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Natural affinors and torsion of connections on Weil like functors on double vector bundles

Abstract. We describe completely all natural affinors on product preserving gauge bundle functors on double vector bundles. Next, we study torsion of double-linear connections.

1. Introduction. We assume that any manifold considered in the paper is Hausdorff, second countable, finite dimensional, without boundary and smooth (i.e. of class C^∞). All maps between manifolds are assumed to be smooth (of class C^∞).

The concept of double vector bundles was introduced in [12] and further studied in [1, 7, 9], etc. The framework of double vector bundles is convenient for many constructions like linear forms, linear Poisson structures, linear connections, etc. The equivalent concept of double vector bundles can be found in [10]. We cite it in Section 2 of the present note.

The general concept of gauge bundle functors can be found in [5]. The concept of product preserving gauge bundle functors (ppgb-functors) on the category of double vector bundles can be found in [10], too. We cite it in Section 3.

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In [10], it is proved that the ppgb-functors F on the category of double vector bundles are in bijection with the A^F-bilinear maps

$$\diamond^F : U^F \times V^F \to W^F,$$

where A^F are Weil algebras and U^F and V^F and W^F are finite dimensional (over $\mathbf{R}$) A^F-modules. Moreover, given a ppgb-functor F on double vector bundles and a point $c \in A^F$, in [10] an affinor (i.e. tensor field of type $(1,1)$)

$$\mathrm{af}(c) : TFK \to TFK$$

on FK is constructed for any double vector bundle K.

The main result of the present note is the following one extending [6].

Theorem 1.1. *Let F be a ppgb-functor on double vector bundles. The canonical affinors* $\mathrm{af}(c)$ *for all $c \in A^F$ are all natural affinors on FK.*

Canonical (later called natural) affinors on some other bundle functors are described in [2, 3, 6, 8], etc. We point out that natural affinors play an important role in differential geometry. For example, natural affinors are useful in the prolongation of vector fields to product preserving bundles, see e.g. [5]. Natural affinors can be also used to define the general concept of a torsion of a connection, [6].

2. On double vector bundles.

Definition 2.1 ([10]). An *almost double vector bundle* is a system $K = (K_r, K_l, E_r, E_l)$ of vector bundles $K_r = (K, \tau_r, E_r)$, $K_l = (K, \tau_l, E_l)$, $E_r = (E_r, \underline{\tau}_l, M)$ and $E_l = (E_l, \underline{\tau}_r, M)$ such that $\underline{\tau}_l \circ \tau_r = \underline{\tau}_r \circ \tau_l$.

If $K' = (K'_r, K'_l, E'_r, E'_l)$ is another almost double vector bundle, an almost double vector bundle map $K \to K'$ is a map $f : K \to K'$ such that there are maps $\underline{f}_r : E_r \to E'_r$, $\underline{f}_l : E_l \to E'_l$ and $\underline{f} : M \to M'$ such that $(f, \underline{f}_r) : K_r \to K'_r$, $(f, \underline{f}_l) : K_l \to K'_l$, $(\underline{f}_r, \underline{f}) : E_r \to E'_r$ and $(\underline{f}_l, \underline{f}) : E_l \to E'_l$ are vector bundle maps.

We call M the basis of K and $\underline{f} : M \to M'$ the base map of f.

We have the trivial almost double vector bundle

$$\mathbf{R}^{m_1,m_2,n_1,n_2} = (K_r, K_l, E_r, E_l),$$

where $K_l = (\mathbf{R}^{m_1} \times \mathbf{R}^{m_2} \times \mathbf{R}^{n_1} \times \mathbf{R}^{n_2}, \tau_l, \mathbf{R}^{m_1} \times \mathbf{R}^{n_1})$, $K_r = (\mathbf{R}^{m_1} \times \mathbf{R}^{m_2} \times \mathbf{R}^{n_1} \times \mathbf{R}^{n_2}, \tau_r, \mathbf{R}^{m_1} \times \mathbf{R}^{m_2})$, $E_r = (\mathbf{R}^{m_1} \times \mathbf{R}^{m_2}, \underline{\tau}_l, \mathbf{R}^{m_1})$ and $E_l = (\mathbf{R}^{m_1} \times \mathbf{R}^{n_1}, \underline{\tau}_r, \mathbf{R}^{m_1})$, and where τ_r, τ_l, $\underline{\tau}_r$, $\underline{\tau}_l$ are the obvious projections.

Definition 2.2 ([10]). A *double vector bundle* is a locally trivial almost double vector bundle K, that is, there are non-negative integers m_1, m_2, n_1, n_2 such that for any $x \in M$ there is an open neighborhood $\Omega \subset M$ of x such that $K_{|\Omega} = \mathbf{R}^{m_1,m_2,n_1,n_2}$ modulo an almost double vector bundle isomorphism.

A very important example of a double vector bundle is the tangent bundle $TE = (TE, TE, E, TM)$ of a vector bundle $E = (E, \pi, M)$, where $\tau_r := p_{TE} : TE \to E$, $\tau_l := T\pi : TE \to TM$, $\underline{\tau_r} := p_{TM} : TM \to M$, $\underline{\tau_l} := \pi : E \to M$. Another such example is the cotangent bundle $T^*E = (T^*E, T^*E, E, E^*)$ of a vector bundle E, see [9]. Double vector bundle structures on TTM and TT^*M make possible the Lagrangian formulation of the dynamics in classical mechanics, see [13].

