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ANDRZEJ WALENDZIAK

On implicative BE algebras

Abstract. We consider some generalizations of BCK algebras (RML, BE, aBE, BE** and aBE** algebras). We investigate the property of implicativity for these algebras. We prove that for any implicative BE** algebra the commutativity property is equivalent to the property of antisymmetry and show that implicative aBE** algebras are commutative BCK algebras. We also show that the class of implicative BE** algebras is a variety.

1. Introduction. In 1966, Y. Imai and K. Iséki [3] introduced a new notion called a BCK algebra. It is an algebraic formulation of the BCK-propositional calculus system of C. A. Meredith [12], which generalizes the concept of implicative algebras (see [1]). Hundred of papers were written on BCK algebras, and the books [11] and [4]. In [10], as a generalization of BCK algebras, H. S. Kim and Y. H. Kim defined BE algebras. In 2008, A. Walendziak [14] defined commutative BE algebras and proved that they are BCK algebras. A. Iorgulescu [5] introduced new generalizations of BCK algebras (RML, aBE, BE**, aBE** algebras and many others).

In 1978, K. Iséki and S. Tanaka [8] introduced the notion of implicativity in the theory of BCK algebras. The present paper is a continuation of the author's paper [15], where the property of implicativity for various generalizations of BCK algebras was studied. Here we consider RML, BE, aBE, BE** and aBE** algebras and investigate the implicative property for these

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algebras. We prove that for any implicative BE** algebra the commutativity property is equivalent to the property of antisymmetry and show that implicative aBE** algebras are commutative BCK algebras. We also show that the class of implicative BE** algebras is a variety.

2. Preliminaries. Let $\mathcal{A} = (A, \to, 1)$ be an algebra of type $(2, 0)$. We consider the following list of properties ([5]) that can be satisfied by $\mathcal{A}$:

(An) (Antisymmetry) $x \to y = 1 = y \to x \Longrightarrow x = y$,
(BB) $(y \to z) \to [(z \to x) \to (y \to x)] = 1$,
(D) $y \to ((y \to x) \to x) = 1$,
(Ex) (Exchange) $x \to (y \to z) = y \to (x \to z)$,
(K) $x \to (y \to x) = 1$,
(L) (Last element) $x \to 1 = 1$,
(M) $1 \to x = x$,
(Re) (Reflexivity) $x \to x = 1$,
(*) $y \to z = 1 \Longrightarrow (x \to y) \to (x \to z) = 1$,
(**) $y \to z = 1 \Longrightarrow (z \to x) \to (y \to x) = 1$,
(Tr) (Transitivity) $x \to y = 1 = y \to z \Longrightarrow x \to z = 1$.

Lemma 2.1 ([5], Proposition 2.1). *Let $\mathcal{A} = (A, \to, 1)$ be an algebra of type $(2, 0)$. Then the following hold:*

(i) (M) + (BB) *imply* (Re), (**),
(ii) (M) + (BB) + (An) *imply* (Ex),
(iii) (M) + (**) *imply* (Tr),
(iv) (Re) + (Ex) *imply* (D),
(v) (Re) + (L) + (Ex) *imply* (K),
(vi) (Re) + (Ex) + (Tr) *imply* (**),
(vii) (M) + (*) *imply* (Tr).

Definition 2.2 ([5]).

1. An *RML algebra* is an algebra $\mathcal{A} = (A, \to, 1)$ verifying (Re), (M), (L).
2. A *BE algebra* is a RML algebra verifying (Ex).
3. An *aBE algebra* is a BE algebra verifying (An).
4. A *BE** algebra* is a BE algebra verifying (**).
5. An *aBE** algebra* is a BE** algebra verifying (An).
6. A *BCK algebra* is an algebra $\mathcal{A} = (A, \to, 1)$ verifying (An), (BB), (M), (L).

Denote by **RML**, **BE**, **aBE**, **BE****, **aBE**** and **BCK** the classes of RML, BE, aBE, BE**, aBE** and BCK algebras, respectively. By definition and Lemma 2.1 (i) and (ii), we have

BCK $\subset$ **aBE**** $\subset$ **aBE** $\subset$ **BE** $\subset$ **RML** and **aBE**** $\subset$ **BE**** $\subset$ **BE**.

The interrelationships between the classes of algebras mentioned before are visualized in Figure 1. (An arrow indicates proper inclusion, that is, if $\mathbf{X}$ and $\mathbf{Y}$ are classes of algebras, then $\mathbf{X} \longrightarrow \mathbf{Y}$ means $\mathbf{X} \subset \mathbf{Y}$.)

