doi: 10.1515/umcsmath-2015-0020

ANNALES
UNIVERSITATIS MARIAE CURIE-SKŁODOWSKA
LUBLIN – POLONIA

VOL. LXIX, NO. 2, 2015 SECTIO A 47–59

S. M. EL-DEEB and M. K. AOUF

Second Hankel determinant for a class of analytic functions of complex order defined by convolution

ABSTRACT. In this paper, we obtain the Fekete–Szegö inequalities for the functions of complex order defined by convolution. Also, we find upper bounds for the second Hankel determinant $\left|a_2a_4 - a_3^2\right|$ for functions belonging to the class $S_\gamma^b\,(g(z); A, B)$.

1. Introduction. Let $\mathcal{A}$ denote the class of analytic functions of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \quad (z \in \mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}) \tag{1.1}$$

and $\mathcal{S}$ be the subclass of $\mathcal{A}$ consisting of univalent functions. Furthermore, let $\mathcal{P}$ be a family of functions $p(z) \in \mathcal{A}$.

Let $g(z) \in \mathcal{S}$ be given by

$$g(z) = z + \sum_{k=2}^{\infty} b_k z^k. \tag{1.2}$$

The Hadamard product (or convolution) of $f(z)$ and $g(z)$ is given by

$$(f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^m = (g * f)(z). \tag{1.3}$$

2010 *Mathematics Subject Classification.* 30C45.

Key words and phrases. Fekete–Szegö inequality, second Hankel determinant, convolution, complex order.

If f and g are analytic functions in $\mathbb{U}$, we say that f is subordinate to g, written $f \prec g$ if there exists a Schwarz function w, which is analytic in $\mathbb{U}$ with $w(0) = 0$ and $|w(z)| < 1$ for all $z \in \mathbb{U}$, such that $f(z) = g(w(z))$. Furthermore, if the function g is univalent in $\mathbb{U}$, then we have the following equivalence (see [6] and [19]):

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \text{ and } f(\mathbb{U}) \subset g(\mathbb{U}).$$

For complex parameters $\alpha_1, \dots, \alpha_q$ and $\beta_1, \dots, \beta_s$ ($\beta_j \notin \mathbb{Z}_0^- = \{0, -1, -2, \dots\}$; $j = 1, 2, \dots, s$), we now define the generalized hypergeometric function ${}_qF_s(\alpha_1, \dots, \alpha_q; \beta_1, \dots, \beta_s; z)$ by (see, for example, [29, p. 19])

$${}_qF_s(\alpha_1, \dots, \alpha_q; \beta_1, \dots, \beta_s; z) = \sum_{k=0}^{\infty} \frac{(\alpha_1)_k \dots (\alpha_q)_k}{(\beta_1)_k \dots (\beta_s)_k} \cdot \frac{z^k}{k!}$$

($q \leq s + 1$; $q, s \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$; $\mathbb{N} = \{1, 2, \dots\}$; $z \in \mathbb{U}$), where $(\theta)_\nu$ is the Pochhammer symbol defined, in terms of the Gamma function Γ, by

$$(1.4) \qquad \begin{aligned} (\theta)_\nu &= \frac{\Gamma(\theta + \nu)}{\Gamma(\theta)} \\ &= \begin{cases} 1 & (\nu = 0;\ \theta \in \mathbb{C}^* = \mathbb{C}\backslash\{0\}), \\ \theta(\theta + 1) \dots (\theta + \nu - 1) & (\nu \in \mathbb{N};\ \theta \in \mathbb{C}). \end{cases} \end{aligned}$$

It corresponds to the function $h_{q,s}(\alpha_1, \beta_1; z) = h(\alpha_1, \dots \alpha_q; \beta_1 \dots, \beta_s; z)$, defined by

$$(1.5) \qquad \begin{aligned} h_{q,s}(\alpha_1, \beta_1; z) &= z {}_qF_s(\alpha_1, \dots, \alpha_q; \beta_1, \dots, \beta_s; z) \\ &= z + \sum_{k=2}^{\infty} \Gamma_k(\alpha_1) z^k, \end{aligned}$$

where

$$(1.6) \qquad \Gamma_k(\alpha_1) = \frac{(\alpha_1)_{k-1} \dots (\alpha_q)_{k-1}}{(\beta_1)_{k-1} \dots (\beta_s)_{k-1} (k-1)!}.$$

In [13] El-Ashwah and Aouf defined the operator $I_{q,s,\lambda}^{m,\ell}(\alpha_1, \beta_1) f(z)$ as follows:

$$\begin{aligned} I_{q,s,\lambda}^{0,\ell}(\alpha_1, \beta_1) f(z) &= f(z) * h_{q,s}(\alpha_1, \beta_1; z); \\ I_{q,s,\lambda}^{1,\ell}(\alpha_1, \beta_1) f(z) &= (1 - \lambda)(f(z) * h_{q,s}(\alpha_1, \beta_1; z)) \\ &\quad + \frac{\lambda}{(1 + \ell) z^{\ell - 1}} \left[z^\ell \left(f(z) * h_{q,s}(\alpha_1, \beta_1; z) \right) \right]'; \end{aligned}$$

and

$$(1.7) \qquad I_{q,s,\lambda}^{m,\ell}(\alpha_1, \beta_1) f(z) = I_{q,s,\lambda}^{1,\ell}(I_{q,s,\lambda}^{m-1,\ell}(\alpha_1, \beta_1) f(z)).$$

If $f \in A$, then from (1.1) and (1.7), we can easily see that

$$I_{q,s,\lambda}^{m,\ell}(\alpha_1,\beta_1)f(z) = z + \sum_{k=2}^{\infty}\left[\frac{1+\ell+\lambda(k-1)}{1+\ell}\right]^m \Gamma_k(\alpha_1)a_k z^k, \tag{1.8}$$

where $m \in \mathbb{Z} = \{0, \pm 1, \dots\}$, $\ell \geq 0$ and $\lambda \geq 0$.

