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Second Hankel determinant
for a class of analytic functions
of complex order defined by convolution

1. Introduction. Let $\mathcal{A}$ denote the class of analytic functions of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \quad (z \in U = \{z : z \in \mathbb{C} \text{ and } |z| < 1\})$$

and $\mathcal{S}$ be the subclass of $\mathcal{A}$ consisting of univalent functions. Furthermore, let $\mathcal{P}$ be a family of functions $p(z) \in \mathcal{A}$.

Let $g(z) \in \mathcal{S}$ be given by

$$g(z) = z + \sum_{k=2}^{\infty} b_k z^k. \quad (1.2)$$

The Hadamard product (or convolution) of $f(z)$ and $g(z)$ is given by

$$(f \ast g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^m = (g \ast f)(z). \quad (1.3)$$

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Furthermore, if the function $U$ with equivalence (see [6] and [19]):

$$f(z) < g(z) \iff f(0) = g(0) \text{ and } f(U) \subset g(U).$$

For complex parameters $\alpha_1, \ldots, \alpha_q$ and $\beta_1, \ldots, \beta_s$ ($\beta_j \notin \mathbb{Z}^-$ = {0, -1, -2, ...}; $j = 1, 2, \ldots, s$), we now define the generalized hypergeometric function $qF_s(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z)$ by (see, for example, [29, p. 19])

$$qF_s(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z) = \sum_{k=0}^{\infty} \frac{(\alpha_1)_k \ldots (\alpha_q)_k}{(\beta_1)_k \ldots (\beta_s)_k} \frac{z^k}{k!}$$

($q \leq s + 1; q, s \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}; \mathbb{N} = \{1, 2, \ldots\}; z \in \mathbb{U}$), where $(\theta)_\nu$ is the Pochhammer symbol defined, in terms of the Gamma function $\Gamma$, by

$$\frac{(\theta + \nu)}{(\theta)} = \begin{cases} 1 & (\nu = 0; \theta \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}), \\ \theta(\theta + 1) \ldots (\theta + \nu - 1) & (\nu \in \mathbb{N}; \theta \in \mathbb{C}). \end{cases}$$

It corresponds to the function $h_{q, s}(\alpha_1, \beta_1; z) = h(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z)$, defined by

$$h_{q, s}(\alpha_1, \beta_1; z) = z qF_s(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z)$$

$$= z + \sum_{k=2}^{\infty} \Gamma_k(\alpha_1) z^k,$$

where

$$\Gamma_k(\alpha_1) = \frac{(\alpha_1)_k \ldots (\alpha_q)_k}{(\beta_1)_k \ldots (\beta_s)_k (k - 1)!}.$$
If \( f \in A \), then from (1.1) and (1.7), we can easily see that

\[
(iii) \quad I_q = 2
\]

For our present discussion, we consider the Hankel determinant in the case

\[
(ii) \quad I_q = \frac{1}{1 + q}
\]

We note that when \( \ell = 0 \), the operator

\[
I_{q,s,\lambda}^m(\alpha_1, \beta_1) f(z) = D_{\lambda}^m(\alpha_1, \beta_1) f(z)
\]

was studied by Selvaraj and Karthikeyan [28]. We also note that:

(i) \( I_{q,s,\lambda}^m(\alpha_1, \beta_1) f(z) = H_{q,s}(\alpha_1, \beta_1) f(z) \) (see Dziok and Srivastava [11, 12]);

(ii) For \( q = s + 1 \), \( \alpha_i = 1 \) (\( i = 1, \ldots, s + 1 \)) and \( \beta_j = 1 \) (\( j = 1, \ldots, s \)), we get the operator \( I(m, \lambda, \ell) \) (see Catas [7], Prajapat [24] and El-Ashwah and Aouf [14]);

(iii) For \( q = s + 1 \), \( \alpha_i = 1 \) (\( i = 1, \ldots, s + 1 \)) and \( \beta_j = 1 \) (\( j = 1, \ldots, s \)) and \( \lambda = 1 \) and \( \ell = 0 \), we obtain the Sa\'alagean operator \( D_{\lambda}^m \) (see Sa\'alagean [27]);

(iv) For \( q = s + 1 \), \( \alpha_i = 1 \) (\( i = 1, \ldots, s + 1 \)), \( \beta_j = 1 \) (\( j = 1, \ldots, s \)) and \( \lambda = 1 \), we get the operator \( I_{q,s}^m \) (see Cho and Srivastava [8] and Cho and Kim [9]).

