{D}(w,\mu)(B)\\&\le\frac{\|B\|\left\|A^{-1}\right\|-1}{\left\|A^{-1}\right\|-\left\|B^{-1}\right\|}\left\|B^{-1}\right\|\\&\quad\times\left[\mathcal{D}(w,\mu)\left(\left\|A^{-1}\right\|^{-1}\right)-\mathcal{D}(w,\mu)\left(\left\|B^{-1}\right\|^{-1}\right)\right].\end{aligned}\tag{2.11}$$

The proof follows by Theorem 3 on taking $\alpha=\left\|A^{-1}\right\|^{-1}$, $\beta=\|A\|$, $\gamma=\left\|B^{-1}\right\|^{-1}$ and $\delta=\|B\|$ and performing the required calculations.

We can state the following result for operator monotone functions on $[0,\infty)$:

Proposition 1. *Assume that* $f:[0,\infty)\to\mathbb{R}$ *is an operator monotone function on* $[0,\infty)$. *If* A, $B>0$ *satisfy condition* (2.4), *then*

$$\begin{aligned}&\frac{\gamma-\beta}{\delta-\beta}\left[f(\beta)\beta^{-1}-f(\delta)\delta^{-1}-f(0)\left(\beta^{-1}-\delta^{-1}\right)\right]\\&\quad\le f(A)A^{-1}-f(B)B^{-1}-f(0)\left(A^{-1}-B^{-1}\right)\\&\quad\le\frac{\delta-\alpha}{\gamma-\alpha}\left[f(\alpha)\alpha^{-1}-f(\gamma)\gamma^{-1}-f(0)\left(\alpha^{-1}-\gamma^{-1}\right)\right].\end{aligned}\tag{2.12}$$

If $f(0)=0$, *then we have the simpler inequality*

$$\begin{aligned}\frac{\gamma-\beta}{\delta-\beta}\left[f(\beta)\beta^{-1}-f(\delta)\delta^{-1}\right]&\le f(A)A^{-1}-f(B)B^{-1}\\&\le\frac{\delta-\alpha}{\gamma-\alpha}\left[f(\alpha)\alpha^{-1}-f(\gamma)\gamma^{-1}\right].\end{aligned}\tag{2.13}$$

Proof. If $f:[0,\infty)\to\mathbb{R}$ is an operator monotone, then by (1.1)

$$\frac{f(t)-f(0)}{t}-b=\mathcal{D}(\ell,\mu)(t),\ t>0,$$

for some $b\ge0$ and a positive measure μ, where $\ell(\lambda)=\lambda$, $\lambda>0$. By applying Theorem 3 for the $\mathcal{D}(\ell,\mu)$ and performing the required calculations, we deduce (2.12). □

Corollary 2. *Assume that* $f:[0,\infty)\to\mathbb{R}$ *is an operator monotone function on* $[0,\infty)$. *If* A, $B>0$ *with* $\|A\|\left\|B^{-1}\right\|<1$, *then*

$$\begin{aligned}0&\le\frac{1-\|A\|\left\|B^{-1}\right\|}{(\|B\|-\|A\|)\left\|B^{-1}\right\|}\\&\quad\times\left[f(\|A\|)\|A\|^{-1}-f(\|B\|)\|B\|^{-1}-f(0)\left(\|A\|^{-1}-\|B\|^{-1}\right)\right]\end{aligned}$$

$$\leq f(A)A^{-1} - f(B)B^{-1} - f(0)\left(A^{-1} - B^{-1}\right)$$
$$\leq \frac{\|B\|\left\|A^{-1}\right\| - 1}{\left\|A^{-1}\right\| - \left\|B^{-1}\right\|}\left\|B^{-1}\right\|$$
$$\times\left[f\left(\left\|A^{-1}\right\|^{-1}\right)\left\|A^{-1}\right\| - f\left(\left\|B^{-1}\right\|^{-1}\right)\left\|B^{-1}\right\|\right.$$
$$\left.-f(0)\left(\left\|A^{-1}\right\| - \left\|B^{-1}\right\|\right)\right].$$