We note that when $\ell = 0$, the operator

$$I_{q,s,\lambda}^{m,0}(\alpha_1,\beta_1)f(z) = D_\lambda^m(\alpha_1,\beta_1)f(z)$$

was studied by Selvaraj and Karthikeyan [28]. We also note that:
(i) $I_{q,s,\lambda}^{0,\ell}f(z) = H_{q,s}(\alpha_1,\beta_1)f(z)$ (see Dziok and Srivastava [11, 12]);
(ii) For $q = s+1$, $\alpha_i = 1$ $(i = 1,\dots,s+1)$ and $\beta_j = 1$ $(j = 1,\dots,s)$, we get the operator $I(m,\lambda,\ell)$ (see Catas [7], Prajapat [24] and El-Ashwah and Aouf [14]);
(iii) For $q = s+1$, $\alpha_i = 1$ $(i = 1,\dots,s+1)$, $\beta_j = 1$ $(j = 1,\dots,s)$, $\lambda = 1$ and $\ell = 0$, we obtain the Sălăgean operator D^m (see Sălăgean [27]);
(iv) For $q = s+1$, $\alpha_i = 1$ $(i = 1,\dots,s+1)$, $\beta_j = 1$ $(j = 1,\dots,s)$ and $\lambda = 1$, we get the operator I_ℓ^m (see Cho and Srivastava [8] and Cho and Kim [9]).
(v) For $q = s+1$, $\alpha_i = 1$ $(i = 1,\dots,s+1)$, $\beta_j = 1$ $(j = 1,\dots,s)$ and $\ell = 0$, we obtain the operator D_λ^m (see Al-Oboudi [2]).

By specializing the parameters m, λ, ℓ, q, s, α_i $(i = 1,\dots,q)$ and β_j $(j = 1,\dots,s)$ we obtain:
(i) $I_{2,1,\lambda}^{m,\ell}(n+1,1;1)f(z) = I_\lambda^{m,\ell}(n)f(z) = z + \sum_{k=2}^{\infty}\left[\frac{1+\ell+\lambda(k-1)}{1+\ell}\right]^m \frac{(n+1)_{k-1}}{(1)_{k-1}}a_k z^k$ $(n > -1)$;
(ii) $I_{2,1,\lambda}^{m,\ell}(a,1;c)f(z) = I_\lambda^{m,\ell}(a;c)f(z) = z + \sum_{k=2}^{\infty}\left[\frac{1+\ell+\lambda(k-1)}{1+\ell}\right]^m \frac{(a)_{k-1}}{(c)_{k-1}}a_k z^k$ $(a \in \mathbb{R};\ c \in \mathbb{R}\setminus\mathbb{Z}_0^-)$;
(iii) $I_{2,1,\lambda}^{m,\ell}(2,1;n+1)f(z) = I_{\lambda,n}^{m,\ell}f(z) = z + \sum_{k=2}^{\infty}\left[\frac{1+\ell+\lambda(k-1)}{1+\ell}\right]^m \frac{(2)_{k-1}}{(n+1)_{k-1}}a_k z^k$ $(n \in \mathbb{Z};\ n > -1)$.

In 1976, Noonan and Thomas [23] discussed the qth Hankel determinant of a locally univalent analytic function $f(z)$ for $q \geq 1$ and $n \geq 1$ which is defined by

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \dots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \dots & a_{n+q} \\ \vdots & \vdots & \dots & \vdots \\ a_{n+q-1} & a_{n+q} & \dots & a_{n+2q-2} \end{vmatrix}.$$

For our present discussion, we consider the Hankel determinant in the case $q = 2$ and $n = 2$, i.e. $H_2(2) = a_2a_4 - a_3^2$. This is popularly known as the second Hankel determinant of f.

In this paper, we define the following class $S_\gamma^b(g(z); A, B)$ $(0 \leq \gamma \leq 1,\ b \in \mathbb{C}^* = \mathbb{C}\setminus\{0\})$ as follows:

Definition 1. Let $0 \leq \gamma \leq 1$, $b \in \mathbb{C}^*$. A function $f(z) \in \mathcal{A}$ is said to be in the class $S_\gamma^b\left(g(z); A, B\right)$ if

$$1+\frac{1}{b}\left((1-\gamma)\frac{(f*g)(z)}{z}+\gamma\left(f*g\right)'(z)-1\right)\prec\frac{1+Az}{1+Bz} \tag{1.9}$$

($b \in \mathbb{C}^*$; $0 \leq \gamma \leq 1$; $-1 \leq B < A \leq 1$; $z \in \mathbb{U}$), which is equivalent to say that

$$\left|\frac{(1-\gamma)\frac{(f*g)(z)}{z}+\gamma\left(f*g\right)'(z)-1}{b(A-B)-B\left[(1-\gamma)\frac{(f*g)(z)}{z}+\gamma\left(f*g\right)'(z)-1\right]}\right|<1.$$