(v) For \( q = s + 1 \), \( \alpha_i = 1 \) (\( i = 1, \ldots, s + 1 \)), \( \beta_j = 1 \) (\( j = 1, \ldots, s \)) and \( \ell = 0 \), we obtain the operator \( D_{\lambda}^m \) (see Al-Oboudi [2]).

By specializing the parameters \( m, \lambda, \ell, q, s, \alpha_i \) (\( i = 1, \ldots, q \)) and \( \beta_j \) (\( j = 1, \ldots, s \)) we obtain:

(i) \( I_{2,1,\lambda}^m(n+1,1;1) f(z) = I_{\lambda}^m(n) f(z) = z + \sum_{k=2}^{\infty} \left[ 1 + \frac{1 + \lambda(k-1)}{1 + \ell} \right]^{m} \frac{n+1}{1} \frac{k-1}{a_k z^k} \) (\( n > -1 \));

(ii) \( I_{2,1,\lambda}^m(a,1;c) f(z) = I_{\lambda}^m(a;c) f(z) = z + \sum_{k=2}^{\infty} \left[ 1 + \frac{1 + \lambda(k-1)}{1 + \ell} \right]^{m} \frac{a k-1}{c k-1} a_k z^k \) (\( a \in \mathbb{R}; c \in \mathbb{R} \setminus \mathbb{Z}^0 \));

(iii) \( I_{2,1,\lambda}^m(2,1;n+1) f(z) = I_{\lambda,n}^m f(z) = z + \sum_{k=2}^{\infty} \left[ 1 + \frac{1 + \lambda(k-1)}{1 + \ell} \right]^{m} \frac{2 k-1}{n+1} a_k z^k \) (\( n \in \mathbb{Z}; n > -1 \)).

In 1976, Noonan and Thomas [23] discussed the \( q \)th Hankel determinant of a locally univalent analytic function \( f(z) \) for \( q \geq 1 \) and \( n \geq 1 \) which is defined by

\[
H_q(n) = \begin{vmatrix}
    a_n & a_{n+1} & \cdots & a_{n+q-1} \\
    a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\
    \vdots & \vdots & \ddots & \vdots \\
    a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2}
\end{vmatrix}
\]

For our present discussion, we consider the Hankel determinant in the case \( q = 2 \) and \( n = 2 \), i.e. \( H_2(2) = a_2 a_4 - a_3^2 \). This is popularly known as the second Hankel determinant of \( f \).

In this paper, we define the following class \( S_\gamma^b(g(z); A, B) \) (\( 0 \leq \gamma \leq 1, b \in \mathbb{C}^* = \mathbb{C} \setminus \{0\} \)) as follows:
Definition 1. Let $0 \leq \gamma \leq 1$, $b \in \mathbb{C}^*$. A function $f(z) \in A$ is said to be in the class $S^b_{\gamma}(g(z); A, B)$ if

$$1 + \frac{1}{b} \left( (1 - \gamma) \frac{(f * g)(z)}{z} + \gamma (f * g)'(z) - 1 \right) < \frac{1 + Az}{1 + Bz}$$

$(b \in \mathbb{C}^*; 0 \leq \gamma \leq 1; -1 \leq B < A \leq 1; z \in \mathbb{U})$, which is equivalent to say that

$$\left| \frac{(1 - \gamma) \frac{(f * g)(z)}{z} + \gamma (f * g)'(z) - 1}{b(A - B) - B \left( (1 - \gamma) \frac{(f * g)(z)}{z} + \gamma (f * g)'(z) - 1 \right)} \right| < 1.$$ 

We note that for suitable choices of $b$, $\gamma$ and $g(z)$ we obtain the following subclasses:

(i) $S^b\left(\frac{z}{z}; A, B\right) = S^b_{\gamma}(A, B)$ $(0 \leq \gamma \leq 1, b \in \mathbb{C}^*, -1 \leq B < A \leq 1)$ (see Bansal [5]);

(ii) $S^0 (1 - \rho) e^{-i \theta} \cos \theta \left( z + \sum_{k=2}^{\infty} \frac{(\alpha + 1)k_{k-1}}{(\beta)_{k-1}} z^k; 1, -1 \right) = R_{a, \beta} (\theta, \rho) \left( \frac{n}{2} < \theta < \frac{n}{2} \right), 0 \leq \rho < 1, \alpha \in \mathbb{C}, \beta \in \mathbb{C} \setminus \mathbb{Z}_0^-$ (see Mishra and Kund [21]);

