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Tomasz Łukasz Żynda

Three types of reproducing kernel Hilbert spaces of polynomials

Abstract. In this paper we will investigate reproducing kernel Hilbert spaces of polynomials of degree at most n with three different inner products: given by an integral with a weight, given by the sum of products of values of a polynomial at $n+1$ points and given by the sum of products of coefficients of the same power. In the first case we will show that the reproducing kernel depends continuously on deformation of an inner product in a precisely defined sense. In the second and third case we will give a formula for the reproducing kernel.

1. Preliminaries. The problem of orthogonal polynomials for an integral inner product with continuous weights of integration and for an inner product defined as a product of values of polynomials at $n+1$ given points was widely investigated (see e.g. [1, 2, 6]). The problem of different reproducing kernels on a Hilbert space of polynomials of degree at most n, however, was not considered. The so-called "discrete polynomial kernel" (see [4]) uses the concept of functions defined on countable domains and therefore is a different idea.

Let $\mathcal{H}$ be a Hilbert space of functions defined on U with values in $\mathbb{K}$, where $\mathbb{K}$ is the field of real or complex numbers. Let $\langle -|\cdot\rangle$ be its inner product (i.e. we assume complex conjugation in the first variable) and $\|\cdot\|$ be its norm.

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Reproducing kernel of $\mathcal{H}$ is a function (if exists) $K : U \times U \to \mathbb{K}$ such that $\overline{K(z,\cdot)} \in \mathcal{H}$ and

$$\left\langle \overline{K(z,\cdot)} \middle| f(\cdot) \right\rangle \left\langle \overline{K(z,\cdot)} \middle| f(\cdot) \right\rangle = f(z)$$

for any $f \in \mathcal{H}$ and any $z \in U$ (**reproducing property**).

Functionals of point evaluation are functionals

$$E_z : U \ni z \mapsto f(z) \in K.$$

It is well known that

Proposition 1.1. *$\mathcal{H}$ is equipped with a reproducing kernel if and only if all functionals of point evaluation are continuous.*

Proof. Indeed, if functionals of point evaluation are continuous, then by the Riesz representation theorem for any E_z there exists $\overline{e_z} \in \mathcal{H}$ such that

$$\langle \overline{e_z} | f \rangle = f(z)$$

for any $f \in \mathcal{H}$ and any $z \in U$. The function $K(z,w) := e_z(w)$ is the reproducing kernel of $\mathcal{H}$.

On the other hand, if K is the reproducing kernel of $\mathcal{H}$, then

$$\begin{aligned} |f(z)| &= \left| \left\langle \overline{K(z,\cdot)} \middle| f \right\rangle \right| \leq \left\| \overline{K(z,\cdot)} \right\| \cdot \|f\| \\ &= \sqrt{\left\langle \overline{K(z,\cdot)} \middle| \overline{K(z,\cdot)} \right\rangle} \cdot \|f\| = \sqrt{K(z,z)} \|f\|, \end{aligned}$$

by the reproducing property, Hölder's inequality and by the reproducing property again, so functionals of point evaluation are continuous. □

In particular all finite-dimensional Hilbert spaces of functions are reproducing kernel Hilbert spaces. Indeed, any linear operator between two finite-dimensional Banach spaces is continuous, so it applies also to any functional of point evaluation. There are, however, infinite-dimensional Hilbert spaces of functions which are not equipped with a reproducing kernel. Some examples can be found in [5] or [7].

Another consequence of the Riesz representation theorem is the fact that the reproducing kernel of $\mathcal{H}$, if it exists, is given by the formula

$$K(z,w) := \sum_{i \in I} \overline{\varphi_i(z)} \varphi_i(w),$$

where

$$\{\varphi_i\}_{i \in I}$$

is an orthonormal basis of $\mathcal{H}$.

It is well known that for any reproducing kernel we have

$$K(z,z) \geq 0$$

for any $z \in U$. Indeed, by the reproducing property and the fact that norm of any element is non-negative, we have

$$\overline{K(z,z)} = \left\langle K(z,\cdot) \middle| \overline{K(z,\cdot)} \right\rangle \geq 0.$$

This fact will be used frequently in what follows without further recalling.