We note that for suitable choices of b, γ and $g(z)$ we obtain the following subclasses:
(i) $S_\gamma^b\big(\frac{z}{1-z}; A, B\big) = S_\gamma^b(A, B)$ $(0 \leq \gamma \leq 1,\ b \in \mathbb{C}^*,\ -1 \leq B < A \leq 1)$ (see Bansal [5]);
(ii) $S_0^{(1-\rho)e^{-i\theta}\cos\theta}\big(z+\sum_{k=2}^{\infty}\frac{(\alpha)_{k-1}}{(\beta)_{k-1}}z^k; 1, -1\big) = \mathcal{R}_{\alpha,\beta}(\theta, \rho)$ $(\frac{-\pi}{2} < \theta < \frac{\pi}{2}$, $0 \leq \rho < 1$, $\alpha \in \mathbb{C}$, $\beta \in \mathbb{C} \setminus \mathbb{Z}_0^-)$ (see Mishra and Kund [21]);
(iii) $S_0^{(1-\rho)e^{-i\alpha}\cos\alpha}\big(z+\sum_{k=2}^{\infty}\frac{(\lambda+1)_{k-1}}{(m)_{k-1}}k^n z^k; 1, -1\big) = S_m^{\lambda,n}(\alpha, \sigma)$ $(m \in \mathbb{N}$; $n, \lambda \in \mathbb{N}_0$; $|\alpha| < \frac{\pi}{2}$; $0 \leq \sigma < 1)$ (see Mohammed and Darus [22]);
(iv) $S_1^1\big(z+\sum_{k=2}^{\infty}\left[1+(\alpha\mu k+\alpha-\mu)(k-1)\right]^\sigma(\rho)_{k-1}z^k; 1, -1\big) = R_{\alpha,\mu}(\sigma, \rho)$ $(0 \leq \mu \leq \alpha \leq 1$; $\rho, \sigma \in \mathbb{N}_0)$ (see Abubaker and Darus [1]);
(v) $S_\gamma^b\left(z+\sum_{k=2}^{\infty}k^m z^k; A, B\right) = G_m(\gamma, b)$ $(b \in \mathbb{C}^*,\ 0 \leq \gamma \leq 1,\ m \in \mathbb{N}_0)$ (see Aouf [3]).

Also, we note that:
(i) $S_\gamma^b\Big(z+\sum_{k=2}^{\infty}\Big[\frac{1+\ell+\lambda(k-1)}{1+\ell}\Big]^m\Gamma_k(\alpha_1)z^k; A, B\Big) = S_\gamma^b(\lambda, \ell, m, q, s, \alpha_1, \beta_1; A, B)$

$$=\Big\{f(z)\in\mathcal{A}: 1+\tfrac{1}{b}\Big((1-\gamma)\tfrac{I_{q,s,\lambda}^{m,\ell}(\alpha_1,\beta_1)f(z)}{z}+\gamma\Big(I_{q,s,\lambda}^{m,\ell}(\alpha_1,\beta_1)f(z)\Big)'-1\Big)$$
$$\prec\tfrac{1+Az}{1+Bz},\ (b\in\mathbb{C}^*;\ 0\leq\gamma\leq1;\ m\in\mathbb{N}_0;\ \ell\geq0;\ \lambda\geq0;\ q\leq s+1;$$
$$q,s\in\mathbb{N}_0;\ z\in\mathbb{U})\Big\};$$

(ii) $S_\gamma^b\Big(z+\sum_{k=2}^{\infty}\Big[\frac{1+\ell}{1+\ell+\lambda(k-1)}\Big]^m z^k; A, B\Big) = S_\gamma^b(\lambda, \ell, m; A, B)$

$$=\Big\{f(z)\in\mathcal{A}: 1+\tfrac{1}{b}\Big((1-\gamma)\tfrac{J^m(\lambda,\ell)f(z)}{z}+\gamma(J^m(\lambda,\ell)f(z))'-1\Big)$$
$$\prec\tfrac{1+Az}{1+Bz},\ (b\in\mathbb{C}^*;\ 0\leq\gamma\leq1;\ m\in\mathbb{N}_0;\ \ell\geq0;\ \lambda\geq0;\ z\in\mathbb{U})\Big\};$$

(iii) $S_{\gamma}^{(1-\rho)\cos\eta e^{-i\eta}}(g(z);A,B) = S^{\gamma}[\rho,\eta,A,B,g(z)]$

$$= \Big\{ f(z) \in \mathcal{A} : e^{i\eta}\Big[(1-\gamma)\tfrac{(f*g)(z)}{z} + \gamma\,(f*g)'(z)\Big]$$
$$\prec (1-\rho)\cos\eta \cdot \tfrac{1+Az}{1+Bz} + \rho\cos\eta + i\sin\eta,$$
$$\big(|\eta| < \tfrac{\pi}{2};\ 0 \le \gamma \le 1;\ 0 \le \rho < 1;\ -1 \le B < A \le 1;\ z \in \mathbb{U}\big)\Big\}$$

In this paper, we obtain the Fekete–Szegö inequalities for the functions in the class $S_{\gamma}^{b}(g(z);A,B)$. We also obtain an upper bound to the functional $H_2(2)$ for $f(z) \in S_{\gamma}^{b}(g(z);A,B)$. Earlier Janteng et al. [16], Mishra and Gochhayat [20], Mishra and Kund [21], Bansal [4] and many other authors have obtained sharp upper bounds of $H_2(2)$ for different classes of analytic functions.