(iii) $S^0 (1 - \rho) e^{-i \sigma} \cos \alpha \left( z + \sum_{k=2}^{\infty} \frac{\lambda + 1)(k+1)_{k-1}}{(m)_{k-1}} z^k; 1, -1 \right) = S^m_{\lambda, \alpha, \beta} (\alpha, \sigma, \rho) (m \in \mathbb{N}; n, \lambda \in \mathbb{N}_0; |\alpha| < \frac{n}{2}; 0 \leq \sigma < 1)$ (see Mohammed and Darus [22]);

(iv) $S^1 (z + \sum_{k=2}^{\infty} [1 + (\alpha k + \alpha - \mu)(k - 1)] \left( \frac{\mu}{k-1} z^k; 1, -1 \right) = R_{a, \mu} (\sigma, \rho) (0 \leq \mu \leq \alpha \leq 1; \rho, \sigma \in \mathbb{N}_0)$ (see Abubaker and Darus [1]);

(v) $S^b (z + \sum_{k=2}^{\infty} k^m z^k; A, B) = G_m (\gamma, b) (b \in \mathbb{C}^*, 0 \leq \gamma \leq 1, m \in \mathbb{N}_0)$ (see Aouf [3]).

Also, we note that:

(i) $S^b\left(\frac{z}{z}; A, B\right) = S^b_{\gamma}(A, B)$ $= \{ f(z) \in A : 1 + \frac{1}{b} \left( (1 - \gamma) \frac{f_{\gamma, \alpha, \beta}}{z} + \gamma (f_{\gamma, \alpha, \beta})' - 1 \right) < \frac{1 + Az}{1 + Bz}, (b \in \mathbb{C}^*; 0 \leq \gamma \leq 1; m \in \mathbb{N}_0; \ell \geq 0; \lambda \geq 0; q \leq s + 1; q, s \in \mathbb{N}_0; z \in \mathbb{U}) \};$

(ii) $S^b\left(\frac{z}{z}; A, B\right) = S^b_{\gamma}(A, B)$ $= \{ f(z) \in A : 1 + \frac{1}{b} \left( (1 - \gamma) \frac{f_{\gamma, \lambda, \ell}}{z} + \gamma (f_{\gamma, \lambda, \ell})' - 1 \right) < \frac{1 + Az}{1 + Bz}, (b \in \mathbb{C}^*; 0 \leq \gamma \leq 1; m \in \mathbb{N}_0; \ell \geq 0; \lambda \geq 0; z \in \mathbb{U}) \};$
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(iii) \( S^\gamma (1-\rho) \cos \eta e^{-i\eta} (g(z); A, B) = S^\gamma [\rho, \eta, A, B, g(z)] \)

\[
= \left\{ f(z) \in A : e^{i\eta} \left[ (1-\gamma) \frac{(f*g)(z)}{z} + \gamma (f*g)'(z) \right] < (1-\rho) \cos \eta \cdot \frac{1+iA}{1+B^2} + \rho \cos \eta + i \sin \eta, \\
\right.
\]
\[
(\eta| < \frac{\pi}{2}; \ 0 \leq \gamma \leq 1; \ 0 \leq \rho < 1; \ -1 \leq B < A \leq 1; \ z \in U) \}
\]

In this paper, we obtain the Fekete–Szegő inequalities for the functions in the class \( S^b_\gamma (g(z); A, B) \). We also obtain an upper bound to the functional \( H_2(2) \) for \( f(z) \in S^b_\gamma (g(z); A, B) \). Earlier Janteng et al. [16], Mishra and Gochhayat [20], Mishra and Kund [21], Bansal [4] and many other authors have obtained sharp upper bounds of \( H_2(2) \) for different classes of analytic functions.

2. Preliminaries. To prove our results, we need the following lemmas.

Lemma 1 ([26]). Let

\[
h(z) = 1 + \sum_{n=1}^{\infty} c_n z^n = 1 + \sum_{n=1}^{\infty} C_n z^n = H(z) \quad (z \in U).
\]

If the function \( H \) is univalent in \( U \) and \( H(U) \) is a convex set, then

\[
|c_n| \leq |C_1|.
\]

Lemma 2 ([10]). Let a function \( p \in \mathcal{P} \) be given by

\[
p(z) = 1 + c_1 z + c_2 z^2 + \ldots \quad (z \in U),
\]

then, we have

\[
|c_n| \leq 2 \quad (n \in \mathbb{N}).
\]

The result is sharp.

