doi: 10.17951/a.2022.76.2.55-58

ANNALES
UNIVERSITATIS MARIAE CURIE-SKŁODOWSKA
LUBLIN – POLONIA

VOL. LXXVI, NO. 2, 2022 SECTIO A 55–58

DENIS ZELENT

Transitivity of implicative aBE algebras

Abstract. We prove that every implicative aBE algebra satisfies the transitivity property. This means that every implicative aBE algebra is a Tarski algebra, and thus is also a commutative BCK algebra.

BCK algebras were introduced by Y. Imai and K. Iséki in [1], and were later generalized by H. S. Kim and Y. H. Kim in the definition of BE algebras [5]. In 2016, A. Iorgulescu defined numerous new generalizations, such as aBE, BE** and aBE** algebras [2]. The definition of implicativity for a BCK algebra was introduced by K. Iséki and S. Tanaka in [3], and was studied for various generalizations of BCK algebras by A. Walendziak in [6, 7]. In [7], the author posed an open problem: whether there exists an implicative aBE algebra that does not satisfy the transitivity property (Open Problem 3.21). In this paper, we prove that this property is satisfied by every implicative aBE algebra, which gives a negative answer to the posed question. This result gives us immediately a few implications about implicative aBE algebras. Firstly, every implicative aBE algebra is a Tarski algebra [7, Corollary 3.19]. This means that every implicative aBE algebra is an implicative aBE** algebra [7, Theorem 3.18], and thus it is also a commutative BCK algebra [7, Corollary 3.13].

2010 *Mathematics Subject Classification.* 03G25.

Key words and phrases. BE algebra, BCK algebra, Tarski algebra, implicativity, transitivity.

We start by introducing the definitions of implicative aBE algebra and transitivity property.

Definition 1. An *implicative aBE algebra* is an algebra of the form (X, →, 1), where X is the non-empty set with a designated element 1 and the arrow as a binary operation satisfying the following axioms:

$$1 \to x = x \tag{1}$$

$$x \to 1 = 1 \tag{2}$$

$$x \to x = 1 \tag{3}$$

$$x \to (y \to z) = y \to (x \to z) \tag{4}$$

$$x \to y = y \to x = 1 \implies x = y \tag{5}$$

$$(x \to y) \to x = x. \tag{6}$$

Definition 2. We say that an (implicative aBE) algebra (X, →, 1) satisfies the *transitivity property* if and only if it satisfies

$$x \to y = y \to z = 1 \implies x \to z = 1. \tag{7}$$

To prove that every implicative aBE algebra satisfies the transitivity property, we will first state and prove some relevant lemmas.

Lemma 1. *If $(X, \to, 1)$ is an implicative aBE algebra, then it satisfies*

$$x = y \to x \text{ or } (y \to x) \to x \neq 1 \tag{8}$$

$$x = (x \to y) \to y \text{ or } ((x \to y) \to y) \to x \neq 1. \tag{9}$$

Proof. Firstly, we note that (5) implies

$$x = y \text{ or } x \to y \neq 1 \text{ or } y \to x \neq 1.$$

To prove (8) we use that fact for x and $y \to x$:

$$x = y \to x \text{ or } x \to (y \to x) \neq 1 \text{ or } (y \to x) \to x \neq 1.$$

Noticing that $x \to (y \to x) \overset{(4)}{=} y \to (x \to x) \overset{(3)}{=} y \to 1 \overset{(2)}{=} 1$, we conclude the proof of the first condition. Proof of (9) is analogous but with x and $(x \to y) \to y$, and the following observation: $x \to ((x \to y) \to y) \overset{(4)}{=} (x \to y) \to (x \to y) \overset{(3)}{=} 1$. □

Lemma 2. *If $(X, \to, 1)$ is an implicative aBE algebra, then it satisfies*

$$x \to y = (z \to x) \to (x \to y). \tag{10}$$

Proof. $x \to y \overset{(6)}{=} ((x \to y) \to (z \to x)) \to (x \to y) \overset{(4)}{=} (z \to ((x \to y) \to x)) \to (x \to y) \overset{(6)}{=} (z \to x) \to (x \to y)$. □

Lemma 3. *If* $(X, \to, 1)$ *is an implicative aBE algebra, then it satisfies*

$$((y \to x) \to z) \to t = ((y \to x) \to z) \to ((x \to z) \to t). \tag{11}$$

Proof. Set $a := ((y \to x) \to z)$. We have $x \to a = x \to ((y \to x) \to z) \overset{(4)}{=} (y \to x) \to (x \to z) \overset{(10)}{=} x \to z$. Then $a \to t \overset{(10)}{=} (x \to a) \to (a \to t) \overset{(4)}{=} a \to ((x \to a) \to t) = a \to ((x \to z) \to t)$. □