All double vector bundles and almost double vector bundle maps between them form a category which we denote by $\mathcal{DVB}$. (In [10], the notion of 2-$\mathcal{VB}$ instead of $\mathcal{DVB}$ is used.) Any $\mathcal{DVB}$-map $f : \mathbf{R}^{m_1,m_2,n_1,n_2} \to \mathbf{R}^{m'_1,m'_2,n'_1,n'_2}$ is of the form

$$\begin{aligned} &f(x,u,v,w) \\ (1)\qquad &= \Big(a(x), \sum_j a_j(x)u^j, \sum_k b_k(x)v^k, \sum_{j,k} c_{jk}(x)u^j v^k + \sum_l d_l(x)w^l\Big) \end{aligned}$$

for some maps $a : \mathbf{R}^{m_1} \to \mathbf{R}^{m'_1}$, $a_j : \mathbf{R}^{m_1} \to \mathbf{R}^{m'_2}$, $b_k : \mathbf{R}^{m_1} \to \mathbf{R}^{n'_1}$, $c_{jk} : \mathbf{R}^{m_1} \to \mathbf{R}^{n'_2}$, $d_l : \mathbf{R}^{m_1} \to \mathbf{R}^{n'_2}$, $j = 1, \dots, m_2$, $k = 1, \dots, n_1$, $l = 1, \dots, n_2$, where $x \in \mathbf{R}^{m_1}$, $u = (u^1, \dots, u^{m_2}) \in \mathbf{R}^{m_2}$, $v = (v^1, \dots, v^{n_1}) \in \mathbf{R}^{n_1}$, $w = (w^1, \dots, w^{n_2}) \in \mathbf{R}^{n_2}$.

By the local description (presented in [7]) of double vector bundles in the sense of [9], the double vector bundles in our sense are equivalent to the one of [9].

3. On ppgb-functors on double vector bundles. Let $\mathcal{FM}$ denote the category of fibred manifolds and fibred maps. The general concept of (gauge) bundle functors can be found in the book [5]. We need the following particular case of it.

Definition 3.1 ([10]). A *gauge bundle functor* on $\mathcal{DVB}$ is a covariant functor $F : \mathcal{DVB} \to \mathcal{FM}$ sending any double vector bundle K with the basis M into fibred manifold $p_K : FK \to M$ over M and any double vector bundle map $f : K \to K'$ with the base map $\underline{f} : M \to M'$ into fibred map $Ff : FK \to FK'$ over $\underline{f} : M \to M'$ and satisfying the following conditions:

(i) (*Localization condition*) For every double vector bundle K with the basis M and any open subset $U \subset M$, the inclusion map $i_{K|U} : K|U \to K$ induces diffeomorphism $Fi_{K|U} : F(K|U) \to p_K^{-1}(U)$, and

(ii) (*Regularity condition*) F transforms smoothly parametrized families of $\mathcal{DVB}$-maps into smoothly parametrized families of $\mathcal{FM}$-maps.

A gauge bundle functor F on $\mathcal{DVB}$ is called a Weil like functor (or ppgb-functor) if $F(K_1 \times K_2) = F(K_1) \times F(K_2)$ for any $\mathcal{DVB}$-objects K_1 and K_2.

An example of a ppgb-functor on $\mathcal{DVB}$ is the tangent functor T sending any $\mathcal{DVB}$-object K with basis M into the tangent bundle TK (treated as the fibred manifold over M) and any $\mathcal{DVB}$-map $f : K \to K'$ into $Tf : TK \to TK'$.

In [10], it is proved that the ppgb-functors F on the category of double vector bundles are in bijection with the A^F-bilinear maps

$$\diamond^F : U^F \times V^F \to W^F,$$

where A^F are Weil algebras and U^F and V^F and W^F are finite dimensional (over $\mathbf{R}$) A^F-modules. We have

$$F\mathbf{R}^{m_1,m_2,n_1,n_2} = (A^F)^{m_1} \times (U^F)^{m_2} \times (V^F)^{n_1} \times (W^F)^{n_2},$$

and if $f : \mathbf{R}^{m_1,m_2,n_1,n_2} \to \mathbf{R}^{m_1',m_2',n_1',n_2'}$ is of the form (1), then

$$\begin{aligned} Ff : (A^F)^{m_1} \times (U^F)^{m_2} \times (V^F)^{n_1} \times (W^F)^{n_2} \to \\ \to (A^F)^{m_1'} \times (U^F)^{m_2'} \times (V^F)^{n_1'} \times (W^F)^{n_2'} \end{aligned}$$

is of the similar form

$$\begin{aligned} &Ff(x,u,v,w) \\ (2) \quad &= \Big(a^{A^F}(x), \sum_j a_j^{A^F}(x)u^j, \sum_k b_k^{A^F}(x)v^k, \sum_{j,k} c_{jk}^{A^F}(x)u^j \diamond^F v^k + \sum_l d_l^{A^F}(x)w^l\Big), \end{aligned}$$

$x \in (A^F)^{m_1}$, $u = (u^1, \dots, u^{m_2}) \in (U^F)^{m_2}$, $v = (v^1, \dots, v^{n_1}) \in (V^F)^{n_1}$, $w = (w^1, \dots, w^{n_2}) \in (W^F)^{n_2}$, where $a^{A^F} = T^{A^F}a : T^{A^F}\mathbf{R}^{m_1} = (A^F)^{m_1} \to T^{A^F}\mathbf{R}^{m_1'} = (A^F)^{m_1'}$, $a_j^{A^F} = T^{A^F}a_j : (A^F)^{m_1} \to (A^F)^{m_2'}$, $b_k^{A^F} = T^{A^F}b_k : (A^F)^{m_1} \to (A^F)^{n_1'}$, $c_{ij}^{A^F} = T^{A^F}c_{jk} : (A^F)^{m_1} \to (A^F)^{n_2'}$, $d_l^{A^F} = T^{A^F}d_l : (A^F)^{m_1} \to (A^F)^{n_2'}$ are the values of a, a_j, b_k, c_{jk}, d_l by the (usual) Weil functor T^{A^F} of Weil algebra A^F. So, F has values in $\mathcal{DVB}$, i.e. $F\colon \mathcal{DVB} \to \mathcal{DVB}$.