If $f(0) = 0$, *then*

$$\begin{aligned} 0 &\leq \frac{1 - \|A\|\left\|B^{-1}\right\|}{(\|B\| - \|A\|)\left\|B^{-1}\right\|}\left[f(\|A\|)\|A\|^{-1} - f(\|B\|)\|B\|^{-1}\right] \\ &\leq f(A)A^{-1} - f(B)B^{-1} \\ &\leq \frac{\|B\|\left\|A^{-1}\right\| - 1}{\left\|A^{-1}\right\| - \left\|B^{-1}\right\|}\left\|B^{-1}\right\| \\ &\quad\times\left[f\left(\left\|A^{-1}\right\|^{-1}\right)\left\|A^{-1}\right\| - f\left(\left\|B^{-1}\right\|^{-1}\right)\left\|B^{-1}\right\|\right]. \end{aligned} \tag{2.14}$$

The proof follows by Proposition 1 on taking $\alpha = \left\|A^{-1}\right\|^{-1}$, $\beta = \|A\|$, $\gamma = \left\|B^{-1}\right\|^{-1}$ and $\delta = \|B\|$.

We can state the following result for operator convex functions on $[0, \infty)$:

Proposition 2. *Assume that* $f : [0, \infty) \to \mathbb{R}$ *is an operator convex function on* $[0, \infty)$. *If* A, $B > 0$ *satisfy condition* (2.4), *then*

$$\begin{aligned} &\frac{\gamma - \beta}{\delta - \beta}\left[f(\beta)\beta^{-2} - f(\delta)\delta^{-2} - f(0)\left(\beta^{-2} - \delta^{-2}\right) - f'_+(0)\left(\beta^{-1} - \delta^{-1}\right)\right] \\ &\leq f(A)A^{-2} - f(B)B^{-2} - f(0)\left(A^{-2} - B^{-2}\right) - f'_+(0)\left(A^{-1} - B^{-1}\right) \\ &\leq \frac{\delta - \alpha}{\gamma - \alpha} \\ &\times\left[f(\alpha)\alpha^{-2} - f(\gamma)\gamma^{-2} - f(0)\left(\alpha^{-2} - \gamma^{-2}\right) - f'_+(0)\left(\alpha^{-1} - \gamma^{-1}\right)\right]. \end{aligned} \tag{2.15}$$

If $f(0) = 0$, *then*

$$\begin{aligned} &\frac{\gamma - \beta}{\delta - \beta}\left[f(\beta)\beta^{-2} - f(\delta)\delta^{-2} - f'_+(0)\left(\beta^{-1} - \delta^{-1}\right)\right] \\ &\leq f(A)A^{-2} - f(B)B^{-2} - f'_+(0)\left(A^{-1} - B^{-1}\right) \\ &\leq \frac{\delta - \alpha}{\gamma - \alpha}\left[f(\alpha)\alpha^{-2} - f(\gamma)\gamma^{-2} - f'_+(0)\left(\alpha^{-1} - \gamma^{-1}\right)\right]. \end{aligned} \tag{2.16}$$

Proof. If $f : [0, \infty) \to \mathbb{R}$ is an operator convex function on $[0, \infty)$, then by (1.3) we have

$$\frac{f(t) - f(0) - f'_+(0)\,t}{t^2} - c = \mathcal{D}(\ell, \mu)(t)$$

for some $c \geq 0$ and a positive measure μ, where $\ell(\lambda) = \lambda$, $\lambda > 0$. By applying Theorem 3 for the $\mathcal{D}(\ell, \mu)$ and performing the required calculations, we deduce (2.15). □