2. Preliminaries. To prove our results, we need the following lemmas.

Lemma 1 ([26]). *Let*

$$h(z) = 1 + \sum_{n=1}^{\infty} c_n z^n \prec 1 + \sum_{n=1}^{\infty} C_n z^n = H(z) \quad (z \in \mathbb{U}). \tag{2.1}$$

If the function H is univalent in $\mathbb{U}$ and $H(\mathbb{U})$ is a convex set, then

$$|c_n| \le |C_1|. \tag{2.2}$$

Lemma 2 ([10]). *Let a function $p \in \mathcal{P}$ be given by*

$$p(z) = 1 + c_1 z + c_2 z^2 + \dots \quad (z \in \mathbb{U}), \tag{2.3}$$

then, we have

$$|c_n| \le 2 \quad (n \in \mathbb{N}). \tag{2.4}$$

The result is sharp.

Lemma 3 ([17, 18]). *Let $p \in \mathcal{P}$ be given by the power series* (2.3)*, then for any complex number ν*

$$\left|c_2 - \nu c_1^2\right| \le 2\max\{1; |2\nu - 1|\}. \tag{2.5}$$

The result is sharp for the functions given by

$$p(z) = \frac{1+z^2}{1-z^2} \quad \text{and} \quad p(z) = \frac{1+z}{1-z} \quad (z \in \mathbb{U}).$$

Lemma 4 ([15]). *Let a function $p \in \mathcal{P}$ be given by the power series* (2.3)*, then*

$$2c_2 = c_1^2 + \varkappa(4 - c_1^2) \tag{2.6}$$

for some $\varkappa$, $|\varkappa| \le 1$, and

$$4c_3 = c_1^3 + 2(4-c_1^2)c_1\varkappa - c_1(4-c_1^2)\varkappa^2 + 2(4-c_1^2)\left(1-|\varkappa|^2\right)z, \tag{2.7}$$

for some z, $|z| \le 1$.

3. Main results. We give the following result related to the coefficient of $f(z) \in S_\gamma^b(g(z); A, B)$.

Theorem 1. *Let $f(z)$ given by (1.1) belong to the class $S_\gamma^b(g(z); A, B)$, $0 \le \gamma \le 1$, $-1 \le B < A \le 1$ and $b \in \mathbb{C}^*$, then*

$$|a_k| \le \frac{(A-B)|b|}{[1+\gamma(k-1)]\, b_k} \quad (k \in \mathbb{N} \setminus \{1\}). \tag{3.1}$$

Proof. If $f(z)$ of the form (1.1) belongs to the class $S_\gamma^b(g(z); A, B)$, then

$$1 + \frac{1}{b}\left((1-\gamma)\frac{(f*g)(z)}{z} + \gamma (f*g)'(z) - 1\right) \prec \frac{1+Az}{1+Bz} = h(z)$$

($b \in \mathbb{C}^*$; $0 \le \gamma \le 1$; $-1 \le B < A \le 1$; $z \in \mathbb{U}$), where $h(z)$ is convex univalent in $\mathbb{U}$ and we have

$$\begin{aligned} &1 + \frac{1}{b}\left((1-\gamma)\tfrac{(f*g)(z)}{z} + \gamma (f*g)'(z) - 1\right) \\ &= 1 + \sum_{k=1}^{\infty} \frac{(1+k\gamma)}{b} b_{k+1} a_{k+1} z^k \prec 1 + (A-B)z - B(A-B)z^2 + \dots \end{aligned} \tag{3.2}$$

($z \in \mathbb{U}$). Now, by applying Lemma 1, we get the desired result. □

Remark 1. Putting $g(z) = \frac{z}{1-z}$ in Theorem 1, we obtain the result obtained by Bansal [5, Theorem 2.1].

It is easy to derive a sufficient condition for $f(z)$ to be in the class $S_\gamma^b(m, \lambda, \ell; A, B)$ using standard techniques (see [25]). Hence we state the following result without proof.

Theorem 2. *Let $f(z) \in \mathcal{A}$, then a sufficient condition for $f(z)$ to be in the class $S_\gamma^b(g(z); A, B)$ is*

$$\sum_{k=2}^{\infty} [1+\gamma(k-1)]\, b_k |a_k| \le \frac{(A-B)|b|}{1+B}. \tag{3.3}$$

In the next two theorems, we obtain the result concerning Fekete–Szegö inequality and an upper bound for the Hankel determinant for the class $S_\gamma^b(g(z); A, B)$.

Remark 2. Putting $g(z) = \frac{z}{1-z}$ in Theorem 2, we obtain the result obtained by Bansal [5, Theorem 2.2].

Theorem 3. *Let $f(z)$ given by (1.1) belong to the class $S_\gamma^b(g(z); A, B)$, $0 \le \gamma \le 1$, $-1 \le B < A \le 1$ and $b \in \mathbb{C}^*$, then*

$$\left|a_3 - \mu a_2^2\right| \le \frac{(A-B)|b|}{(1+2\gamma)\, b_3} \cdot \max\left\{1, \left|B + \frac{\mu b b_3 (A-B)(1+2\gamma)}{(1+\gamma)^2 b_2^2}\right|\right\}. \tag{3.4}$$

This result is sharp.