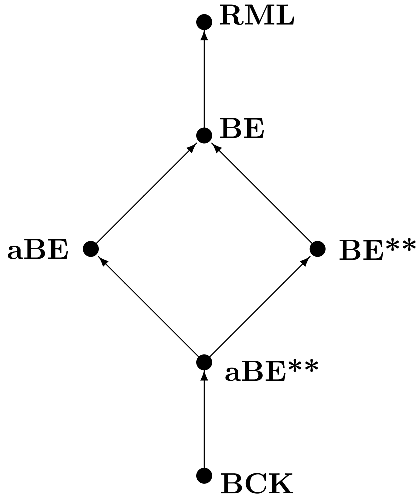

Figure 1

Let $\mathcal{A} = (A, \to, 1)$ be an algebra of type $(2, 0)$. We define the binary relation $\leq$ by: for all $x, y \in A$,

$$x \leq y \Longleftrightarrow x \to y = 1.$$

It is known that $\leq$ is an order relation in BCK algebras. By definition, in RML and BE algebras, $\leq$ is a reflexive relation; in aBE algebras, $\leq$ is reflexive and antisymmetric. By Lemma 2.1 (iii), in BE** algebras, $\leq$ is reflexive and transitive (i.e., it is a pre-order relation). Lastly, in aBE** algebras, $\leq$ is an order relation.

In [13], S. Tanaka introduced the notion of commutativity in the theory of BCK algebras. A BCK algebra $\mathcal{A} = (A, \to, 1)$ is called *commutative* if, for all $x, y \in A$,

(Com) $\quad (x \to y) \to y = (y \to x) \to x.$

H. Yutani [17] proved that the class of commutative BCK algebras is equationally definable. Commutative BCI and BE algebras were considered in [7] and [14, 2], respectively. K. Iséki [7] proved that any commutative BCI algebra is a BCK algebra. For BE algebras, this was shown in [14]. The property of commutativity for other generalizations of BCK algebras was investigated in [16].

As in the case of BCK algebras, we define:

Definition 2.3. An RML algebra $\mathcal{A} = (A, \to, 1)$ is called *commutative* if it satisfies (Com).

Lemma 2.4. *Let* $\mathcal{A} = (A, \to, 1)$ *be an algebra of type* $(2, 0)$*. Then:*

(i) (Com) + (M) *imply* (An),
(ii) (Com) + (K) *imply* (D),
(iii) (Com) + (Re) + (L) + (Ex) *imply* (BB).

Proof. The statements (i) and (ii) are proved in [16] (see Proposition 3.3).

(iii) Let $\mathcal{A}$ verify (Com), (Re), (L) and (Ex). By Lemma 2.1 (v), $\mathcal{A}$ verifies (K). Let $x, y, z \in A$. We have

$$\begin{aligned}
&(y \to z) \to [(z \to x) \to (y \to x)] \\
&\qquad \overset{\text{(Ex)}}{=} (y \to z) \to [y \to ((z \to x) \to x)] \\
&\qquad \overset{\text{(Com)}}{=} (y \to z) \to [y \to ((x \to z) \to z)] \\
&\qquad \overset{\text{(Ex)}}{=} (y \to z) \to [(x \to z) \to (y \to z)] \overset{\text{(K)}}{=} 1,
\end{aligned}$$

that is, (BB) holds in $\mathcal{A}$. □

3. Implicative BE algebras. The well-known implicative and positive implicative BCK algebras were introduced by K. Iséki and S. Tanaka [8].

Let $\mathcal{A} = (A, \to, 1)$ be an algebra of type $(2, 0)$. We first consider the following properties:

(Im) $(x \to y) \to x = x$,

(pi) $y \to (y \to x) = y \to x$,

(pimpl) $x \to (y \to z) = (x \to y) \to (x \to z)$.

Remark 3.1. Note that from Theorem 8 of [8] it follows that for BCK algebras, (pimpl) and (pi) are equivalent. By Theorem 9 of [8], in commutative BCK algebras, we have (Im) $\Longleftrightarrow$ (pi) $\Longleftrightarrow$ (pimpl).

Proposition 3.2. *Let* $\mathcal{A} = (A, \to, 1)$ *be an algebra of type* $(2, 0)$*. Then*

(i) (Re) + (Im) *imply* (M),
(ii) (M) + (Im) *imply* (L),
(iii) (Im) *implies* (pi),
(iv) (Re) + (M) + (pimpl) + (An) *imply* (Ex),
(v) (Re) + (Im) + (BB) + (An) *imply* (pimpl),
(vi) (Re) + (pimpl) *imply* (L),
(vii) (M) + (K) + (Com) + (pimpl) *imply* (Im).