Corollary 3. *Assume that $f : [0, \infty) \to \mathbb{R}$ is an operator monotone function on $[0, \infty)$. If A, $B > 0$ with $\|A\| \left\|B^{-1}\right\| < 1$, then*

$$\begin{aligned}
0 &\leq \frac{1 - \|A\| \left\|B^{-1}\right\|}{(\|B\| - \|A\|) \|B^{-1}\|} \\
&\quad \times \Big[f(\|A\|) \|A\|^{-2} - f(\|B\|) \|B\|^{-2} - f(0) \left(\|A\|^{-2} - \|B\|^{-2} \right) \\
&\qquad - f'_{+}(0) \left(\|A\|^{-1} - \|B\|^{-1} \right) \Big] \\
&\leq f(A) A^{-2} - f(B) B^{-2} - f(0) \left(A^{-2} - B^{-2} \right) - f'_{+}(0) \left(A^{-1} - B^{-1} \right) \\
&\leq \frac{\|B\| \left\|A^{-1}\right\| - 1}{\|A^{-1}\| - \|B^{-1}\|} \left\|B^{-1}\right\| \\
&\quad \times \Big[f\left(\left\|A^{-1}\right\|^{-1} \right) \left\|A^{-1}\right\|^{2} - f\left(\left\|B^{-1}\right\|^{-1} \right) \left\|B^{-1}\right\|^{2} \\
&\qquad - f(0) \left(\left\|A^{-1}\right\|^{2} - \left\|B^{-1}\right\|^{2} \right) - f'_{+}(0) \left(\left\|A^{-1}\right\| - \left\|B^{-1}\right\| \right) \Big].
\end{aligned}$$

If $f(0) = 0$, then

(2.17)

$$\begin{aligned}
0 &\leq \frac{1 - \|A\| \left\|B^{-1}\right\|}{(\|B\| - \|A\|) \|B^{-1}\|} \\
&\quad \times \Big[f(\|A\|) \|A\|^{-2} - f(\|B\|) \|B\|^{-2} - f'_{+}(0) \left(\|A\|^{-1} - \|B\|^{-1} \right) \Big] \\
&\leq f(A) A^{-2} - f(B) B^{-2} - f'_{+}(0) \left(A^{-1} - B^{-1} \right) \\
&\leq \frac{\|B\| \left\|A^{-1}\right\| - 1}{\|A^{-1}\| - \|B^{-1}\|} \left\|B^{-1}\right\| \\
&\quad \times \Big[f\left(\left\|A^{-1}\right\|^{-1} \right) \left\|A^{-1}\right\|^{2} - f\left(\left\|B^{-1}\right\|^{-1} \right) \left\|B^{-1}\right\|^{2} \\
&\qquad - f'_{+}(0) \left(\left\|A^{-1}\right\| - \left\|B^{-1}\right\| \right) \Big].
\end{aligned}$$

3. Some examples. Consider the operator monotone function $f(t) = t^r$, $r \in (0, 1]$. If the condition (2.4) is satisfied, then by (2.13) we get the power inequalities

$$\frac{\gamma - \beta}{\delta - \beta} \left(\beta^{r-1} - \delta^{r-1} \right) \leq A^{r-1} - B^{r-1} \leq \frac{\delta - \alpha}{\gamma - \alpha} \left(\alpha^{r-1} - \gamma^{r-1} \right).$$

If A, $B > 0$ with $\|A\| \left\|B^{-1}\right\| < 1$, then by (2.14) we obtain

$$0 \leq \frac{1 - \|A\| \left\|B^{-1}\right\|}{(\|B\| - \|A\|) \|B^{-1}\|} \left(\|A\|^{r-1} - \|B\|^{r-1} \right) \leq A^{r-1} - B^{r-1}$$

$$\leq \frac{\|B\| \left\|A^{-1}\right\| - 1}{\|A^{-1}\| - \|B^{-1}\|} \left\|B^{-1}\right\| \left(\left\|A^{-1}\right\|^{1-r} - \left\|B^{-1}\right\|^{1-r} \right).$$