Proof. Let $f(z) \in S_{\gamma}^{b}(g(z); A, B)$, then there is a Schwarz function $w(z)$ in U with $w(0) = 0$ and $|w(z)| < 1$ in U and such that

$$1 + \frac{1}{b}\left((1-\gamma)\frac{(f*g)(z)}{z} + \gamma(f*g)'(z) - 1\right) = \Phi(w(z)) \tag{3.5}$$

($z \in \mathbb{U}$), where

$$\begin{aligned}\Phi(z) &= \frac{1+Az}{1+Bz} = 1 + (A-B)z - B(A-B)z^2 + B^2(A-B)z^3 - \dots \\ &= 1 + B_1 z + B_2 z^2 + B_3 z^3 + \dots\end{aligned} \tag{3.6}$$

($z \in \mathbb{U}$). If the function $p_1(z)$ is analytic and has positive real part in U and $p_1(0) = 1$, then

$$p_1(z) = \frac{1+w(z)}{1-w(z)} = 1 + c_1 z + c_2 z^2 + \dots \tag{3.7}$$

($z \in \mathbb{U}$), since $w(z)$ is a Schwarz function. Define

$$\begin{aligned}h(z) &= 1 + \frac{1}{b}\left((1-\gamma)\frac{(f*g)(z)}{z} + \gamma(f*g)'(z) - 1\right) \\ &= 1 + d_1 z + d_2 z^2 + \dots\end{aligned} \tag{3.8}$$

($z \in \mathbb{U}$). In view of the equations (3.5) and (3.7), we have

$$p(z) = \Phi\left(\frac{p_1(z)-1}{p_1(z)+1}\right).$$

Since

$$\frac{p_1(z)-1}{p_1(z)+1} = \frac{1}{2}\left[c_1 z + \left(c_2 - \frac{c_1^2}{2}\right)z^2 + \left(c_3 + \frac{c_1^3}{4} - c_1 c_2\right)z^3 + \dots\right], \tag{3.9}$$

we have

$$\Phi\left(\frac{p_1(z)-1}{p_1(z)+1}\right) = 1 + \frac{1}{2}B_1 c_1 z + \left[\frac{1}{2}B_1\left(c_2 - \frac{c_1^2}{2}\right) + \frac{1}{4}B_2 c_1^2\right]z^2 + \dots, \tag{3.10}$$

and from this equation and (3.8), we obtain

$$d_1 = \frac{1}{2}B_1 c_1, \quad d_2 = \frac{1}{2}B_1\left(c_2 - \frac{c_1^2}{2}\right) + \frac{1}{4}B_2 c_1^2 \tag{3.11}$$

and

$$d_3 = \frac{B_1}{2}\left(c_3 - c_1 c_2 + \frac{c_1^3}{4}\right) + \frac{B_2 c_1}{2}\left(c_2 - \frac{c_1^2}{2}\right) + \frac{B_3 c_1^3}{8}. \tag{3.12}$$

Then, from (3.6), we see that

$$d_1 = \frac{(1+\gamma)\, b_2 a_2}{b} \quad \text{and} \quad d_2 = \frac{(1+2\gamma)\, b_3 a_3}{b}. \tag{3.13}$$

Now from (3.6), (3.8) and (3.13), we have

$$a_2 = \frac{(A-B)\,bc_1}{2(1+\gamma)\,b_2}, \quad a_3 = \frac{b(A-B)}{4(1+2\gamma)\,b_3}\left\{2c_2 - c_1^2(1+B)\right\} \tag{3.14}$$

and

$$a_4 = \frac{b(A-B)}{8(1+3\gamma)\,b_4}\left\{4c_3 - 4c_1c_2(1+B) + c_1^3(1+B)^2\right\} \tag{3.15}$$

Therefore, we have

$$a_3 - \mu a_2^2 = \frac{b(A-B)}{2(1+2\gamma)\,b_3}\left\{c_2 - \nu c_1^2\right\}, \tag{3.16}$$

where

$$\nu = \frac{1}{2}\left[1 + B + \frac{\mu b(A-B)(1+2\gamma)\,b_3}{(1+\gamma)^2\,b_2^2}\right]. \tag{3.17}$$

Our result now follows by an application of Lemma 3. The result is sharp for the functions

$$1 + \frac{1}{b}\left((1-\gamma)\frac{(f*g)(z)}{z} + \gamma\,(f*g)'(z) - 1\right) = \Phi(z^2) \tag{3.18}$$

and

$$1 + \frac{1}{b}\left((1-\gamma)\frac{(f*g)(z)}{z} + \gamma\,(f*g)'(z) - 1\right) = \Phi(z). \tag{3.19}$$

This completes the proof of Theorem 3. □

Remark 3. Putting $g(z) = \frac{z}{1-z}$ in Theorem 3, we obtain the result due to Bansal [5, Theorem 2.3].