Theorem 1.1. *Let $\mathcal{H}$ be equipped with the reproducing kernel. If $K(z,z) \neq 0$, then*

$$k_z(\cdot) := \frac{\overline{K(z,\cdot)}}{K(z,z)}$$

is the only element of $\mathcal{H}$ with the following properties:

(i) $k_z(z) = 1$;

(ii) if $m_z \in \mathcal{H}$, $m_z(z) = 1$ and $\|m_z\| \leq \|k_z\|$, then $m_z = k_z$. Moreover,

$$\|k_z\| = \frac{1}{\sqrt{K(z,z)}}.$$

Proof. By the reproducing property and the Cauchy–Schwarz inequality, for any $f \in \mathcal{H}$, $z \in U$ we have

$$|f(z)| = |\langle K(z,\cdot)|f\rangle| \leq \|K(z,\cdot)\| \cdot \|f\|,$$

i.e.

$$|f(z)| \leq \sqrt{K(z,z)}\|f\|. \tag{1}$$

Moreover, $\sqrt{K(z,z)}$ is the smallest possible constant for which inequality (1) holds. Indeed, let $E_z : \mathcal{H} \ni f \mapsto f(z) \in \mathbb{C}$ be the functional of point evaluation. By the Riesz correspondence theorem,

$$\|E_z\|^* = \left\|\overline{K(z,\cdot)}\right\|_\mu,$$

but

$$\left\|\overline{K(z,\cdot)}\right\| = \sqrt{K(z,z)}.$$

Clearly, by definition, $\left\|\overline{K_\mu(z,\cdot)}\right\|_\mu$ is the smallest constant for which inequality (1) holds.

Now we have

$$\frac{1}{\sqrt{K(z,z)}} \leq \frac{\|f\|}{|f(z)|} = \left\|\frac{f}{f(z)}\right\|.$$

But

$$\left\|\frac{\overline{K(z,w)}}{K(z,z)}\right\| = \frac{1}{\sqrt{K(z,z)}},$$

by the reproducing property. To end the proof we need only to show that if $\|m_z\|_\mu = \|k_z\|_\mu$, then $m_z = k_z$. Note that for $f_z := \frac{1}{2}(m_z + k_z)$ we have $f_z(z) = 1$ and

$$\|f_z\| = \left\|\frac{1}{2}(m_z + k_z)\right\| \leq \frac{1}{2}(\|m_z\| + \|k_z\|) = \|k_z\|.$$

On the other hand, we have shown above that

$$\|f_z\| \geq \|k_z\|,$$

so $\|f_z\| = \|k_z\|$. Since in our case the triangle inequality is in fact the equality and each Hilbert space is strictly convex, there exists $\alpha \in \mathbb{C}$ such that $m_z = \alpha k_z$. Thus

$$\left\| \frac{1}{2}(m_z + k_z) \right\| = \frac{1}{2}(\alpha + 1)\|k_z\|.$$

Since

$$\|f_z\| = \|k_z\|,$$

we see that $\alpha = 1$ and in conclusion $m_z = k_z$. □

Now we investigate the case in which $f(z) = 0$ for each $f \in \mathcal{H}$ and some $z \in U$.

Theorem 1.2. *The following conditions are equivalent for a point $z \in U$:*
(i) $f(z) = 0$ for any $f \in \mathcal{H}$;
(ii) $K(z, z) = 0$;
(iii) $K(z, \cdot) \equiv 0$.

Proof. (i) $\Rightarrow$ (ii) If for some $z \in U$ we have $f(z) = 0$ for any $f \in \mathcal{H}$, then in particular for $g(\cdot) = \overline{K(z, \cdot)}$ we have $g(z) = 0$.

(ii) $\Rightarrow$ (iii) Because

$$\|K(z, \cdot)\| = \sqrt{K(z, z)} = 0,$$

we conclude that $K_\mu(z, \cdot) \equiv 0$.

(iii) $\Rightarrow$ (i) By the reproducing property, for any $f \in \mathcal{H}$ we have

$$f(z) = \left\langle \overline{K(z, \cdot)} \middle| f \right\rangle = \langle 0 | f \rangle = 0. \qquad \square$$

Note that since for any $z \in \mathbb{R}$ there exist polynomials f such that $f(z) \neq 0$, if $\mathcal{H}$ is the space of polynomials of degree at most n and K is its reproducing kernel, then $K(z, z) > 0$, no matter what the inner product is. That property will be used in what follows without further recalling.