4. Tangent bundle of a ppgb-functor on double vector bundles. It is observed that any ppgb-functor F on $\mathcal{DVB}$ has values in $\mathcal{DVB}$. So, we can compose ppgb-functors F_1 and F and obtain ppgb-functor F_1F on $\mathcal{DVB}$. In particular, the composition TF of the tangent functor T and a ppgb-functor F on $\mathcal{DVB}$ is again a ppgb-functor on $\mathcal{DVB}$. We have

$$A^{TF} = A^F \times A^F,\ U^{TF} = U^F \times U^F,\ V^{TF} = V^F \times V^F,\ W^{TF} = W^F \times W^F$$

and the algebra multiplication (of A^{TF}) and the module multiplications (of U^{TF} and V^{TF} and W^{TF}) and the A^{TF}-bilinear map $\diamond^{TF}$ satisfy

$$\begin{aligned} (a_1,a_2)(b_1,b_2) &= (a_1b_1, a_2b_1 + a_1b_2)\,, \\ (a_1,a_2)(u_1,u_2) &= (a_1u_1, a_2u_1 + a_1u_2)\,, \\ (3) \qquad (a_1,a_2)(v_1,v_2) &= (a_1v_1, a_2v_1 + a_1v_2)\,, \\ (a_1,a_2)(w_1,w_2) &= (a_1w_1, a_2w_1 + a_1w_2)\,, \\ (u_1,u_2) \diamond^{TF} (v_1,v_2) &= (u_1 \diamond^F v_1, u_2 \diamond^F v_1 + u_1 \diamond^F v_2) \end{aligned}$$

for any $a_1, a_2, b_1, b_2 \in A^F$, $u_1, u_2 \in U^F$, $v_1, v_2 \in V^F$, $w_1, w_2 \in W^F$.

In [10], for any $c \in A^F$, it is constructed a $\mathcal{DVB}$-invariant affinor

$$\mathrm{af}(c) : TFK \to TFK$$

on FK for any $\mathcal{DVB}$-object K. If $K = \mathbf{R}^{m_1,m_2,n_1,n_2}$, then

$$\mathrm{af}(c)((a_1, u_1, v_1, w_1), (a_2, u_2, v_2, w_2)) = ((a_1, u_1, v_1, w_1), c(a_2, u_2, v_2, w_2))$$

for any $a_1, a_2 \in (A^F)^{m_1}$, $u_1, u_2 \in (U^F)^{m_2}$, $v_1, v_2 \in (V^F)^{n_1}$, $w_1, w_2 \in (W^F)^{n_2}$, where the standard identification $TX = X \times X$ for vector spaces X is applied. The invariance means that if $f : K \to K_1$ is a $\mathcal{DVB}$-map, then $TFf \circ \mathrm{af}(c) = \mathrm{af}(c) \circ TFf$.

5. The natural affinors on ppgb-functors on double vector bundles. Let $\mathcal{DVB}_{m_1,m_2,n_1,n_2}$ be the category of all $\mathcal{DVB}$-objects K being locally isomorphic with $\mathbf{R}^{m_1,m_2,n_1,n_2}$ with local $\mathcal{DVB}$-isomorphisms between them as morphisms.

Definition 5.1. A $\mathcal{DVB}_{m_1,m_2,n_1,n_2}$-natural affinor on F is a $\mathcal{DVB}_{m_1,m_2,n_1,n_2}$-invariant family $B : TF \to TF$ of affinors

$$B : TFK \to TFK$$

on FK for any $\mathcal{DVB}_{m_1,m_2,n_1,n_2}$-object K. It means that $TFf \circ B = B \circ TFf$ for any $\mathcal{DVB}_{m_1,m_2,n_1,n_2}$-map $f : K \to K'$.

Theorem 5.2. *If $m_1 \geq 2$, then the natural affinors*

$$\mathrm{af}(c) : TF \to TF$$

for $c \in A^F$ are all $\mathcal{DVB}_{m_1,m_2,n_1,n_2}$-natural affinors on a ppgb-functor F.

Proof. Let B be a $\mathcal{DVB}_{m_1,m_2,n_1,n_2}$-natural affinor on a ppgb-functor F on $\mathcal{DVB}$. Clearly, B is determined by the affinor

$$B : TF\mathbf{R}^{m_1,m_2,n_1,n_2} \to TF\mathbf{R}^{m_1,m_2,n_1,n_2}$$

on $F\mathbf{R}^{m_1,m_2,n_1,n_2} = (A^F)^{m_1} \times (U^F)^{m_2} \times (V^F)^{n_1} \times (W^F)^{n_2}$. Then

$$B : F\mathbf{R}^{m_1,m_2,n_1,n_2} \times F\mathbf{R}^{m_1,m_2,n_1,n_2} \to F\mathbf{R}^{m_1,m_2,n_1,n_2} \times F\mathbf{R}^{m_1,m_2,n_1,n_2}$$

modulo the standard identification. So, we can write

$$B(x, y) = (x, \tilde{B}(x, y))$$

for all $x, y \in F\mathbf{R}^{m_1,m_2,n_1,n_2}$, where $\tilde{B}(x, y) \in F\mathbf{R}^{m_1,m_2,n_1,n_2}$ is linear in y.

Because of the invariance of B with respect to the homotheties $t \cdot \mathrm{id}_{\mathbf{R}^{m_1,m_2,n_1,n_2}}$ for $t > 0$, $\tilde{B}(tx, ty) = t\tilde{B}(x, y)$, i.e. $\tilde{B}(tx, y) = \tilde{B}(x, y)$. Consequently, $\tilde{B}(x, y)$ is independent of x. So, we can write

$$\begin{aligned} B((a_1, u_1, v_1, w_1), (a_2, u_2, v_2, w_2)) \\ = ((a_1, u_1, v_1, w_1), (\alpha(a_2, u_2, v_2, w_2), \beta(a_2, u_2, v_2, w_2), \\ \gamma(a_2, u_2, v_2, w_2), \delta(a_2, u_2, v_2, w_2))) \end{aligned}$$

for all $a_1, a_2 \in (A^F)^{m_1}$, $u_1, u_2 \in (U^F)^{m_2}$, $v_1, v_2 \in (V^F)^{n_1}$ and $w_1, w_2 \in (W^F)^{n_2}$, where $\alpha(a_2, u_2, v_2, w_2) \in (A^F)^{m_1}$ is linear in (a_2, u_2, v_2, w_2) and $\beta(a_2, u_2, v_2, w_2) \in (U^F)^{m_2}$ is linear in (a_2, u_2, v_2, w_2) and $\gamma(a_2, u_2, v_2, w_2) \in (V^F)^{n_1}$ is linear in (a_2, u_2, v_2, w_2) and $\delta(a_2, u_2, v_2, w_2) \in (W^F)^{n_2}$ is linear in (a_2, u_2, v_2, w_2).