Putting $g(z) = z + \sum_{k=2}^{\infty}\left[\frac{1+\ell+\lambda(k-1)}{1+\ell}\right]^m \Gamma_k(\alpha_1)$ ($m \in \mathbb{N}_0$, $\ell \geq 0$, $\lambda \geq 0$, $q \leq s+1$, $q, s \in \mathbb{N}_0$), where $\Gamma_k(\alpha_1)$ is given by (1.6) in Theorem 3, we obtain the following corollary.

Corollary 1. *Let $f(z)$ given by (1.1) belong to the class $S_\gamma^b(\lambda, \ell, m, q, s, \alpha_1, \beta_1; A, B)$, $0 \leq \gamma \leq 1$, $-1 \leq B < A \leq 1$, $m \in \mathbb{N}_0$, $\ell \geq 0$, $\lambda \geq 0$, $q \leq s+1$, $q, s \in \mathbb{N}_0$ and $b \in \mathbb{C}^*$, then*

$$\begin{aligned}\left|a_3 - \mu a_2^2\right| \leq\; & \frac{(A-B)(1+\ell)^m\,|b|}{(1+2\gamma)(1+\ell+2\lambda)^m\,\Gamma_3(\alpha_1)} \\ & \times \max\left\{1, \left|B + \frac{\mu b\left[\frac{1+\ell+2\lambda}{1+\ell}\right]^m \Gamma_3(\alpha_1)(A-B)(1+2\gamma)}{(1+\gamma)^2\left[\frac{1+\ell+\lambda}{1+\ell}\right]^{2m}\Gamma_2^2(\alpha_1)}\right|\right\}.\end{aligned} \tag{3.20}$$

This result is sharp.

Putting $g(z) = z + \sum_{k=2}^{\infty}\left[\frac{1+\ell}{1+\ell+\lambda(k-1)}\right]^m z^k$ ($m \in \mathbb{N}_0$; $\ell \geq 0$; $\lambda \geq 0$) in Theorem 3, we obtain the following corollary.

Corollary 2. *Let $f(z)$ given by* (1.1) *belong to the class $S_{\gamma}^{b}(\lambda,\ell,m;A,B)$, $0\le\gamma\le 1$, $-1\le B<A\le 1$, $m\in\mathbb{N}_0$, $\ell\ge 0$, $\lambda\ge 0$ and $b\in\mathbb{C}^*$, then*

$$\left|a_3-\mu a_2^2\right|\le\frac{(A-B)\,|b|}{(1+2\gamma)}\left[\tfrac{1+\ell+2\lambda}{1+\ell}\right]^m \times\max\left\{1,\left|B+\frac{\mu b\left[\frac{1+\ell}{1+\ell+2\lambda}\right]^m(A-B)(1+2\gamma)}{(1+\gamma)^2\left[\frac{1+\ell}{1+\ell+\lambda}\right]^{2m}}\right|\right\}. \tag{3.21}$$

This result is sharp.

Putting $b=(1-\rho)\,e^{-i\eta}\cos\eta$ $(|\eta|<\frac{\pi}{2},\ 0\le\rho<1)$ in Theorem 3, we obtain the following corollary.

Corollary 3. *Let $f(z)$ given by* (1.1) *belong to the class $S^{\gamma}[\rho,\eta,A,B,g(z)]$, $0\le\gamma\le 1$, $-1\le B<A\le 1$ and $b\in\mathbb{C}^*$, then*

$$\left|a_3-\mu a_2^2\right|\le\frac{(A-B)(1-\rho)\cos\eta}{(1+2\gamma)\,b_3} \times\max\left\{1,\left|B+\frac{\mu b_3(A-B)(1+2\gamma)(1-\rho)\,e^{-i\eta}\cos\eta}{(1+\gamma)^2\,b_2^2}\right|\right\}. \tag{3.22}$$

This result is sharp.

Theorem 4. *Let $f(z)$ given by* (1.1) *belong to the class $S_{\gamma}^{b}(g(z);A,B)$, $0\le\gamma\le 1$, $-1\le B<A\le 1$ and $b\in\mathbb{C}^*$, then*

$$\left|a_2a_4-a_3^2\right|\le\frac{(A-B)^2\,|b|^2}{(1+2\gamma)^2\,b_3^2}. \tag{3.23}$$

Proof. Using (3.14) and (3.15), we have

$$\begin{aligned}\left|a_2a_4-a_3^2\right|&=\frac{(A-B)^2\,|b|^2}{16(1+\gamma)(1+3\gamma)\,b_2b_4}\Bigg|4c_1c_3-4c_1^2c_2(1+B)+c_1^4(1+B)^2\\&\quad-\frac{(1+\gamma)(1+3\gamma)\,b_2b_4}{(1+2\gamma)^2\,b_3^2}\left[4c_2^2-4c_1^2c_2(1+B)+c_1^4(1+B)^2\right]\Bigg|\\&=M\,\big|4c_1c_3-4c_1^2c_2(1+B)+c_1^4(1+B)^2\\&\quad-N\left[4c_2^2-4c_1^2c_2(1+B)+c_1^4(1+B)^2\right]\big|,\end{aligned} \tag{3.24}$$

where

$$M=\frac{(A-B)^2\,|b|^2}{16(1+\gamma)(1+3\gamma)\,b_2b_4}\quad\text{and}\quad N=\frac{(1+\gamma)(1+3\gamma)\,b_2b_4}{(1+2\gamma)^2\,b_3^2}. \tag{3.25}$$