Lemma 3 ([17, 18]). Let \( p \in \mathcal{P} \) be given by the power series (2.3), then for any complex number \( \nu \)

\[
|c_2 - \nu c_2^2| \leq 2 \max\{1; |2\nu - 1|\}.
\]

The result is sharp for the functions given by

\[
p(z) = \frac{1 + z^2}{1 - z^2} \quad \text{and} \quad p(z) = \frac{1 + z}{1 - z} \quad (z \in U).
\]

Lemma 4 ([15]). Let a function \( p \in \mathcal{P} \) be given by the power series (2.3), then

\[
2c_2 = c_1^2 + \kappa (4 - c_1^2)
\]

for some \( \kappa \), \( |\kappa| \leq 1 \), and

\[
4c_3 = c_1^3 + 2(4-c_1^2)c_1 \kappa - c_1(4-c_1^2)\kappa^2 + 2(4-c_1^2) \left( 1 - |\kappa|^2 \right) z,
\]

for some \( \kappa \), \( |z| \leq 1 \).
3. Main results. We give the following result related to the coefficient of \( f(z) \in S_b^\gamma (g(z); A, B) \).

**Theorem 1.** Let \( f(z) \) given by (1.1) belong to the class \( S_b^\gamma (g(z); A, B) \), \( 0 \leq \gamma \leq 1 \), \(-1 \leq B < A \leq 1 \) and \( b \in \mathbb{C}^* \), then

\[
|a_k| \leq \frac{(A - B)|b|}{[1 + \gamma (k - 1)] b_k} \quad (k \in \mathbb{N} \setminus \{1\}).
\]

**Proof.** If \( f(z) \) of the form (1.1) belongs to the class \( S_b^\gamma (g(z); A, B) \), then

\[
1 + \frac{1}{b} \left( (1 - \gamma) \frac{(f * g)(z)}{z} + \gamma (f * g)'(z) - 1 \right) \prec 1 + \frac{Az}{1 + Bz} = h(z)
\]

\((b \in \mathbb{C}^*; 0 \leq \gamma \leq 1; -1 \leq B < A \leq 1; z \in U)\), where \( h(z) \) is convex univalent in \( U \) and we have

\[
1 + \frac{1}{b} \left( (1 - \gamma) \frac{(f * g)(z)}{z} + \gamma (f * g)'(z) - 1 \right)
\]

\[
= 1 + \sum_{k=1}^{\infty} \frac{(1 + k\gamma)}{b} b_{k+1} a_k z^k < 1 + (A - B)z - B(A - B)z^2 + \ldots
\]

\((z \in U)\). Now, by applying Lemma 1, we get the desired result. \(\square\)

**Remark 1.** Putting \( g(z) = \frac{z}{1 - z} \) in Theorem 1, we obtain the result obtained by Bansal [5, Theorem 2.1].

It is easy to derive a sufficient condition for \( f(z) \) to be in the class \( S_b^\gamma (m, \lambda, \ell; A, B) \) using standard techniques (see [25]). Hence we state the following result without proof.

**Theorem 2.** Let \( f(z) \in A \), then a sufficient condition for \( f(z) \) to be in the class \( S_b^\gamma (g(z); A, B) \) is

\[
\sum_{k=2}^{\infty} [1 + \gamma (k - 1)] b_k |a_k| \leq \frac{(A - B)|b|}{1 + B}.
\]

In the next two theorems, we obtain the result concerning Fekete–Szegö inequality and an upper bound for the Hankel determinant for the class \( S_b^\gamma (g(z); A, B) \).

**Remark 2.** Putting \( g(z) = \frac{z}{1 - z} \) in Theorem 2, we obtain the result obtained by Bansal [5, Theorem 2.2].

**Theorem 3.** Let \( f(z) \) given by (1.1) belong to the class \( S_b^\gamma (g(z); A, B) \), \( 0 \leq \gamma \leq 1 \), \(-1 \leq B < A \leq 1 \) and \( b \in \mathbb{C}^* \), then

\[
|a_3 - \mu a_2^3| \leq \frac{(A - B)|b|}{(1 + 2\gamma) b_3} \cdot \max \left\{ 1, \left| B + \frac{\mu b b_3 (A - B)(1 + 2\gamma)}{(1 + \gamma)^2 b_3^2} \right| \right\}.
\]

This result is sharp.
Proof. Let \( f(z) \in S^b_\gamma \left( g(z); A, B \right) \), then there is a Schwarz function \( w(z) \) in \( U \) with \( w(0) = 0 \) and \( |w(z)| < 1 \) in \( U \) and such that

\[
(3.5) \quad 1 + \frac{1}{b} \left( (1 - \gamma) \left( \frac{f \ast g}{z} \right) + \gamma (f \ast g)'(z) - 1 \right) = \Phi(w(z))
\]

\((z \in U)\), where

\[
(3.6) \quad \Phi(z) = \frac{1 + Az}{1+Bz} = 1 + (A - B)z - B(A - B)z^2 + B^2(A - B)z^3 - \ldots
\]