Proof. (i)–(iii) follow from Proposition 3.5 of [15].

(iv) follows from Theorem 6.16 of [5].

(v) By above (i)–(iii), $\mathcal{A}$ satisfies (M), (L) and (pi). By Lemma 2.1 (ii) and (v), $\mathcal{A}$ also satisfies (Ex) and (K). Let $x, y, z \in A$. We have

$$\begin{aligned}
&(x \to (y \to z)) \to [(x \to y) \to (x \to z)] \\
&\overset{\text{(Ex)}}{=} (x \to y) \to [(x \to (y \to z)) \to (x \to z)] \\
&\overset{\text{(Ex)}}{=} (x \to y) \to [(y \to (x \to z)) \to (x \to z)] \\
&\overset{\text{(pi)}}{=} (x \to y) \to [(y \to (x \to z)) \to (x \to (x \to z))] \overset{\text{(BB)}}{=} 1.
\end{aligned}$$

Then

$$x \to (y \to z) \leq (x \to y) \to (x \to z). \tag{3.1}$$

On the other hand, from (K) we see that $y \to (x \to y) = 1$, and we obtain

$$\begin{aligned}
&[(x \to y) \to (x \to z)] \to (x \to (y \to z)) \\
&\overset{\text{(Ex)}}{=} [(x \to y) \to (x \to z)] \to (y \to (x \to z)) \\
&\overset{\text{(M)}}{=} [y \to (x \to y)] \to [((x \to y) \to (x \to z)) \to (y \to (x \to z))] \\
&\overset{\text{(BB)}}{=} 1.
\end{aligned}$$

Hence

$$(x \to y) \to (x \to z) \leq x \to (y \to z). \tag{3.2}$$

Applying (An), from (3.1) and (3.2), we get (pimpl).

(vi) follows from Proposition 6.4 (i) of [5].

(vii) By Lemma 2.4, $\mathcal{A}$ satisfies (An) and (D). Let $x, y, z \in A$. We have

$$\begin{aligned}
((x \to y) \to x) \to x &\overset{\text{(Com)}}{=} (x \to (x \to y)) \to (x \to y) \\
&\overset{\text{(pimpl)}}{=} x \to ((x \to y) \to y) \overset{\text{(D)}}{=} 1.
\end{aligned}$$

Therefore, $(x \to y) \to x \leq x$. On the other hand, from (K) we see that $x \leq (x \to y) \to x$. Applying (An), we get (Im). □

Lemma 3.3 ([15], Lemma 3.8). *Let* $\mathcal{A} = (A, \to, 1)$ *be an algebra of type* $(2, 0)$. *Then*

$$\text{(Re)} + \text{(Com)} + \text{(Im)} + \text{(BB)} \iff \text{(Re)} + \text{(Com)} + \text{(Im)} + \text{(Ex)}.$$

As in the case of BCK algebras, we now define:

Definition 3.4. An RML algebra $\mathcal{A}$ is called *implicative* if it satisfies (Im).

Example 3.5 ([15], Example 3.24). Consider the set $A = \{a, b, c, d, 1\}$ and the operation $\to$ given by the following table:

$\to$	a	b	c	d	1
a	1	b	b	d	1
b	a	1	a	a	1
c	1	1	1	1	1
d	a	1	1	1	1
1	a	b	c	d	1

We can observe that the properties (Im), (Re), (M) and (L) are satisfied. Hence, $(A, \to, 1)$ is an implicative RML algebra. It does not satisfy (An) for $(x, y) = (c, d)$; (Ex) for $(x, y, z) = (a, b, d)$; (Tr) for $(x, y, z) = (d, c, a)$; (**) for $(x, y, z) = (a, d, c)$.

Example 3.6. Let $A = \{a, b, c, 1\}$ and $\to$ be defined as follows:

$\to$	a	b	c	1
a	1	b	b	1
b	a	1	1	1
c	1	1	1	1
1	a	b	c	1

It is easy to check that the properties (Im), (Re), (M), (L) and (Ex) are satisfied; (An) is not satisfied for $x = b$, $y = c$; (**) is not satisfied for $x = a$, $y = b$, $z = c$. Therefore, $(A, \to, 1)$is an implicative BE algebra without (An) and (**).

Example 3.7. Consider the set $A = \{a, b, c, 1\}$ and the operation $\to$ given by the following table:

$\to$	a	b	c	1
a	1	b	c	1
b	a	1	1	1
c	a	1	1	1
1	a	b	c	1

The algebra $\mathcal{A} = (A, \to, 1)$ satisfies properties (Im), (Re), (M), (L), (Ex), (**). It does not satisfy (An) for $(x, y) = (b, c)$. Hence, $\mathcal{A}$ is an implicative BE** algebra without (An).