The above equation (3.24) is equivalent to

$$\left|a_2a_4-a_3^2\right|=M\left|4c_1c_3+d_2c_1^2c_2+d_3c_2^2+d_4c_1^4\right|, \tag{3.26}$$

where

$$d_1 = 4, \;\; d_2 = -4(1+B)(1-N), \;\; d_3 = -4N, \;\; d_4 = (1-N)(1+B)^2. \tag{3.27}$$

Since the functions $p(z)$ and $p(re^{i\theta})$ $(\theta \in \mathbb{R})$ are members of the class $\mathcal{P}$ simultaneously, we assume without loss of generality that $c_1 > 0$. For convenience of notation, we take $c_1 = c$ $(c \in [0,2]$, see (2.4)). Also, substituting the values of c_2 and c_3, respectively, from (2.6) and (2.7) in (3.26), we have

$$\begin{aligned}\left|a_2a_4 - a_3^2\right| = \frac{M}{4} &\left|c^4(d_1 + 2d_2 + d_3 + 4d_4) + 2\varkappa c^2(4-c^2)(d_1 + d_2 + d_3)\right. \\ &\left. + (4-c^2)\varkappa^2(-d_1c^2 + d_3(4-c^2)) + 2d_1c(4-c^2)\left(1 - |\varkappa|^2 z\right)\right|.\end{aligned}$$

An application of triangle inequality, replacement of $|\varkappa|$ by ν and substituting the values of $d_1, \; d_2, \; d_3$ and d_4 from (3.27), we have

$$\begin{aligned}\left|a_2a_4 - a_3^2\right| &\le \frac{M}{4}\left[4c^4(1-N)B^2 + 8|B|(1-N)\nu c^2(4-c^2)\right. \\ &\quad \left. + (4-c^2)\nu^2\left(4c^2 + 4N(4-c^2)\right) + 8c(4-c^2)\left(1-\nu^2\right)\right] \\ &= M\left[c^4(1-N)B^2 + 2c(4-c^2) + 2\nu|B|(1-N)c^2(4-c^2)\right. \\ &\quad \left. + \nu^2(4-c^2)\left(c^2(1-N) - 2c + 4N\right)\right] \\ &= F(c,\nu).\end{aligned} \tag{3.28}$$

Next, we assume that the upper bound for (3.28) occurs at an interior point of the rectangle $[0,2]\times[0,1]$. Differentiating $F(c,\nu)$ in (3.28) partially with respect to ν, we have

$$\begin{aligned}\frac{\partial F(c,\nu)}{\partial \nu} &= M\left[2|B|(1-N)c^2(4-c^2)\right. \\ &\quad \left. + 2\nu(4-c^2)\left(c^2(1-N) - 2c + 4N\right)\right].\end{aligned} \tag{3.29}$$

For $0 < \nu < 1$ and for any fixed c with $0 < c < 2$, from (3.29), we observe that $\frac{\partial F}{\partial \nu} > 0$. Therefore, $F(c,\nu)$ is an increasing function of ν, which contradicts our assumption that the maximum value of $F(c,\nu)$ occurs at an interior point of the rectangle $[0,2]\times[0,1]$. Moreover, for fixed $c \in [0,2]$,

$$\max F(c,\nu) = F(c,1) = G(c). \tag{3.30}$$

Thus

$$\begin{aligned}G(c) = M&\left[c^4(1-N)\left(B^2 - 2|B| - 1\right)\right. \\ &\left. + 4c^2(2|B|(1-N) + 1 - 2N) + 16N\right].\end{aligned} \tag{3.31}$$

Next,

$$\begin{aligned}G'(c) &= 4Mc\left[c^2(1-N)\left(B^2 - 2|B| - 1\right) + 2(2|B|(1-N) + 1 - 2N\right] \\ &= 4Mc\left[c^2(1-N)\left(B^2 - 2|B| - 1\right) + 2\left\{(1-N)(2|B|+1) - N\right\}\right].\end{aligned}$$

So $G^{'}(c) < 0$ for $0 < c < 2$ and has a real critical point at $c = 0$. Also $G(c) > G(2)$. Therefore, maximum of $G(c)$ occurs at $c = 0$. Therefore, the upper bound of $F(c, \nu)$ corresponds to $\nu = 1$ and $c = 0$. Hence,

$$\left|a_2 a_4 - a_3^2\right| \leq 16MN = \frac{(A-B)^2 |b|^2}{(1+2\gamma)^2 b_3^2}.$$

This completes the proof of Theorem 4. □

Remark 4. (i) Putting $g(z) = \frac{z}{1-z}$ in Theorem 4, we obtain the result due to Bansal [5, Theorem 2.4];
(ii) Putting

$$g(z) = z + \sum_{k=2}^{\infty} \frac{(\alpha)_{k-1}}{(\beta)_{k-1}} z^k$$

($\alpha \in \mathbb{C}$, $\beta \in \mathbb{C} \setminus \mathbb{Z}_0^-$), $b = (1-\rho)e^{-i\theta} \cos\theta$ ($|\theta| < \frac{\pi}{2}$, $0 \leq \rho < 1$), $\gamma = 0$, $A = 1$ and $B = -1$ in Theorem 4, we obtain the result due to Mishra and Kund [21, Theorem 3.1];
(iii) Putting

$$g(z) = z + \sum_{k=2}^{\infty} \frac{(\lambda+1)_{k-1}}{(m)_{k-1}} k^n z^k$$

($m \in \mathbb{N}$; $\lambda, n \in \mathbb{N}_0$), $b = (1-\rho)e^{-i\alpha} \cos\alpha$ ($|\alpha| < \frac{\pi}{2}$; $0 \leq \sigma < 1$), $\gamma = 0$, $A = 1$ and $B = -1$ in Theorem 4, we obtain the result due to Mohammed and Darus [22, Theorem 2.1];
(iv) Putting

$$g(z) = z + \sum_{k=2}^{\infty} \left[1 + (\alpha\mu k + \alpha - \mu)(k-1)\right]^{\sigma} (\rho)_{k-1} z^k$$