\((z \in U)\). If the function \( p_1(z) \) is analytic and has positive real part in \( U \) and \( p_1(0) = 1 \), then

\[
(3.7) \quad p_1(z) = \frac{1 + w(z)}{1 - w(z)} = 1 + c_1z + c_2z^2 + \ldots
\]

\((z \in U)\), since \( w(z) \) is a Schwarz function. Define

\[
(3.8) \quad h(z) = 1 + \frac{1}{b} \left( (1 - \gamma) \left( \frac{f \ast g}{z} \right) + \gamma (f \ast g)'(z) - 1 \right)
\]

\((z \in U)\). In view of the equations (3.5) and (3.7), we have

\[
(p_1(z)) = \Phi \left( \frac{p_1(z) - 1}{p_1(z) + 1} \right).
\]

Since

\[
(3.9) \quad \frac{p_1(z) - 1}{p_1(z) + 1} = \frac{1}{2} \left[ c_1z + \left( c_2 - \frac{c_1^2}{2} \right) z^2 + \left( c_3 + \frac{c_1^2}{4} - c_1c_2 \right) z^3 + \ldots \right],
\]

we have

\[
(3.10) \quad \Phi \left( \frac{p_1(z) - 1}{p_1(z) + 1} \right) = 1 + \frac{1}{2} B_1 c_1z + \left[ \frac{1}{2} B_1 \left( c_2 - \frac{c_1^2}{2} \right) + \frac{1}{4} B_2 c_1^2 \right] z^2 + \ldots,
\]

and from this equation and (3.8), we obtain

\[
(3.11) \quad d_1 = \frac{1}{2} B_1 c_1, \quad d_2 = \frac{1}{2} B_1 \left( c_2 - \frac{c_1^2}{2} \right) + \frac{1}{4} B_2 c_1^2
\]

and

\[
(3.12) \quad d_3 = \frac{B_1}{2} \left( c_3 - c_1c_2 + \frac{c_1^3}{4} \right) + \frac{B_2 c_1}{2} \left( c_2 - \frac{c_1^2}{2} \right) + \frac{B_3 c_1^3}{8}.
\]

Then, from (3.6), we see that

\[
(3.13) \quad d_1 = \frac{(1 + \gamma) b_2 a_2}{b} \quad \text{and} \quad d_2 = \frac{(1 + 2\gamma) b_3 a_3}{b}.
\]
Now from (3.6), (3.8) and (3.13), we have

\[ a_2 = \frac{(A - B) bc_1}{2(1 + \gamma) b_2}, \quad a_3 = \frac{b(A - B)}{4(1 + 2\gamma) b_3} \left\{ 2c_2 - c_1^2 (1 + B) \right\} \]

and

\[ a_4 = \frac{b(A - B)}{8(1 + 3\gamma) b_4} \left\{ 4c_3 - 4c_1c_2 (1 + B) + c_1^3(1 + B)^2 \right\} \]

Therefore, we have

\[ a_3 - \mu a_2^2 = \frac{b(A - B)}{2(1 + 2\gamma) b_3} \left\{ c_2 - \nu c_1^2 \right\}, \]

where

\[ \nu = \frac{1}{2} \left[ 1 + B + \frac{\mu b(A - B)(1 + 2\gamma) b_3}{(1 + \gamma)^2 b_2^2} \right]. \]

Our result now follows by an application of Lemma 3. The result is sharp for the functions

\[ 1 + \frac{1}{b} \left( (1 - \gamma) \frac{(f \ast g)(z)}{z} + \gamma (f \ast g)'(z) - 1 \right) = \Phi(z^2) \]

and

\[ 1 + \frac{1}{b} \left( (1 - \gamma) \frac{(f \ast g)(z)}{z} + \gamma (f \ast g)'(z) - 1 \right) = \Phi(z). \]

This completes the proof of Theorem 3.

**Remark 3.** Putting \( g(z) = \frac{z}{1-z} \) in Theorem 3, we obtain the result due to Bansal [5, Theorem 2.3].

Putting \( g(z) = z + \sum_{k=2}^{\infty} \left[ \frac{1 + \ell + \lambda(k-1)}{1+\ell} \right]^m \Gamma_k(\alpha_1) \) \((m \in \mathbb{N}_0, \ell \geq 0, \lambda \geq 0, q \leq s + 1, q, s \in \mathbb{N}_0), \) where \( \Gamma_k(\alpha_1) \) is given by (1.6) in Theorem 3, we obtain the following corollary.