Proposition 1.2. *Let $\mathcal{H}$ be a reproducing kernel Hilbert space and the function equal to 1 everywhere be its element. Then*

$$K(z, z) \geq \frac{1}{\|1\|^2}.$$

Proof. By the reproducing property, Cauchy's inequality and by the reproducing property again,

$$1 = |\langle K(z, \cdot) | 1 \rangle| \leq \|K(z, \cdot)\| \cdot \|1\| = \sqrt{K(z, z)}\|1\|. \qquad \square$$

2. RKHS of polynomials with an integral inner product. Let $\mu : \mathbb{R} \to \mathbb{R}$ be a measurable function such that:

(i) $\mu(w) \geq 0$ a.e.;

(ii) μ is bounded from above;

(iii) there exists a set X with the Lebesgue measure greater than 0 such that $\mu(w) > 0$ for any $w \in X$;

(iv) μ has the compact support.

Such a function will be called a **weight**.

In this section we will consider the Hilbert space $P_n(\mathbb{R})$ of **all** polynomials of degree at most n equipped with the inner product

$$\langle f|g\rangle_\mu := \int_{\mathbb{R}} f(w)g(w)\mu(w)\mathrm{d}w.$$

We will use the symbol

$$\|f\|_\mu$$

for the norm generated by this inner product.

For example, for the weight μ being the indicator function of the interval $[0,1]$,

$$\left\{1, 2\sqrt{3}x - \sqrt{3}\right\}$$

is an orthonormal basis of $P_1(\mathbb{R})$ equipped with the inner product $\langle\cdot|\cdot\rangle_\mu$ and

$$K(z,w) = 12zw - 6z - 6w + 4$$

is the reproducing kernel of this space.

We could also consider the case of weights not bounded from above, but in such a situation some polynomials may not be elements of our space. Note also that in the case of weighted Bergman and Szegö spaces (see [5, 7]), if a weight "goes to zero" at some point too quickly, then there is no reproducing kernel of the corresponding weighted space. In our case any weight is admissible, i.e. for any weight there exists the reproducing kernel of the corresponding weighted space.

Theorem 2.1. *Let $W(\mathbb{R})$ be the set of weights on $\mathbb{R}$ equipped with the topology of almost everywhere convergence. The map*

$$W(\mathbb{R}) \ni \mu \mapsto K_\mu(z,\cdot) \in P_n(\mathbb{R})$$

is continuous.

Note that since all topologies generated by norms on a finite-dimensional space are the same, we can conclude that reproducing kernels converge in $\sup_{x\in[a,b]} |\cdot|$ or $L^p([a,b])$ metric.

Proof. First let us note that the map

$$\Phi : P_n(\mathbb{R}) \times P_n(\mathbb{R}) \times W(\mathbb{R}) \ni (f,g,\mu) \mapsto \langle f|g\rangle_\mu \in \mathbb{R}$$

is continuous. Indeed, if $f_n \to f$, $g_n \to g$ in norm, then in particular these sequences converge pointwise and if also $\mu_n \to \mu$ almost everywhere, then by Lebesgue's dominated convergence theorem

$$\int_{\mathbb{R}} f_n g_n \mu_n \mathrm{d}w \to \int_{\mathbb{R}} f g \mu \mathrm{d}w.$$

(Remember that all weights are bounded from above and have compact supports.)

The orthogonal projection of $f \neq 0 \in P_n(\mathbb{R})$ onto $g \in P_n(\mathbb{R})$ with respect to the inner product $\langle \cdot | \cdot \rangle_\mu$ is given by

$$P(f, g, \mu) := \frac{\langle f|g\rangle_\mu}{\langle g|g\rangle_\mu} g = \frac{\Phi(f, g, \mu)}{\Phi(g, g, \mu)} g,$$

so

$$P : (f, g, \mu) \mapsto P(f, g, \mu) \in P_n(\mathbb{R})$$

is continuous, because scalar multiplication is continuous in any normed space.