Let $\varphi_{t,t_1,t_2,t_3} : \mathbf{R}^{m_1,m_2,n_1,n_2} \to \mathbf{R}^{m_1,m_2,n_1,n_2}$ be given by

$$\varphi_{t,t_1,t_2,t_3}(x, y_1, y_2, y_3) = (tx, t_1 y_1, t_2 y_2, t_3 y_3)$$

for all $x \in \mathbf{R}^{m_1}$ and $y_1 \in \mathbf{R}^{m_2}$ and $y_2 \in \mathbf{R}^{n_1}$ and $y_3 \in \mathbf{R}^{n_2}$, where t, t_1, t_2, t_3 are positive real numbers. It is a $\mathcal{DVB}_{m_1,m_2,n_1,n_2}$-map. Then, by the invariance of B with respect to φ_{t,t_1,t_2,t_3}, we derive

$$\alpha(ta_2, t_1 u_2, t_2 v_2, t_3 w_2) = t\alpha(a_2, u_2, v_2, w_2)\,.$$

Consequently, $\alpha(a_2, u_2, v_2, w_2)$ is linear in a_2 and independent of u_2, v_2, w_2. Similarly, $\beta(a_2, u_2, v_2, w_2)$ is linear in u_2 and independent of a_2, v_2, w_2, and $\gamma(a_2, u_2, v_2, w_2)$ is linear in v_2 and independent od a_2, u_2, w_2, and $\delta(a_2, u_2, v_2, w_2)$ is linear in w_2 and independent of b, u_2, v_2. Hence we can write

$$\begin{aligned} &B((a_1, u_1, v_1, w_1), (a_2, u_2, v_2, w_2)) \\ &\qquad = ((a_1, u_1, v_1, w_1), (\alpha(a_2), \beta(u_2), \gamma(v_2), \delta(w_2))) \end{aligned}$$

for all $a_1, a_2 \in (A^F)^{m_1}$, $u_1, u_2 \in (U^F)^{m_2}$, $v_1, v_2 \in (V^F)^{n_1}$, $w_1, w_2 \in (W^F)^{n_2}$, where $\alpha(a_2) \in (A^F)^{m_1}$ is linear in a_2 and $\beta(u_2) \in (U^F)^{m_2}$ is linear in u_2 and $\gamma(v_2) \in (V^F)^{n_1}$ is linear in v_2 and $\delta(w_2) \in (W^F)^{n_2}$ is linear in w_2.

Let $f : \mathbf{R}^{m_1,m_2,n_1,n_2} \to \mathbf{R}^{m_1,m_2,n_1,n_2}$ be given by

$$f(x, y_1, y_2, y_3) = (x + x^1 x, y_1 + x^1 y_1, y_2 + x^1 y_2, y_3 + x^1 y_3)$$

for all $x = (x^1, \dots, x^{m_1}) \in \mathbf{R}^{m_1}$ and $y_1 \in \mathbf{R}^{m_2}$ and $y_2 \in \mathbf{R}^{n_1}$ and $y_3 \in \mathbf{R}^{n_2}$. It is a $\mathcal{DVB}_{m_1,m_2,n_1,n_2}$-map on the open and dense subset in $\mathbf{R}^{m_1,m_2,n_1,n_2}$, $x^1 \neq -1$. Then, by the invariance of B with respect to f and (in particular) formula (2) for TF instead of F and formulas (3), we get

$$\begin{aligned} &((a_1 + a_1^1 a_1, u_1 + a_1^1 u_1, \dots), (\alpha(a_2 + a_1^1 a_2 + a_2^1 a_1), \beta(u_2 + a_1^1 u_2 + a_2^1 u_1), \dots)) \\ &\qquad = ((a_1 + a_1^1 a_1, u_1 + a_1^1 u_1, \dots), (\alpha(a_2) + a_1^1 \alpha(a_2) + \alpha^1(a_2) a_1, \\ &\qquad\qquad \beta(u_2) + a_1^1 \beta(u_2) + \alpha^1(a_2) u_1, \dots)) \end{aligned}$$

for all $a_1, a_2 \in (A^F)^{m_1}$ and $u_1, u_2 \in (U^F)^{m_2}$ and ..., where $(\alpha^1(b), \dots, \alpha^{m_1}(b)) = \alpha(b) \in (A^F)^{m_1}$ and $(b^1, \dots, b^{m_1}) = b \in (A^F)^{m_1}$. Then

$$\begin{aligned} \alpha(a_1^1 a_2) + \alpha(a_2^1 a_1) &= a_1^1 \alpha(a_2) + \alpha^1(a_2) a_1\,, \\ \beta(a_1^1 u_2) + \beta(a_2^1 u_1) &= a_1^1 \beta(u_2) + \alpha^1(a_2) u_1\,, \\ \gamma(a_1^1 v_2) + \gamma(a_2^1 v_1) &= a_1^1 \gamma(v_2) + \alpha^1(a_2) v_1\,, \\ \delta(a_1^1 w_2) + \delta(a_2^1 w_1) &= a_1^1 \delta(w_2) + \alpha^1(a_2) w_1\,. \end{aligned}$$