**Corollary 1.** Let \( f(z) \) given by (1.1) belong to the class \( S^b_{\gamma} (\lambda, \ell, m, q, s, \alpha_1, \beta_1; A, B), 0 \leq \gamma \leq 1, -1 \leq B < A \leq 1, m \in \mathbb{N}_0, \ell \geq 0, \lambda \geq 0, q \leq s + 1, q, s \in \mathbb{N}_0 \) and \( b \in \mathbb{C}^*, \) then

\[ |a_3 - \mu a_2^2| \leq \frac{(A - B) (1 + \ell)^m |b|}{(1 + 2\gamma)(1 + \ell + 2\lambda)^m \Gamma_3(\alpha_1)} \times \max \left\{ 1, \left| B + \frac{\mu b(1 + \ell + 2\lambda)^m \Gamma_3(\alpha_1)(A - B)(1 + 2\gamma)}{(1 + \gamma)^2 (1 + \ell + 2\lambda)^m \Gamma_2(\alpha_1)} \right| \right\}. \]

This result is sharp.

Putting \( g(z) = z + \sum_{k=2}^{\infty} \left[ \frac{1 + \ell}{1+\ell+\lambda(k-1)} \right]^m z^k \) \((m \in \mathbb{N}_0; \ell \geq 0; \lambda \geq 0) \) in Theorem 3, we obtain the following corollary.
Corollary 2. Let \( f(z) \) given by (1.1) belong to the class \( S^b_\gamma (\lambda, \ell, m; A, B) \), \( 0 \leq \gamma \leq 1, -1 \leq B < A \leq 1, m \in \mathbb{N}_0, \ell \geq 0, \lambda \geq 0 \) and \( b \in \mathbb{C}^* \), then

\[
|a_3 - \mu a_2|^2 \leq \frac{(A - B) |b|^2 (1 + 2\gamma)^m}{(1 + 2\gamma)^{1+\ell+2\lambda}} \max \left\{ 1, \left| B + \frac{\mu b^3 (A - B) (1 + 2\gamma) (1 - \rho) e^{-i\eta} \cos \eta}{(1 + \gamma)^2 b_2^2} \right| \right\}.
\]

(3.21)

This result is sharp.

Putting \( b = (1 - \rho) e^{-i\eta} \cos \eta \) \( (|\eta| < \frac{\pi}{2}, 0 \leq \rho < 1) \) in Theorem 3, we obtain the following corollary.

Corollary 3. Let \( f(z) \) given by (1.1) belong to the class \( S^7 [\rho, \eta, A, B, g(z)] \), \( 0 \leq \gamma \leq 1, -1 \leq B < A \leq 1 \) and \( b \in \mathbb{C}^* \), then

\[
|a_3 - \mu a_2|^2 \leq \frac{(A - B) (1 - \rho) \cos \eta}{(1 + 2\gamma)^{1+\ell+2\lambda}} \max \left\{ 1, \left| B + \frac{\mu b^3 (A - B) (1 + 2\gamma) (1 - \rho) e^{-i\eta} \cos \eta}{(1 + \gamma)^2 b_2^2} \right| \right\}.
\]

(3.22)

This result is sharp.

Theorem 4. Let \( f(z) \) given by (1.1) belong to the class \( S^b_\gamma (g(z); A, B) \), \( 0 \leq \gamma \leq 1, -1 \leq B < A \leq 1 \) and \( b \in \mathbb{C}^* \), then

\[
|a_2 a_4 - a_3^2| \leq \frac{(A - B)^2 |b|^2}{(1 + 2\gamma)^2 b_2^2}.
\]

(3.23)

Proof. Using (3.14) and (3.15), we have

\[
|a_2 a_4 - a_3^2| = \frac{(A - B)^2 |b|^2}{16 (1 + \gamma) (1 + 3\gamma) b_2 b_4} \left| 4c_1 c_3 - 4c_1^2 c_2 (1 + B) + c_1^2 (1 + B)^2 \right.
\]

\[
- \frac{(1 + \gamma) (1 + 3\gamma) b_2 b_4}{(1 + 2\gamma)^2 b_2^2} \left[ 4c_2^2 - 4c_1^2 c_2 (1 + B) + c_1^4 (1 + B)^2 \right] 
\]

\[
+ M \left| 4c_1 c_3 - 4c_1^2 c_2 (1 + B) + c_1^2 (1 + B)^2 \right.
\]

\[
- N \left[ 4c_2^2 - 4c_1^2 c_2 (1 + B) + c_1^4 (1 + B)^2 \right],
\]

where

\[
M = \frac{(A - B)^2 |b|^2}{16 (1 + \gamma) (1 + 3\gamma) b_2 b_4} \quad \text{and} \quad N = \frac{(1 + \gamma) (1 + 3\gamma) b_2 b_4}{(1 + 2\gamma)^2 b_2^2}.
\]