Now let $\{\Psi_1, \ldots, \Psi_n\}$ be an arbitrary orthonormal basis for $P_n(\mathbb{R})$ equipped with the inner product $\langle \cdot | \cdot \rangle_{\mu_0}$ for some weight μ_0. In this case

$$\phi_1 := \frac{\Psi_1}{\|\Psi_1\|_\mu} = \frac{\Psi_1}{\sqrt{\Phi(\Psi_1, \Psi_1, \mu)}},$$

$$\phi_k := \frac{\Psi_k - \sum_{i=1}^{k-1} P(\Psi_k, \phi_i, \mu)}{\|\Psi_k - \sum_{i=1}^{k-1} P(\Psi_k, \phi_i, \mu)\|_\mu}$$

is an orthonormal basis for $P_n(\mathbb{R})$ equipped with the inner product $\langle \cdot | \cdot \rangle_\mu$. Each element of this basis depends continuously on deformation of an inner product. Indeed, for

$$\phi_1 : W(\mathbb{R}) \ni \mu \mapsto \phi_1(\mu) \in P_n(\mathbb{R})$$

we have

$$\phi_1(\mu) := \frac{\Psi_1}{\|\Psi_1\|_\mu} = \frac{\Psi_1}{\sqrt{\Phi(\Psi_1, \Psi_1, \mu)}},$$

so $\phi_1(\mu)$ is continuous, because scalar multiplication in any normed space is continuous.

Now let $k > 1$. Then

$$\phi_k : W(\mathbb{R}) \ni \mu \mapsto \phi_k(\mu) \in P_n(\mathbb{R})$$

is a continuous function. Indeed, by induction if ϕ_{k-1} is continuous, then also the value of

$$Q(\mu) := \Phi\left(\Psi_k - \sum_{i=1}^{k-1} P(\Psi_i, \phi_i, \mu), \Psi_k - \sum_{i=1}^{k-1} P(\Psi_i, \phi_i, \mu), \mu\right)$$

depends continuously on μ as a composition and difference of continuous functions. Therefore also

$$\phi_k(\mu) = \frac{\Psi_k - \sum_{i=1}^{k-1} P(\Psi_i, \phi_i, \mu)}{Q(\mu)}$$

is continuous, because addition and scalar multiplication are continuous in any normed space.

Finally

$$K_\mu(z, \cdot) := \sum_{i=1}^{n} \phi_i(\mu)(z)\phi_i(\mu)(\cdot)$$

depends continuously on μ, because addition and scalar multiplication are continuous in any normed space. $\square$

Note that for any $z \in \mathbb{R}$ there exists a polynomial f of degree at most n such that $f(z) \neq 0$. Therefore by Theorem 1.2, for any weight μ it is always true that $K_\mu(z, z) > 0$.

Theorem 2.2. *Let μ_1, μ_2 be weights on $\mathbb{R}$, such that $\mu_1 \leq \mu_2$ a.e. Then for any $z \in \mathbb{R}$ we have*

$$K_{\mu_2}(z, z) \leq K_{\mu_1}(z, z).$$

Proof. First let us recall that $K_{\mu_1}(z, z)$ and $K_{\mu_2}(z, z)$ are greater than 0. By Theorem 1.1, it is true that

$$\frac{1}{K_{\mu_1}(z, z)} = \int_{\mathbb{R}} \left| \frac{K_{\mu_1}(z, w)}{K_{\mu_1}(z, z)} \right|^2 \mu_1(w)\mathrm{d}w \leq \int_{\mathbb{R}} \left| \frac{K_{\mu_2}(z, w)}{K_{\mu_2}(z, z)} \right|^2 \mu_1(w)\mathrm{d}w.$$

Since $\mu_1 \leq \mu_2$,

$$\int_{\mathbb{R}} \left| \frac{K_{\mu_2}(z, w)}{K_{\mu_2}(z, z)} \right|^2 \mu_1(w)\mathrm{d}w \leq \int_{\mathbb{R}} \left| \frac{K_{\mu_2}(z, w)}{K_{\mu_2}(z, z)} \right|^2 \mu_2(w)\mathrm{d}w.$$