Consider the operator convex function $f(t) = -\ln(t+1)$. If the condition (2.4) is satisfied, then by (2.16) we get the logarithmic inequalities

$$\begin{aligned}
&\frac{\gamma - \beta}{\delta - \beta} \left[\delta^{-2} \ln(\delta + 1) - \beta^{-2} \ln(\beta + 1) + \beta^{-1} - \delta^{-1}\right] \\
&\leq B^{-2} \ln(B + 1) - A^{-2} \ln(A + 1) + A^{-1} - B^{-1} \\
&\leq \frac{\delta - \alpha}{\gamma - \alpha} \left[\gamma^{-2} \ln(\gamma + 1) - \alpha^{-2} \ln(\alpha + 1) + \alpha^{-1} - \gamma^{-1}\right].
\end{aligned}$$

If A, $B > 0$ with $\|A\| \left\|B^{-1}\right\| < 1$, then by (2.17) we derive

$$\begin{aligned}
0 &\leq \frac{1 - \|A\| \left\|B^{-1}\right\|}{(\|B\| - \|A\|) \|B^{-1}\|} \\
&\quad \times \left[\|B\|^{-2} \ln(\|B\| + 1) - \|A\|^{-2} \ln(\|A\| + 1) + \|A\|^{-1} - \|B\|^{-1}\right] \\
&\leq B^{-2} \ln(B + 1) - A^{-2} \ln(A + 1) + A^{-1} - B^{-1} \\
&\leq \frac{\|B\| \left\|A^{-1}\right\| - 1}{\|A^{-1}\| - \|B^{-1}\|} \left\|B^{-1}\right\| \\
&\quad \times \left[\left\|B^{-1}\right\|^{2} \ln\left(\left\|B^{-1}\right\|^{-1} + 1\right) - \left\|A^{-1}\right\|^{2} \ln\left(\left\|A^{-1}\right\|^{-1} + 1\right)\right. \\
&\quad \left. + \left\|A^{-1}\right\| - \left\|B^{-1}\right\|\right].
\end{aligned}$$

Assume that $a < 1$. By taking

$$\mathcal{D}\left(w_{\cdot^{-a}e^{-\cdot}}\right)(T) = \int_0^\infty \lambda^{-a} e^{-\lambda} (T + \lambda)^{-1} d\lambda = \Gamma(1-a) T^{-a} \exp(T) \Gamma(a, T)$$

in (2.5), we obtain

$$\begin{aligned}
0 &\leq \frac{\gamma - \beta}{\delta - \beta} \left[\beta^{-a} \exp(\beta) \Gamma(a, \beta) - \delta^{-a} \exp(\delta) \Gamma(a, \delta)\right] \\
&\leq A^{-a} \exp(A) \Gamma(a, A) - B^{-a} \exp(B) \Gamma(a, B) \\
&\leq \frac{\delta - \alpha}{\gamma - \alpha} \left[\alpha^{-a} \exp(\alpha) \Gamma(a, \alpha) - \gamma^{-a} \exp(\gamma) \Gamma(a, \gamma)\right]
\end{aligned}$$

provided that the positive operators A, B satisfy condition (2.4).

In particular, we have

$$\begin{aligned}
0 &\leq \frac{\gamma - \beta}{\delta - \beta} \left[\exp(\beta) E_1(\beta) - \exp(\delta) E_1(\delta)\right] \\
&\leq \exp(A) E_1(A) - \exp(B) E_1(B) \\
&\leq \frac{\delta - \alpha}{\gamma - \alpha} \left[\exp(\alpha) E_1(\alpha) - \exp(\gamma) E_1(\gamma)\right].
\end{aligned}$$