If $a_1^1 = 1$, then

$$\alpha(a_2^1 a_1) = \alpha^1(a_2)a_1\,,\ \beta(a_2^1 u_1) = \alpha^1(a_2)u_1\,,$$
$$\gamma(a_2^1 v_1) = \alpha^1(a_2)v_1\,,\ \delta(a_2^1 w_1) = \alpha^1(a_2)w_1\,.$$

If $a_2 = (1, 0, \dots, 0) \in (A^F)^{m_1}$, we get

$$\alpha(a) = c_1 a\,,\ \beta(u) = c_1 u\,,\ \gamma(v) = c_1 v\,,\ \delta(w) = c_1 w$$

for any $a = (a^1, \dots, a^{m_1}) \in (A^F)^{m_1}$ with $a^1 = 1$ and $u \in (U^F)^{m_2}$ and $v \in (V^F)^{n_1}$ and $w \in (W^F)^{n_2}$, where $c_1 := \alpha^1(1, 0, \dots, 0) \in A^F$.

Similarly, replacing 1 by $i \in \{1, \dots, m_1\}$, we derive

$$\alpha(a) = c_i a\,,\ \beta(u) = c_i u\,,\ \gamma(v) = c_i v\,,\ \delta(w) = c_i w$$

for any $a = (a^1, \dots, a^{m_1}) \in (A^F)^{m_1}$ with $a^i = 1$ and $u \in (U^F)^{m_2}$ and $v \in (V^F)^{n_1}$ and $w \in (W^F)^{n_2}$, where $c_i := \alpha^i(0, \dots, 1, \dots, 0) \in A^F$ (1 in i-th position).

From the linearity of α and $m_1 \geq 2$ we obtain

$$\alpha(a) = ca\,,\ \beta(u) = cu\,,\ \gamma(v) = cv\,,\ \delta(w) = cw$$

for any $a = (a^1, \dots, a^m) \in (A^F)^{m_1}$ and $u \in (U^F)^{m_2}$ and $v \in (V^F)^{n_1}$ and $w \in (W^F)^{n_2}$, where $c := c_1 = \dots = c_m \in A^F$. That $c_1 = \dots = c_m$ is a simple consequence of the invariance of B with respect to the permutations of the base coordinates.

Then

$$B((a_1, u_1, v_1, w_1), (a_2, u_2, v_2, w_2)) = ((a_1, u_1, v_1, w_1), c(a_2, u_2, v_2, w_2))$$

for all $a_1, a_2 \in (A^F)^{m_1}$, $u_1, u_2 \in (U^F)^{m_2}$, $v_1, v_2 \in (V^F)^{n_1}$, $w_1, w_2 \in (W^F)^{n_2}$, where $c \in A^F$ is as above. Then $B = \mathrm{af}(c)$, as well and the proof is complete. Q.E.D. □

6. On double-linear vector fields. Let K be a double vector bundle with basis M. A vector field Z on K is called double-linear if the flow of Z is formed by local $\mathcal{DVB}$-isomorphisms.

Let $x^1, \dots, x^{m_1}$, $u^1, \dots, u^{m_2}$, $v^1, \dots, v^{n_1}$, $w^1, \dots, w^{n_2}$ be (local) $\mathcal{DVB}$-coordinates on K. A map $f : K \to K$ is a $\mathcal{DVB}$-map if and only if it is of the form (1). Consequently, a vector field Z on K is double linear if and only if it is of the form

$$\begin{aligned} Z = &\sum_{i=1}^{m_1} a^i(x)\frac{\partial}{\partial x^i} + \sum_{j,j^1=1}^{m_2} b_j^{j_1}(x)u^j\frac{\partial}{\partial u^{j_1}} + \sum_{k,k_1=1}^{n_1} c_k^{k_1}(x)v^k\frac{\partial}{\partial v^{k_1}} \\ &+ \sum_{l,l^1=1}^{n_2} e_l^{l_1}(x)w^l\frac{\partial}{\partial w^{l_1}} + \sum_{j_2=1}^{m_2}\sum_{k_2=1}^{n_1}\sum_{l_2=1}^{n_2} f_{j_2k_2}^{l_2}(x)u^{j_2}v^{k_2}\frac{\partial}{\partial w^{l_2}}\,. \end{aligned} \tag{4}$$

So, we have:

Lemma 6.1 ([11]). *The space of all double-linear vector fields on K is the Lie subalgebra in the Lie algebra of vector fields on K.*

Let F be a ppgb-functor on $\mathcal{DVB}$. Then FK is again a $\mathcal{DVB}$-object (see Section 3).

Lemma 6.2. *Let Z be a double-linear vector field on FK and $c \in A^F$ be a point. Then the vector field* $\mathrm{af}(c)(Z)$ *on FK is also double-linear.*

Proof. We may assume that $K = \mathbf{R}^{m_1,m_2,n_1,n_2}$. Then $FK = A^{m_1} \times U^{m_2} \times V^{n_1} \times W^{n_2}$. Fixing the bases (over $\mathbf{R}$) of A^{m_1} and U^{m_2} and V^{n_1} and W^{n_2}, we can write $FK = \mathbf{R}^{M_1,M_2,N_1,N_2}$. Let $x^1, \dots, x^{M_1}, u^1, \dots, u^{M_2}, v^1, \dots, v^{N_1}, w^1, \dots, w^{N_2}$ be the usual coordinates on $\mathbf{R}^{M_1,M_2,N_1,N_2}$. Then Z is of the form

$$
\begin{aligned}
Z = {} & \sum_{i=1}^{M_1} a^i(x)\frac{\partial}{\partial x^i} + \sum_{j,j^1=1}^{M_2} b_j^{j_1}(x)u^j\frac{\partial}{\partial u^{j_1}} + \sum_{k,k_1=1}^{N_1} c_k^{k_1}(x)v^k\frac{\partial}{\partial v^{k_1}} \\
& + \sum_{l,l^1=1}^{N_2} e_l^{l_1}(x)w^l\frac{\partial}{\partial w^{l_1}} + \sum_{j_2=1}^{M_2}\sum_{k_2=1}^{N_1}\sum_{l_2=1}^{N_2} f_{j_2k_2}^{l_2}(x)u^{j_2}v^{k_2}\frac{\partial}{\partial w^{l_2}}\,.
\end{aligned}
\tag{5}
$$