Because

$$\int_{\mathbb{R}} \left| \frac{K_{\mu_2}(z, w)}{K_{\mu_2}(z, z)} \right|^2 \mu_2(w)\mathrm{d}w = \frac{1}{K_{\mu_2}(z, z)},$$

in conclusion we have

$$\frac{1}{K_{\mu_1}(z, z)} \leq \frac{1}{K_{\mu_2}(z, z)},$$

which ends the proof. $\square$

Theorem 2.3. *Let K_μ be the reproducing kernel of $P_n(\mathbb{R})$ equipped with the inner product $\langle \cdot | \cdot \rangle_\mu$. Then for any $z \in \mathbb{R}$ we have*

$$K_\mu(z, z) \geq \frac{1}{\int_{\mathbb{R}} \mu(w)\mathrm{d}w}.$$

Proof. It is just a simple consequence of Proposition 1.2. $\square$

3. Inner product as a sum of products of values in n points. It is well known that values at any n points define a polynomial of degree at most $n-1$ uniquely. Moreover, a non-zero polynomial of degree at most $n-1$ can have no more than $n-1$ zeroes. Therefore the following formula

$$\langle f|g\rangle_{x_1,x_2,\ldots,x_n} := \sum_{i=1}^{n} f(x_i)g(x_i), \tag{2}$$

for any pairwise different $x_1, x_2, \ldots, x_n \in \mathbb{R}$ is an inner product on $P_{n-1}(\mathbb{R})$. The norm generated by this inner product will be denoted by $\|\cdot\|_{x_1,x_2,\ldots,x_n}$.

Note that although only values at points $x_1, x_2, \ldots, x_n$ matter for polynomials, the reproducing kernel is a polynomial of two variables defined for any $(z,w) \in \mathbb{R}^2$. In particular we can "reproduce" values of polynomials at any point $z \in \mathbb{R}$, not only at points $z \in \{x_1, x_2, \ldots, x_n\}$.

For example,

$$\varphi_1(w) := \frac{\sqrt{2}}{2},$$
$$\varphi_2(w) := \frac{\sqrt{2}}{|x_1 - x_2|}w - \frac{\sqrt{2}}{2}\frac{(x_1+x_2)}{|x_1-x_2|}$$

is an orthonormal basis for $P_1(\mathbb{R})$ equipped with $\langle\cdot|\cdot\rangle_{x_1,x_2}$ and therefore the reproducing kernel of $P_1(\mathbb{R})$ is equal to

$$\begin{aligned} K_{x_1,x_2}(z,w) &= \sum_{i=1}^{2} \varphi_i(z)\varphi_i(w) \\ &= \frac{1}{2} + \frac{1}{|x_1-x_2|^2}\left(2zw - (x_1+x_2)(z+w) + \frac{1}{2}(x_1+x_2)^2\right). \end{aligned}$$

In fact we can prove

Theorem 3.1. *The reproducing kernel of $P_{n-1}(\mathbb{R})$ equipped with the inner product (2) is given by*

$$K_{x_1,x_2,\ldots,x_n}(z,w) = \sum_{i=1}^{n} \prod_{1\le j\le n,\, j\ne i} \frac{(z-x_j)(w-x_j)}{(x_i-x_j)^2}.$$

Before we proceed, we will show some lemmas. We will also simplify notation $\langle\cdot|\cdot\rangle_{x_1,x_2,\ldots x_n}$ to $\langle\cdot|\cdot\rangle$ and $K_{x_1,x_2,\ldots,x_n}$ to K when it is not misleading.

Lemma 3.1. *Let $z, w \in \{x_1, x_2, \ldots, x_n\}$. Then*

$$K_{x_1,x_2,\ldots,x_n}(z,w) = \begin{cases} 1 & \text{for } z = w; \\ 0 & \text{for } z \ne w. \end{cases}$$

Note that since only values at n points matter for polynomials, we can treat our space as $\mathbb{R}^n$. The reproducing kernel of $\mathbb{R}^n$ is given by the very same formula. The reproducing kernel of $P_{n-1}(\mathbb{R})$, however, is a polynomial defined for any $z, w \in \mathbb{R}$, not only for $z, w \in \{x_1, x_2, \dots, x_n\}$.

Proof of Lemma 3.1. Let $z = x_j$ for some j. By the reproducing property, for a polynomial f which takes value 0 at points x_i for $i \neq j$ and non-zero value at x_j, we have

$$\langle K(z,\cdot)|f\rangle = f(x_j).$$

On the other hand,

$$\langle K(z,\cdot)|f\rangle = \sum_{i=1}^{n} K(z,x_i)f(x_i) = K(z,x_j)f(x_j).$$

We conclude that $K(z,z) = 1$.