Example 3.8 ([15], Example 3.34). Let $A = \{a, b, c, 1\}$ and $\to$ be defined as follows:

$\to$	a	b	c	1
a	1	b	b	1
b	a	1	a	1
c	1	1	1	1
1	a	b	c	1

We can observe that the properties (Im), (An), (Re), (M), (L), (BB) and (Ex) are satisfied. Therefore, $\mathcal{A} = (A, \to, 1)$ is an implicative BCK algebra.

Denote by **im-RML** the class of all implicative RML algebras. Similarly, if **X** is a subclass of the class **RML**, then **im-X** denotes the class of all implicative algebras belonging to **X**. Examples 3.5–3.8 show that

$$\textbf{im-BCK} \subseteq \textbf{im-aBE**} \subset \textbf{im-BE**} \subset \textbf{im-BE} \subset \textbf{im-RML}.$$

Proposition 3.9 ([15], Proposition 3.14). *Let $\mathcal{A} = (A, \to, 1)$ be an algebra verifying* (Re), (D), (**) *and* (Im). *Then*

$$y \leq x \Longrightarrow (x \to y) \to y \leqslant x \tag{3.3}$$

for all $x, y \in A$.

Theorem 3.10. *If $\mathcal{A} = (A, \to, 1)$ is an implicative BE** algebra, then $\mathcal{A}$ is commutative if and only if it satisfies* (An).

Proof. Let $\mathcal{A}$ be an implicative BE** algebra. If $\mathcal{A}$ is commutative, then $\mathcal{A}$ satisfies (An) by Lemma 2.4 (i). Conversely, suppose that (An) holds in $\mathcal{A}$. From Lemma 2.1 (iii)–(v) it follows that $\mathcal{A}$ satisfies (Tr), (D) and (K). By Proposition 3.9, $\mathcal{A}$ satisfies (3.3). Let $x, y \in A$. From (K) we have $x \leq (y \to x) \to x$. Applying (**) twice, we obtain

$$(x \to y) \to y \leq (((y \to x) \to x) \to y) \to y. \tag{3.4}$$

By (D), $y \leq (y \to x) \to x$, and hence, using (3.3), we get

$$(((y \to x) \to x) \to y) \to y \leq (y \to x) \to x. \tag{3.5}$$

Since $\mathcal{A}$ satisfies (Tr), from inequalities (3.4) and (3.5) we have

$$(x \to y) \to y \leq (y \to x) \to x.$$

Hence, by (An), we obtain (Com). □

Proposition 3.11. *In aBE** algebras, we have*

$$\text{(Im)} \iff \text{(Com)} + \text{(pimpl)}.$$

Proof. Let $\mathcal{A} = (A, \to, 1)$ be an aBE** algebra. Assume that $\mathcal{A}$ satisfies (Im). From Theorem 3.10 we conclude that (Com) holds in $\mathcal{A}$. Applying Lemma 2.4 (iii), we see that $\mathcal{A}$ satisfies (BB). By Proposition 3.2 (v), (pimpl) holds in $\mathcal{A}$.

Conversely, suppose that $\mathcal{A}$ satisfies (Com) and (pimpl). By Lemma 2.1 (v), (Re) + (L) + (Ex) imply (K). Then, from Proposition 3.2 (vii) we deduce that $\mathcal{A}$ satisfies (Im). □

Remark 3.12. Since every BCK algebra is an aBE** algebra, from Proposition 3.11 we obtain Theorem 9 of [8].

Corollary 3.13. *Any implicative aBE** algebra is a commutative BCK algebra.*

Proof. Let $\mathcal{A}$ be an implicative aBE** algebra. By Proposition 3.11, $\mathcal{A}$ is commutative. From Lemma 3.3 we see that $\mathcal{A}$ verifies (BB). Hence $\mathcal{A}$ is a BCK algebra. □

Remark 3.14. Note that from Corollary 3.13 we deduce that **im-BCK** = **im-aBE****.

By Proposition 3.2 (i) and (ii), we obtain:

Proposition 3.15. *An algebra* $\mathcal{A} = (A, \to, 1)$ *is an implicative BE algebra if and only if it satisfies the equations* (Re), (Ex) *and* (Im)*.*

Lemma 3.16. *If* $\mathcal{A} = (A, \to, 1)$ *is an implicative BE** algebra, then it satisfies the following condition:*

(W) $(((y \to x) \to x) \to z) \to (y \to z) = 1.$

Proof. By Lemma 2.1 (iv), $\mathcal{A}$ satisfies (D). Let $x, y, z \in A$. From (D) it follows that $y \leq (y \to x) \to x$. Applying (**), we obtain $((y \to x) \to x) \to z \leq y \to z$. Hence we have (W). □

The next theorem shows that the class of implicative BE** algebras is a variety.