The above equation (3.24) is equivalent to

\[
|a_2 a_4 - a_3^2| = M \left| 4c_1 c_3 + d_2 c_1^2 c_2 + d_3 c_2^2 + d_4 c_1^4 \right|
\]

(3.26)
that
\[ \frac{\partial F}{\partial \nu} \]
For
0

(3.28)
\[ \text{max} \]
\[ \exists \text{ } \nu, \text{ } \nu \in \mathbb{R} \]
algebraic that \( c_1 > 0 \). For convenience of notation, we take \( c_1 = c \) (\( c \in [0, 2] \), see (2.4)). Also, substituting the values of \( c_2 \) and \( c_3 \), respectively, from (2.6) and (2.7) in (3.26), we have
\[ \left| a_2a_4 - a_3^2 \right| = \frac{M}{4} \left| c^4(d_1 + 2d_2 + d_3 + 4d_4) + 2 \gamma c^2(4 - c^2)(d_1 + d_2 + d_3) \right. \]
\[ + \left. (4 - c^2) \gamma^2(-d_1c^2 + d_3(4 - c^2)) + 2d_1c(4 - c^2) \left( 1 - |\gamma|^2 z \right) \right| . \]

An application of triangle inequality, replacement of \( |\gamma| \) by \( \nu \) and substituting the values of \( d_1 \), \( d_2 \), \( d_3 \) and \( d_4 \) from (3.27), we have
\[ \left| a_2a_4 - a_3^2 \right| \leq \frac{M}{4} \left[ 4c^4(1 - N)B^2 + 8 |B| (1 - N)\nu c^2(4 - c^2) \right. \]
\[ + \left. (4 - c^2)\nu^2(4c^2 + 4N(4 - c^2)) + 8c(4 - c^2)(1 - \nu^2) \right] \]
(3.28)
\[ = M \left[ c^4(1 - N)B^2 + 2c(4 - c^2) + 2\nu |B| (1 - N)c^2(4 - c^2) \right. \]
\[ + \left. \nu^2(4 - c^2) (c^2 (1 - N) - 2c + 4N) \right] \]
\[ = F(c, \nu). \]

Next, we assume that the upper bound for (3.28) occurs at an interior point of the rectangle \([0, 2] \times [0, 1]\). Differentiating \( F(c, \nu) \) in (3.28) partially with respect to \( \nu \), we have
\[ \frac{\partial F(c, \nu)}{\partial \nu} = M \left[ 2 |B| (1 - N)c^2(4 - c^2) \right. \]
\[ + \left. 2\nu(4 - c^2) (c^2 (1 - N) - 2c + 4N) \right] . \]

For \( 0 < \nu < 1 \) and for any fixed \( c \) with \( 0 < c < 2 \), from (3.29), we observe that \( \frac{\partial F}{\partial \nu} > 0 \). Therefore, \( F(c, \nu) \) is an increasing function of \( \nu \), which contradicts our assumption that the maximum value of \( F(c, \nu) \) occurs at an interior point of the rectangle \([0, 2] \times [0, 1]\). Moreover, for fixed \( c \in [0, 2] \),
(3.30)
\[ \max F(c, \nu) = F(c, 1) = G(c). \]
Thus
\[ G(c) = M \left[ c^4(1 - N) (B^2 - 2 |B| - 1) \right. \]
\[ + \left. 4c^2(2 |B| (1 - N) + 1 - 2N) + 16N \right] . \]
(3.31)

Next,
\[ G'(c) = 4Mc \left[ c^2(1 - N) (B^2 - 2 |B| - 1) + 2(2 |B| (1 - N) + 1 - 2N) \right. \]
\[ = 4Mc \left[ c^2(1 - N) (B^2 - 2 |B| - 1) + 2 \{(1 - N) (2 |B| + 1) - N\} \right] . \]
So \( G'(c) < 0 \) for \( 0 < c < 2 \) and has a real critical point at \( c = 0 \). Also \( G(c) > G(2) \). Therefore, maximum of \( G(c) \) occurs at \( c = 0 \). Therefore, the upper bound of \( F(c, \nu) \) corresponds to \( \nu = 1 \) and \( c = 0 \). Hence,

\[
|a_2a_4 - a_3^2| \leq 16MN = \frac{(A - B)^2 |b|^2}{(1 + 2\gamma)^2 b_3^2}.
\]

This completes the proof of Theorem 4. \( \square \)