Now let us take a polynomial g such that $g(x_i) \neq 0$ for any x_i. By the reproducing property,

$$\langle K(z,\cdot)|g\rangle = g(x_j).$$

On the other hand, since $K(z,z) = 1$,

$$\langle K(z,\cdot)|g\rangle = \sum_{i=1}^{n} K(z,x_i)g(x_i) = \sum_{i=1,\, i\neq j}^{n} K(z,x_i)g(x_i) + g(x_j)$$

and since the result does not depend on choice of a polynomial g, we conclude that

$$K(x_i,x_j) = 0$$

for $i \neq j$. $\square$

Lemma 3.2 (Lagrange interpolation formula). *The unique polynomial of degree at most $n-1$ such that $f(x_1) = y_1, \dots, f(x_n) = y_n$ for pairwise different $x_i \in \mathbb{R}$ is given by*

$$f(w) := \sum_{i=1}^{n} \left(\prod_{1\leq j\leq n,\, j\neq i} \frac{w - x_j}{x_i - x_j} \right) y_i.$$

For more details see e.g. [3].

Proof of Theorem 3.1. For a given x_i, $K(x_i,\cdot)$ is a polynomial of degree at most $n-1$. By Lemma 3.1, we know its values at n points $\{x_1, x_2, \dots, x_n\}$. By the Lagrange interpolation formula

$$P_{x_i}(w) := K(x_i,w) = \prod_{1\leq j\leq n,\, j\neq i} \frac{w - x_j}{x_i - x_j}.$$

Now we can think of $K(z,w)$ as a polynomial of variable z with values $P_z(w)$ dependent on w. Since we know values $P_z(w)$ for n different values of z, we can find the unique polynomial with these properties. By the Lagrange interpolation formula again:

$$K(z,w) = \sum_{i=1}^{n} \left(\prod_{1\le j\le n,\, j\ne i} \frac{z-x_j}{x_i-x_j} \right) P_{x_i}(w)$$
$$= \sum_{i=1}^{n} \prod_{1\le j\le n,\, j\ne i} \frac{(z-x_j)(w-x_j)}{(x_i-x_j)^2}. \qquad \square$$

Theorem 3.2. *Let $\{x_1, x_2, \ldots, x_n, x_{n+1}\} \subset \mathbb{R}$ be a set of pairwise different numbers. Let $K_{x_1,x_2,\ldots,x_n}$ be the reproducing kernel of $P_{n-1}(\mathbb{R})$ equipped with the inner product $\langle\cdot|\cdot\rangle_{x_1,x_2,\ldots,x_n}$ and $K_{x_1,x_2,\ldots,x_{n+1}}$ be the reproducing kernel of $P_n(\mathbb{R})$ equipped with the inner product $\langle\cdot|\cdot\rangle_{x_1,x_2,\ldots,x_{n+1}}$. Then for any $z \in \mathbb{R}$*

$$K_{x_1,x_2,\ldots,x_n}(z,z) \ge K_{x_1,x_2,\ldots,x_{n+1}}(z,z).$$

Proof. By the reproducing property,

$$\frac{1}{\sqrt{K_{x_1,x_2,\ldots,x_n}(z,z)}} = \sum_{i=1}^{n} \left| \frac{K_{x_1,x_2,\ldots,x_n}(z,x_i)}{K_{x_1,x_2,\ldots,x_n}(z,z)} \right|^2.$$

By Theorem 1.1, we have

$$\sum_{i=1}^{n} \left| \frac{K_{x_1,x_2,\ldots,x_n}(z,x_i)}{K_{x_1,x_2,\ldots,x_n}(z,z)} \right|^2 \le \sum_{i=1}^{n} \left| \frac{K_{x_1,x_2,\ldots,x_{n+1}}(z,x_i)}{K_{x_1,x_2,\ldots,x_{n+1}}(z,z)} \right|^2.$$

Obviously

$$\sum_{i=1}^{n} \left| \frac{K_{x_1,x_2,\ldots,x_{n+1}}(z,x_i)}{K_{x_1,x_2,\ldots,x_{n+1}}(z,z)} \right|^2 \le \sum_{i=1}^{n+1} \left| \frac{K_{x_1,x_2,\ldots,x_{n+1}}(z,x_i)}{K_{x_1,x_2,\ldots,x_{n+1}}(z,z)} \right|^2.$$