If A, $B > 0$ with $\|A\| \left\|B^{-1}\right\| < 1$, then

$$\begin{aligned}
0 &\leq \frac{1 - \|A\| \left\|B^{-1}\right\|}{\left(\|B\| - \|A\|\right) \|B^{-1}\|} \\
&\quad \times \left[\|A\|^{-a} \exp\left(\|A\|\right) \Gamma(a, \|A\|) - \|B\|^{-a} \exp\left(\|B\|\right) \Gamma(a, \|B\|)\right] \\
&\leq A^{-a} \exp\left(A\right) \Gamma(a, A) - B^{-a} \exp\left(B\right) \Gamma(a, B) \\
&\leq \frac{\|B\| \left\|A^{-1}\right\| - 1}{\|A^{-1}\| - \|B^{-1}\|} \left\|B^{-1}\right\| \\
&\quad \times \Big[\left\|A^{-1}\right\|^{a} \exp\left(\left\|A^{-1}\right\|^{-1}\right) \Gamma(a, \left\|A^{-1}\right\|^{-1}) \\
&\quad - \left\|B^{-1}\right\|^{a} \exp\left(\left\|B^{-1}\right\|^{-1}\right) \Gamma(a, \left\|B^{-1}\right\|^{-1})\Big].
\end{aligned}$$

In particular,

$$\begin{aligned}
0 &\leq \frac{1 - \|A\| \left\|B^{-1}\right\|}{\left(\|B\| - \|A\|\right) \|B^{-1}\|} \\
&\quad \times \left[\exp\left(\|A\|\right) E_1\left(\|A\|\right) - \exp\left(\|B\|\right) E_1\left(\|B\|\right)\right] \\
&\leq \exp\left(A\right) E_1(A) - \exp\left(B\right) E_1(B) \\
&\leq \frac{\|B\| \left\|A^{-1}\right\| - 1}{\|A^{-1}\| - \|B^{-1}\|} \left\|B^{-1}\right\| \\
&\quad \times \Big[\exp\left(\left\|A^{-1}\right\|^{-1}\right) E_1\left(\left\|A^{-1}\right\|^{-1}\right) \\
&\quad - \exp\left(\left\|B^{-1}\right\|^{-1}\right) E_1\left(\left\|B^{-1}\right\|^{-1}\right)\Big].
\end{aligned}$$

The interested author may state other similar inequalities by using the examples of operator monotone functions from [2], [4] and the references therein.

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References

[1] Bhatia, R., *Matrix Analysis*, Springer-Verlag, New York, 1997.
[2] Fujii, J. I., Seo, Y., *On parametrized operator means dominated by power ones*, Sci. Math. **1** (1998), 301–306.
[3] Furuichi, S., *Refined Young inequalities with Specht's ratio*, J. Egypt. Math. Soc. **20**(1) (2011), 46–49.
[4] Furuta, T., *Concrete examples of operator monotone functions obtained by an elementary method without appealing to Löwner integral representation*, Linear Algebra Appl. **429** (2008), 972–980.
[5] Furuta, T., *Precise lower bound of $f(A) - f(B)$ for $A > B > 0$ and non-constant operator monotone function f on* $[0, \infty)$, J. Math. Inequal. **9**(1) (2015), 47–52.

[6] *Generalized Exponential Integral*, Digital Library of Mathematical Functions, NIST. [Online https://dlmf.nist.gov/8.19#E1].

[7] Heinz, E., *Beiträge zur Störungsteorie der Spektralzerlegung*, Math. Ann. **123** (1951), 415–438.

[8] *Incomplete Gamma and Related Functions*, Definitions, Digital Library of Mathematical Functions, NIST. [Online https://dlmf.nist.gov/8.2].

[9] *Incomplete Gamma and Related Functions*, Integral Representations, Digital Library of Mathematical Functions, NIST. [Online https://dlmf.nist.gov/8.6].

[10] Löwner, K., *Über monotone MatrixFunktionen*, Math. Z. **38** (1934), 177–216.

[11] Moslehian, M. S., Najafi, H., *An extension of the Löwner–Heinz inequality*, Linear Algebra Appl., **437** (2012), 2359–2365.

[12] Zuo, H., Duan, G., *Some inequalities of operator monotone functions*, J. Math. Inequal. **8**(4) (2014), 777–781.

[13] Zuo, H., Shi, G., Fujii, M., R*efined Young inequality with Kantorovich constant*, J. Math. Inequal. **5**(4) (2011), 551–556.

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