To prove that $\mathrm{af}(c)(Z)$ is double-linear, it is sufficient to show that $\mathrm{af}(c)(Z)$ is of the form (5), too. Of course, it is sufficient to show that $\mathrm{af}(c)(\frac{\partial}{\partial x^i})$ is the linear combination of $\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^{M_1}}$ with real coefficients and that $\mathrm{af}(c)(\frac{\partial}{\partial u^j})$ is the linear combination of $\frac{\partial}{\partial u^1}, \dots, \frac{\partial}{\partial u^{M_2}}$ with real coefficients and that $\mathrm{af}(c)(\frac{\partial}{\partial v^k})$ is the linear combination of $\frac{\partial}{\partial v^1}, \dots, \frac{\partial}{\partial v^{N_1}}$ with real coefficients and that $\mathrm{af}(c)(\frac{\partial}{\partial w^l})$ is the linear combination of $\frac{\partial}{\partial w^1}, \dots, \frac{\partial}{\partial w^{N_2}}$ with real coefficients.

For example, we prove that $\mathrm{af}(c)(\frac{\partial}{\partial u^1})$ is the linear combination of $\frac{\partial}{\partial u^1}, \dots, \frac{\partial}{\partial u^{M_2}}$ with real coefficients. Let $(x,u,v,w) \in A^{m_1} \times U^{m_2} \times V^{n_1} \times W^{n_2}$. Let $e_1, \dots, e_{M_2}$ be the usual basis in $\mathbf{R}^{M_2} \tilde{=} U^{m_2}$. We can write $\frac{\partial}{\partial u^j}{}_{|(x,u,v,w)} = ((x,u,v,w),(0,e_j,0,0))$. Then

$$
\mathrm{af}(c)\left(\frac{\partial}{\partial u^1}\right)_{|(x,u,v,w)} = ((x,u,v,w),(0,c\cdot e_1,0,0))\,.
$$

On the other hand, $c \cdot e_1 \in U^{m_2}$ (as $e_1 \in U^{m_2}$), and then $c \cdot e_1$ is the linear combination of $e_1, \dots, e_{M_2}$ with real coefficients. The proof of the proposition is complete. Q.E.D. □

7. The F-N-bracket and double-linear (semi-basic) tangent valued *p*-forms. If $K \to M$ is a fibred manifold, a projectable semi-basic tangent valued p-form on K is a section $\varphi : K \to \wedge^p T^*M \otimes TK$ such that $\varphi(X_1, \dots, X_p)$ is a projectable vector field on K for any vector fields $X_1, \dots, X_p$ on M.

Given a projectable semi-basic tangent valued p-form $\varphi : K \to \wedge^p T^*M \otimes TK$ we have the underlying tangent valued p-form $\underline{\varphi} : M \to \wedge^p T^*M \otimes TM$ on M such that $\underline{\varphi}(X_1, \dots, X_p)$ is the underlying vector field of $\varphi(X_1, \dots, X_p)$ for any vector fields $X_1, \dots, X_p$ on M.

Lemma 7.1. *Let $K \to M$ be a fibred manifold. Given a projectable semi-basic tangent valued p-form $\varphi : K \to \wedge^p T^*M \otimes TK$ on K and a projectable semi-basic tangent valued q-form $\psi : K \to \wedge^q T^*M \otimes TK$ on K, the Frolicher–Nijenhuis bracket (F-N-bracket) $[[\varphi, \psi]]$ is (again) a projectable semi-basic tangent valued $(p+q)$-form $[[\varphi, \psi]] : K \to \wedge^{p+q} T^*M \otimes TK$ on K satisfying*

$$
\begin{aligned}
&[[\varphi,\psi]](X_1,\dots,X_{p+q})\\
&= \frac{1}{p!q!}\sum_\sigma \operatorname{sign}\sigma[\varphi(X_{\sigma 1},\dots,X_{\sigma p}),\psi(X_{\sigma(p+1)},\dots,X_{\sigma(p+q)})]\\
&\quad+\frac{-1}{p!(q-1)!}\sum_\sigma \operatorname{sign}\sigma\psi([\underline{\varphi}(X_{\sigma 1},\dots,X_{\sigma p}),X_{\sigma(p+1)}],X_{\sigma(p+2)},\dots)\\
&\quad+\frac{(-1)^{pq}}{(p-1)q!}\sum_\sigma \operatorname{sign}\sigma\varphi([\underline{\psi}(X_{\sigma 1},\dots,X_{\sigma q}),X_{\sigma(q+1)}],X_{\sigma(q+2)},\dots)\\
&\quad+\frac{(-1)^{p-1}}{(p-1)!(q-1)!2!}\sum_\sigma \operatorname{sign}\sigma\psi(\underline{\varphi}([X_{\sigma 1},X_{\sigma 2}],X_{\sigma 3},\dots),X_{\sigma(p+2)},\dots)\\
&\quad+\frac{(-1)^{(p-1)q}}{(p-1)!(q-1)!2!}\sum_\sigma \operatorname{sign}\sigma\varphi(\underline{\psi}([X_{\sigma_1},X_{\sigma 2}],X_{\sigma 3},\dots),X_{\sigma(q+2)},\dots)
\end{aligned}
\tag{6}
$$

for any vector fields $X_1, \dots, X_{p+q}$ on M, where sums are over all permutations $\sigma : \{1, \dots, p+q\} \to \{1, \dots, p+q\}$.