By the reproducing property again,

$$\sum_{i=1}^{n+1} \left| \frac{K_{x_1,x_2,\ldots,x_{n+1}}(z,x_i)}{K_{x_1,x_2,\ldots,x_{n+1}}(z,z)} \right|^2 = \frac{1}{\sqrt{K_{x_1,x_2,\ldots,x_{n+1}}(z,z)}}. \qquad \square$$

Theorem 3.3. *Let K be the reproducing kernel of $P_{n-1}(\mathbb{R})$ equipped with the inner product $\langle\cdot|\cdot\rangle_{x_1,x_2,\ldots,x_n}$. Then for any $z \in \mathbb{R}$ we have*

$$K(z,z) \ge \frac{1}{n}.$$

Proof. It is just a simple consequence of Proposition 1.2. $\square$

4. Inner product as a sum of products of coefficients of the same power. On the vector space $P_n(\mathbb{R})$ of polynomials of degree at most n over $\mathbb{R}$ we can define the inner product

$$\langle f|g\rangle := \sum_{i=0}^{n} a_i b_i,$$

where

$$f(x)=\sum_{i=0}^{n} a_i x^i$$

and

$$g(x)=\sum_{i=0}^{n} b_i x^i.$$

As any finite-dimensional topological vector space, this space is complete. From the definition it follows that

$$\left\langle a_i x^i | b_j x^j \right\rangle = \begin{cases} a_i b_j, & i=j; \\ 0, & i \neq j. \end{cases}$$

In particular, the set

$$\left\{1, x, x^2, \ldots, x^n\right\}$$

is an orthonormal basis and

$$K(z,w) := 1 + zw + z^2w^2 + \cdots + z^n w^n$$

is the reproducing kernel of $P_n(\mathbb{R})$.

We can also consider weighted inner product, i.e. the inner product

$$\left\langle \sum_{i=0}^{n} a_i x^i \middle| \sum_{i=0}^{n} b_i x^j \right\rangle_{\mu} := \sum_{i=0}^{n} a_i b_i \mu_i$$

for any vector $\mu = (\mu_0, \ldots, \mu_1) \in \mathbb{R}^{n+1}$ for which $\mu_0, \mu_1, \ldots, \mu_n$ are positive numbers. In this case the set

$$\left\{\frac{1}{\sqrt{\mu_0}}, \frac{x}{\sqrt{\mu_1}}, \frac{x^2}{\sqrt{\mu_2}}, \ldots, \frac{x^n}{\sqrt{\mu_n}}\right\}$$

is an orthonormal basis and

$$K(z,w) := \frac{1}{\mu_0} + \frac{1}{\mu_1} zw + \frac{1}{\mu_2} z^2 w^2 + \cdots + \frac{1}{\mu_n} z^n w^n$$

is the reproducing kernel of such a space.

5. Concluding remarks. In the paper we have analyzed the Hilbert space of polynomials of degree at most n with three different inner products. All these inner products generate the same topology — a sequence of polynomials converges in any one of them if and only if coefficients converge to corresponding coefficients. Moreover, since in the second and third considered Hilbert space a polynomial is identified with n numbers, these Hilbert spaces can be treated as $\mathbb{R}^n$. That is an additional structure — a reproducing kernel — which makes these three spaces different.

Note also that in any set

$$A_x := \{f \in P_n(\mathbb{R}) \mid f(x) = c\}, \ x \in \mathbb{R}$$

there is exactly one element with the minimal norm, no matter what the inner product is. (See Theorem 1.1 and remember that for any $c \neq 0$ the linear operator $A_c := cf$ is a bijection with the property $\|A_c f\| = |c| \cdot \|f\|$.) Moreover, such an element "does not change too much" with deformation of an inner product.

To "reproduce" values of a polynomial using the integral inner product we need to know its values almost everywhere. To "reproduce" values of a polynomial of degree at most $n-1$ using the inner product defined as the sum of products of values of polynomials at n points we need only to know its values at n points. To "reproduce" values of a polynomial using the last inner product we need to know its coefficients.

Note also that for any point $z \in \mathbb{R}$ there exists a polynomial f of any non-negative degree such that $f(z) \neq 0$ and therefore by Theorem 1.2, $K(z, z) > 0$, no matter what the inner product is.

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Tomasz Łukasz Żynda
Warsaw Military University of Technology
Cybernetics Faculty
Gen. Sylwestra Kaliskiego 2
00-908 Warsaw 46
Poland
e-mail: `tomasz.zynda@wat.edu.pl`

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