Theorem 3.17. *An algebra* $\mathcal{A} = (A, \to, 1)$ *is an implicative BE** algebra if and only if it satisfies the equations* (Re)*,* (Ex), (Im) *and* (W)*.*

Proof. If $\mathcal{A}$ is an implicative BE** algebra, then it satisfies (Re), (Ex), (Im) and, by Lemma 3.16, (W).

Conversely, let $\mathcal{A}$ satisfy (Re), (Ex), (Im) and (W). Obviously, $\mathcal{A}$ is an implicative BE algebra. To prove (**), let $x, y, z \in A$ and $y \leq x$. By (M) and (W), $x \to z \leq y \to z$, that is, $\mathcal{A}$ satisfies (**). Thus $\mathcal{A}$ is an implicative BE** algebra. □

Recall the definition of Tarski algebras. A *Tarski algebra* is an algebra $\mathcal{A} = (A, \to, 1)$ of type $(2, 0)$ satisfying the following axioms ([9]): for all $x, y, z \in A$,

(T1) $1 \to x = x$,
(T2) $x \to x = 1$,
(T3) $(x \to y) \to y = (y \to x) \to x$,
(T4) $x \to (y \to z) = (x \to y) \to (x \to z)$.

Note that (T1) is (M), (T2) is (Re), (T3) is (Com) and (T4) is (pimpl).

Theorem 3.18. *Let* $\mathcal{A} = (A, \to, 1)$ *be an algebra of type* $(2, 0)$*. The following conditions are equivalent:*

(i) $\mathcal{A}$ *is an implicative aBE** algebra,*
(ii) $\mathcal{A}$ *satisfies* (Re) + (Com) + (Im) + (Ex)*,*
(iii) $\mathcal{A}$ *satisfies* (Re) + (Com) + (Im) + (BB)*,*
(iv) $\mathcal{A}$ *is a Tarski algebra.*

Proof. (i) implies (ii) and (ii) implies (iii) by Proposition 3.11 and Lemma 3.3, respectively. Now let $\mathcal{A}$ satisfy (Re), (Com), (Im) and (BB). By Proposition 3.2 (i), (Re) + (Im) imply (M), thus, (T1) = (M) holds. By Lemma 2.4 (i), (Com) + (M) imply (An); then, by Proposition 3.2 (v), (Re) + (Im) + (BB) + (An) imply (pimpl); thus, (T4) = (pimpl) holds. Consequently, $\mathcal{A}$ is a Tarski algebra.

Finally, let $\mathcal{A}$ satisfy (M), (Re), (Com) and (pimpl). By Lemma 2.4 (i), (Com) + (M) imply (An); thus, (An) holds. By Proposition 3.2 (vi), (Re) + (pimpl) imply (L); thus, (L) holds. By Proposition 3.2 (iv), (Re) + (M) + (pimpl) + (An) imply (Ex); thus, (Ex) holds.

By Lemma 2.4 (iii), (Com) + (Re) + (L) + (Ex) imply (BB); then, by Lemma 2.1 (i), (M) + (BB) imply (**); thus, (**) holds. By Lemma 2.1 (v), (Re) + (L) + (Ex) imply (K); then, by Proposition 3.2 (vii), (M) + (K) + (Com) + (pimpl) imply (Im); thus (Im) holds.

Consequently, $\mathcal{A}$ is an implicative aBE** algebra, that is, (i) holds. □

Corollary 3.19. *Let $\mathcal{A}$ be an implicative aBE algebra satisfying* (Tr)*. Then $\mathcal{A}$ is a Tarski algebra.*

Proof. By Lemma 2.1 (vi), $\mathcal{A}$ satisfies (**). Then $\mathcal{A}$ is an implicative aBE** algebra. From Theorem 3.18 we see that $\mathcal{A}$ is a Tarski algebra. □

Remark 3.20. If $\mathcal{A}$ is an algebra satisfying (M) and (*), then it also satisfies (Tr) by Lemma 2.1 (vii). Therefore, implicative aBE algebras with (*) are Tarski algebras.

Open Problem 3.21. Is there an implicative aBE algebra not satisfying (Tr)?

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Andrzej Walendziak
Faculty of Exact and Natural Scieces
Siedlce University
3 Maja 54, 08-110 Siedlce
Poland
e-mail: `walent@interia.pl`

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