Proof. It is well-known fact, see e.g. [4]. Q.E.D. □

Let F be a ppgb-functor on $\mathcal{DVB}$ and K be a $\mathcal{DVB}$-object with basis M. Then we have the fibred manifold $FK \to M$. We have also the $\mathcal{DVB}$-object FK with basis FM.

Definition 7.2. A double-linear semi-basic tangent valued p-form on $FK \to M$ is a projectable semi-basic tangent valued p-form $\varphi : FK \to \wedge^p T^*M \otimes TFK$ on (fibered manifold) FK (with basis M) such that (additionally) $\varphi(X_1, \dots, X_p)$ is a double-linear vector field on $\mathcal{DVB}$-object FK (with basis FM) for any vector fields $X_1, \dots, X_p$ on M.

Remark 7.3. If F is the identity functor and $K = \mathbf{R}^{m_1,m_2,n_1,n_2}$, the space of double-linear semi-basic tangent valued p-forms on $\mathbf{R}^{m_1,m_2,n_1,n_2}$ is (in obvious way) a module over the ring of smooth maps $\mathbf{R}^{m_1} \to \mathbf{R}$. This module is free and the basis is (for example) the collection consisting of $dx^{i_1} \wedge \ldots \wedge dx^{i_p} \otimes \frac{\partial}{\partial x^i}$ and $u^{j_1} dx^{i_1} \wedge \ldots \wedge dx^{i_p} \otimes \frac{\partial}{\partial u^j}$ and $du^{j_1} \wedge dx^{i'_1} \wedge \ldots \wedge dx^{i'_{p-1}} \otimes \frac{\partial}{\partial u^j}$ and $v^{k_1} dx^{i_1} \wedge \ldots \wedge dx^{i_p} \otimes \frac{\partial}{\partial v^k}$ and $dv^{k_1} \wedge dx^{i'_1} \wedge \ldots \wedge dx^{i'_{p-1}} \otimes \frac{\partial}{\partial v^k}$ and $w^{l_1} dx^{i_1} \wedge \ldots \wedge dx^{i_p} \otimes \frac{\partial}{\partial w^l}$ and $dw^{l_1} \wedge dx^{i'_1} \wedge \ldots \wedge dx^{i'_{p-1}} \otimes \frac{\partial}{\partial w^l}$ and $u^j v^k dx^{i_1} \wedge \ldots \wedge dx^{i_p} \otimes \frac{\partial}{\partial w^l}$ and $v^k du^j \wedge dx^{i'_1} \wedge \ldots \wedge dx^{i'_{p-1}} \otimes \frac{\partial}{\partial w^l}$ and $u^j dv^k \wedge dx^{i'_1} \wedge \ldots \wedge dx^{i'_{p-1}} \otimes \frac{\partial}{\partial w^l}$ and $du^j \wedge dv^k \wedge dx^{i''_1} \wedge \ldots \wedge dx^{i''_{p-2}} \otimes \frac{\partial}{\partial w^l}$ for all integers $i, i_1, \ldots, i_p, i'_1, \ldots, i'_{p-1}, i''_1, \ldots, i''_{p-2}, j, j_1, k, k_1, l, l_1$ with $1 \leq i_1 < \ldots < i_p \leq m_1$ and $1 \leq i'_1 < \ldots < i'_{p-1} \leq m_1$ and $1 \leq i''_1 < \ldots < i''_{p-2} \leq m_1$ and $1 \leq i \leq m_1$ and $1 \leq j \leq m_2$ and $1 \leq j_2 \leq m_2$ and $1 \leq k \leq n_1$ and $1 \leq k_1 \leq n_1$ and $1 \leq l \leq n_2$ and $1 \leq l_1 \leq n_2$, where $x^1, \ldots, x^{m_1}$, $u^1, \ldots, u^{m_2}$, $v^1, \ldots, v^{n_1}$, $w^1, \ldots, w^{n_2}$ are the usual coordinates on $\mathbf{R}^{m_1,m_2,n_1,n_2}$.

Proposition 7.4. *Let $\varphi : FK \to \wedge^p T^*M \otimes TFK$ be a double-linear (then projectable) semi-basic tangent valued p-form on $FK \to M$ and $\psi : FK \to \wedge^q T^*M \otimes TFK$ be a double-linear semi-basic tangent valued q-form on $FK \to M$. Then the F-N bracket $[[\varphi, \psi]] : FK \to \wedge^{p+q} T^*M \otimes TFK$ of φ and ψ is a double-linear semi-basic tangent valued $(p+q)$-form on $FK \to M$.*

Proof. It is a simple consequence of formula (6) (with $FK \to M$ playing the role of $K \to M$) and Lemma 6.1. Q.E.D. □

8. An application to torsion of double-linear connections in $FK \to M$. Let F be a ppgb-functor on $\mathcal{DVB}$ and let K be a $\mathcal{DVB}_{m_1,m_2,n_1,n_2}$-object with basis M.

Definition 8.1. A double-linear connection in $FK \to M$ is a double-linear semi-basic tangent valued 1-form $\Gamma : FK \to T^*M \otimes TFK$ on $FK \to M$ such that the underlying vector field of $\Gamma(X)$ is equal to X for any vector field X on basis M.

Assume $m_1 \geq 2$. Let $\Gamma : FK \to T^*M \otimes TFK$ be a double-linear connection in $FK \to M$ and let $B : TFK \to TFK$ be a $\mathcal{DVB}_{m_1,m_2,n_1,n_2}$-natural affinor on FK. If $m_1 \geq 2$, then $B = \mathrm{af}(c)$ for some $c \in A^F$, see Theorem 5.2. If $c = \lambda + n$, where $\lambda \in \mathbf{R}$ and n is nilpotent, then given a vector field X on M, the vector field $B \circ \Gamma(X)$ on FK is projectable with the underlying vector field λX. Now, because of Lemma 6.2, $B \circ \Gamma$ is a double-linear semi-basic tangent-valued 1-form on $FK \to M$, where $(B \circ \Gamma)(X) := B \circ \Gamma(X)$ for any vector field X on M.