Lemma 4. *If* $(X, \to, 1)$ *is an implicative aBE algebra, then it satisfies*

$$(((x \to y) \to z) \to y) \to (x \to y) = 1. \tag{12}$$

Proof. Set $a := (x \to y) \to z$. We get $(a \to y) \to (x \to y) \overset{(6)}{=} (a \to y) \to (a \to (x \to y)) \overset{(4)}{=} (a \to y) \to (x \to (a \to y)) \overset{(4)}{=} x \to ((a \to y) \to (a \to y)) \overset{(3)}{=} x \to 1 \overset{(2)}{=} 1$. □

Lemma 5. *If* $(X, \to, 1)$ *is an implicative aBE algebra, then it satisfies*

$$(((((x \to y) \to y) \to x) \to y) \to y) = 1. \tag{13}$$

Proof. $(((((x \to y) \to y) \to x) \to y) \to y) \overset{(11)}{=} ((((x \to y) \to y) \to x) \to y) \to ((x \to y) \to y) \overset{(12)}{=} 1$. □

Lemma 6. *If* $(X, \to, 1)$ *is an implicative aBE algebra, then it satisfies*

$$y = (((x \to y) \to y) \to x) \to y. \tag{14}$$

Proof. Follows directly from (8) and (13). □

Lemma 7. *If* $(X, \to, 1)$ *is an implicative aBE algebra, then it satisfies*

$$x \to y = x \to (((x \to y) \to z) \to y). \tag{15}$$

Proof. $x \to y \overset{(6)}{=} ((x \to y) \to z) \to (x \to y) \overset{(4)}{=} x \to (((x \to y) \to z) \to y)$. □

Lemma 8. *If* $(X, \to, 1)$ *is an implicative aBE algebra, then it satisfies*

$$((x \to y) \to y) \to x = ((x \to y) \to y) \to (y \to x). \tag{16}$$

Proof. $((x \to y) \to y) \to x \overset{(15)}{=} ((x \to y) \to y) \to ((((x \to y) \to y) \to x) \to y) \to x) \overset{(14)}{=} ((x \to y) \to y) \to (y \to x)$. □

Lemma 9. *If* $(X, \to, 1)$ *is an implicative aBE algebra, then it satisfies*

$$y \to x = ((x \to y) \to y) \to x. \tag{17}$$

Proof. Follows directly from (16) and (10). □

Lemma 10. *If* $(X, \to, 1)$ *is an implicative aBE algebra, then it satisfies*

$$(x \to y) \to y = x \text{ or } y \to x \neq 1. \tag{18}$$

Proof. This lemma is just a combination of (9) together with (17). □

Theorem 1 (Transitivity of implicative aBE algebras). *Every implicative aBE algebra satisfies the transitivity property.*

Proof. Assume by contradiction that there exists an algebra for which this condition does not hold, i.e., there exist a, b, c such that $a \to b = 1$, $b \to c = 1$, but $a \to c \neq 1$. Then from Lemma 10, since $b \to c = 1$, we must have $c = (c \to b) \to b$. But multiplying both sides from the left by a, we obtain

$$a \to c = a \to ((c \to b) \to b) \overset{(4)}{=} (c \to b) \to (a \to b) = (c \to b) \to 1 \overset{(2)}{=} 1,$$

which is a contradiction with $a \to c \neq 1$. □

References

[1] Imai, Y., Iséki, K., *On Axiom Systems of Propositional Calculi. XIV*, Proc. Japan Acad. **42** (1966), 19–22. https://doi.org/10.3792/pja/1195522169.

[2] Iorgulescu, A., *New generalizations of BCI, BCK and Hilbert algebras – Part I*, J. Mult.-Valued Logic Soft Comput. **27** (2016), 353–406.

[3] Iséki, K., Tanaka, S., *An introduction to the theory of BCK-algebras*, Math. Japon. **23** (1) (1978/79), 1–26.

[4] Jun, Y. B., Kang, M. S., *Fuzzifications of generalized Tarski filters in Tarski algebras*, Comp. Math. Appl. **61** (2011), 1–7.

[5] Kim, H. S., Kim, Y. H., *On BE-algebras*, Sci. Math. Jpn. **66** (2007), 113–128

[6] Walendziak, A., *The implicative property for some generalizations of BCK algebras*, J. Mult.-Valued Logic Soft Comput. **31** (2018), 591–611.

[7] Walendziak, A., *On implicative BE algebras*, Ann. Univ. Mariae Curie-Skłodowska Sect. A **76** (2) (2022), 45–54.

Denis Zelent
Department of Mathematical Sciences
Norwegian University of Science and Technology
7491 Trondheim
Norway
e-mail: `zelden99@zohomail.eu`

Received December 9, 2022