**Remark 4.** (i) Putting \( g(z) = \frac{z}{1 - z^2} \) in Theorem 4, we obtain the result due to Bansal [5, Theorem 2.4];
(ii) Putting

\[
g(z) = z + \sum_{k=2}^{\infty} \frac{(\alpha)_{k-1}}{(\beta)_{k-1}} z^k
\]

\((\alpha \in \mathbb{C}, \beta \in \mathbb{C} \setminus \mathbb{Z}^0), b = (1 - \rho) e^{-i\theta} \cos \theta (|\theta| < \frac{\pi}{2}, 0 \leq \rho < 1), \gamma = 0, A = 1 \) and \( B = -1 \) in Theorem 4, we obtain the result due to Mishra and Kund [21, Theorem 3.1];
(iii) Putting

\[
g(z) = z + \sum_{k=2}^{\infty} \frac{(\lambda + 1)_{k-1}k^m}{(m)_{k-1}} z^k
\]

\((m \in \mathbb{N}; \lambda, n \in \mathbb{N}_0), b = (1 - \rho) e^{-i\alpha} \cos \alpha (|\alpha| < \frac{\pi}{2}; 0 \leq \sigma < 1), \gamma = 0, A = 1 \) and \( B = -1 \) in Theorem 4, we obtain the result due to Mohammed and Darus [22, Theorem 2.1];
(iv) Putting

\[
g(z) = z + \sum_{k=2}^{\infty} [1 + (\alpha \mu k + \alpha - \mu)(k - 1)]^\sigma (\rho)_{k-1} z^k
\]

\((0 \leq \mu \leq \alpha \leq 1, \rho, \sigma \in \mathbb{N}_0), b = \gamma = A = 1 \) and \( B = -1 \) in Theorem 4, we obtain the result due to Abubaker and Darus [1, Theorem 3.1].

Putting \( g(z) = z + \sum_{k=2}^{\infty} \frac{[1 + k\alpha(k-1)]^m}{\Gamma_k(\alpha_1)} (m \in \mathbb{N}_0, \ell \geq 0, \lambda \geq 0, q \leq s + 1, q, s \in \mathbb{N}_0), \) where \( \Gamma_k(\alpha_1) \) is given by (1.6) in Theorem 4, we obtain the following corollary.

**Corollary 4.** Let \( f(z) \) given by (1.1) belong to the class \( \mathcal{S}_\gamma^b(\lambda, \ell, m, q, s, \alpha_1, \beta_1; A, B) \), \( 0 \leq \gamma \leq 1, -1 \leq B < A \leq 1, m \in \mathbb{N}_0, \ell \geq 0, \lambda \geq 0, q \leq s + 1, q, s \in \mathbb{N}_0 \) and \( b \in \mathbb{C}^* \), then

\[
|a_2a_4 - a_3^2| \leq \frac{(A - B)^2 |b|^2}{(1 + 2\gamma)^2 [\frac{1 + \ell + 2\lambda}{1 + \ell}]^{2m} \Gamma_3^2(\alpha_1)}.
\]

Putting \( g(z) = z + \sum_{k=2}^{\infty} \frac{[1 + k\ell\lambda + \ell(\lambda - 1)]^m}{\Gamma_k(\alpha_1)} z^k \) \((m \in \mathbb{N}_0; \ell \geq 0; \lambda \geq 0)\) in Theorem 4, we obtain the following corollary.
Corollary 5. Let $f(z)$ given by (1.1) belong to the class $S^b_\gamma (\lambda, \ell, m; A, B)$, $0 \leq \gamma \leq 1$, $-1 \leq B < A \leq 1$, $m \in \mathbb{N}_0$, $\ell \geq 0$, $\lambda \geq 0$ and $b \in \mathbb{C}^*$, then

$$|a_2a_4 - a_3^2| \leq \frac{(A - B)^2 |b|^2}{(1 + 2\gamma)^2 \left[ \frac{1 + \ell}{1 + \ell + 2\gamma} \right]^{2m}}.$$  

(3.33)

Putting $b = (1 - \rho)e^{-i\eta}\cos \eta$ ($|\eta| < \frac{\pi}{2}$, $0 \leq \rho < 1$) in Theorem 4, we obtain the following corollary.

Corollary 6. Let $f(z)$ given by (1.1) belong to the class $S^\gamma [\rho, \eta, A, B, g(z)]$, $0 \leq \gamma \leq 1$, $-1 \leq B < A \leq 1$ and $b \in \mathbb{C}^*$, then

$$|a_2a_4 - a_3^2| \leq \frac{(A - B)^2 (1 - \rho)^2 \cos^2 \eta}{(1 + 2\gamma)^2 b_3^2}.$$  

(3.34)

References


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