Definition 8.2. The torsion $\tau^B(\Gamma)$ of type B of Γ is by definition the F-N bracket of Γ and $B \circ \Gamma$, i.e. $\tau^B(\Gamma) := [[\Gamma, B \circ \Gamma]]$.

Theorem 8.3. *Let F and Γ and B be as above. Assume m_1, m_2, n_1, n_2 are non-negative integers with $m_1 \geq 2$. The torsion $\tau^B(\Gamma)$ of type B of Γ is a double-linear semi-basic tangent valued 2-form on FK. If $B = \mathrm{af}(c)$, where $c = \lambda + n$, $\lambda \in \mathbf{R}$, $n \in A^F$ is a nilpotent, then*

$$\tau^B(\Gamma)(X,Y) = 2\lambda\mathcal{R}_\Gamma(X,Y) + [\Gamma(X), \mathrm{af}(n) \circ \Gamma(Y)] - [\Gamma(Y), \mathrm{af}(n) \circ \Gamma(X)] - \mathrm{af}(n) \circ \Gamma([X,Y])$$

for any vector fields X and Y on M, where $\mathcal{R}_\Gamma = \frac{1}{2}[[\Gamma,\Gamma]] : FK \to \wedge^2 T^*M \otimes VFK$ (i.e. $\mathcal{R}_\Gamma(X,Y) = [\Gamma(X),\Gamma(Y)] - \Gamma([X,Y])$) is the curvature of Γ. Thus $\tau^B(\Gamma) : FK \to \wedge^2 T^*M \otimes VFK$, where V is the vertical functor.

Proof. Since Γ is a double-linear semi-basic tangent valued 1-form on $FK \to M$, then so is $B \circ \Gamma$, see Lemma 6.2, and then $[[\Gamma, B \circ \Gamma]]$ is a double-linear semi-basic tangent valued 2-form on $FK \to M$, see Proposition 7.4. To obtain the formula, we propose to apply the one of the F-N-bracket, see (6) for $FK \to M$ instead of $K \to M$ and Γ and $B \circ \Gamma$ instead of φ and ψ. Q.E.D. □

Remark 8.4. If $F = T$ is the tangent functor and $B = J$ is the almost tangent structure (i.e. $A^F = \mathbf{D}$, $c = n = (0,1) \in \mathbf{D}$, $\lambda = 0$), then

$$\tau^J(\Gamma)(X,Y) = [\Gamma(X), J \circ \Gamma(Y)] - [\Gamma(Y), J \circ \Gamma(X)] - J \circ \Gamma([X,Y])$$

for any vector fields X and Y on M. If additionally $K = (M,M,M,M)$, then $\tau^J(\Gamma)$ is (almost) the usual torsion of a usual linear connection Γ on M. Indeed, if $x^1,\dots,x^m$ are local coordinates on M and $x^1,\dots,x^m,y^1,\dots,y^m$ are the induced coordinates on TM, then $J = \sum_{i=1}^m dx^i \otimes \frac{\partial}{\partial y^i}$. If $\Gamma(\frac{\partial}{\partial x^i}) = \frac{\partial}{\partial x^i} - \Gamma^k_{ij}(x)y^j\frac{\partial}{\partial y^k}$, then $J \circ \Gamma(\frac{\partial}{\partial x^i}) = \frac{\partial}{\partial y^i}$. Then $\tau^J_\Gamma(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}) = (\Gamma^k_{ij} - \Gamma^k_{ji})\frac{\partial}{\partial y^k}$ (the Einstein summation convention). Therefore our torsion generalizes the classical torsion of classical linear connection.

Example 8.5. Let $V : \mathcal{DVB} \to \mathcal{FM}$ be the vertical functor sending any double vector bundle K with basis M into the usual vertical bundle $VK = \cup_{x\in M} T(K_x) \to M$ of K. Then $A^V = \mathbf{R}$ and $\diamond^V : \mathbf{D} \times \mathbf{D} \to \mathbf{D}$ is the usual multiplication of the dual numbers. By our theorem, any natural affinor B on VK is $B = \mathrm{af}(\lambda) = \lambda\mathrm{Id}$, $\lambda \in \mathbf{R}$. On the other hand, the torsion (in our sense) of the type $B = \lambda\mathrm{Id}$ of double linear connection Γ on $VK \to M$ is of the form $\tau^B(\Gamma) = [[\Gamma, B \circ \Gamma]] = \lambda[[\Gamma,\Gamma]] = 2\lambda\mathcal{R}_\Gamma$. Consequently, any torsion (in our sense) of double linear connection Γ on $VK \to M$ is the constant multiple of the curvature $\mathcal{R}_\Gamma$ of Γ.

Example 8.6. If we replace in the previous example the algebra of dual numbers $\mathbf{D}$ by the arbitrary Weil algebra A, we get the A-vertical bundle $V^AK = \cup_{x\in M} T^A(K_x) \to M$ of a double vector bundle $K \to M$. Clearly, $A^{V^A} = \mathbf{R}$ and $\diamond^{V^A} : A \times A \to A$ is the algebra multiplication of A. Thus any natural affinor B on V^AK is proportional to the identity one, i.e. $B = \lambda\mathrm{Id}$.

Then (similarly to the previous example) $\tau^B(\Gamma) = 2\lambda\mathcal{R}_\Gamma$ for any double linear connection Γ on $V^AK \to M$. Hence any torsion (in our sense) of a double linear connection Γ on $V^AK \to M$ is a constant multiple of the curvature $\mathcal{R}_\Gamma$ of Γ.

By the arguments of the above examples we have:

Corollary 8.7. *If $A^F = \mathbf{R}$, then any natural affinor on FK is the constant multiple of the identity one and any torsion (in our sense) of a double linear connection Γ on $FK \to M$ is a constant multiple of the curvature $\mathcal{R}_\Gamma$ of Γ.*

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