($0 \leq \mu \leq \alpha \leq 1$, $\rho, \sigma \in \mathbb{N}_0$), $b = \gamma = A = 1$ and $B = -1$ in Theorem 4, we obtain the result due to Abubaker and Darus [1, Theorem 3.1].

Putting $g(z) = z + \sum_{k=2}^{\infty} \left[\frac{1+\ell+\lambda(k-1)}{1+\ell}\right]^m \Gamma_k(\alpha_1)$ ($m \in \mathbb{N}_0$, $\ell \geq 0$, $\lambda \geq 0$, $q \leq s+1$, $q, s \in \mathbb{N}_0$), where $\Gamma_k(\alpha_1)$ is given by (1.6) in Theorem 4, we obtain the following corollary.

Corollary 4. *Let $f(z)$ given by (1.1) belong to the class $S_\gamma^b(\lambda, \ell, m, q, s, \alpha_1, \beta_1; A, B)$, $0 \leq \gamma \leq 1$, $-1 \leq B < A \leq 1$, $m \in \mathbb{N}_0$, $\ell \geq 0$, $\lambda \geq 0$, $q \leq s+1$, $q, s \in \mathbb{N}_0$ and $b \in \mathbb{C}^*$, then*

$$\left|a_2 a_4 - a_3^2\right| \leq \frac{(A-B)^2 |b|^2}{(1+2\gamma)^2 \left[\frac{1+\ell+2\lambda}{1+\ell}\right]^{2m} \Gamma_3^2(\alpha_1)}. \tag{3.32}$$

Putting $g(z) = z + \sum_{k=2}^{\infty} \left[\frac{1+\ell}{1+\ell+\lambda(k-1)}\right]^m z^k$ ($m \in \mathbb{N}_0$; $\ell \geq 0$; $\lambda \geq 0$) in Theorem 4, we obtain the following corollary.

Corollary 5. *Let $f(z)$ given by (1.1) belong to the class $S_{\gamma}^{b}(\lambda, \ell, m; A, B)$, $0 \leq \gamma \leq 1$, $-1 \leq B < A \leq 1$, $m \in \mathbb{N}_0$, $\ell \geq 0$, $\lambda \geq 0$ and $b \in \mathbb{C}^*$, then*

$$\left|a_2 a_4 - a_3^2\right| \leq \frac{(A-B)^2 |b|^2}{(1+2\gamma)^2 \left[\frac{1+\ell}{1+\ell+2\lambda}\right]^{2m}}. \tag{3.33}$$

Putting $b = (1-\rho)e^{-i\eta}\cos\eta$ ($|\eta| < \frac{\pi}{2}$, $0 \leq \rho < 1$) in Theorem 4, we obtain the following corollary.

Corollary 6. *Let $f(z)$ given by (1.1) belong to the class $S^{\gamma}[\rho, \eta, A, B, g(z)]$, $0 \leq \gamma \leq 1$, $-1 \leq B < A \leq 1$ and $b \in \mathbb{C}^*$, then*

$$\left|a_2 a_4 - a_3^2\right| \leq \frac{(A-B)^2 (1-\rho)^2 \cos^2\eta}{(1+2\gamma)^2 b_3^2}. \tag{3.34}$$

References

[1] Abubaker, A., Darus, M., *Hankel determinant for a class of analytic functions involving a generalized linear differential operator*, Internat. J. Pure Appl. Math. **69** (3) (2011), 429–435.

[2] Al-Oboudi, F. M., *On univalent functions defined by a generalized Salagean operator*, Internat. J. Math. Math. Sci. **27** (2004), 1429–1436.

[3] Aouf, M. K., *Subordination properties for a certain class of analytic functions defined by the Salagean operator*, Appl. Math. Lett. **22** (2009), 1581–1585.

[4] Bansal, D., *Upper bound of second Hankel determinant for a new class of analytic functions*, Appl. Math. Lett. **26** (1) (2013), 103–107.

[5] Bansal, D., *Fekete–Szegö problem and upper bound of second Hankel determinant for a new class of analytic functions*, Kyungpook Math. J. **54** (2014), 443–452.

[6] Bulboacă, T., *Differential Subordinations and Superordinations, Recent Results*, House of Scientific Book Publ., Cluj-Napoca, 2005.

[7] Cătaş, A., *On certain classes of p-valent functions defined by multiplier transformations*, in *Proceedings of the International Symposium on Geometric Function Theory and Applications*, (Istanbul, Turkey, August 2007), Istanbul, 2008, 241–250.

[8] Cho, N. E., Srivastava, H. M., *Argument estimates of certain analytic functions defined by a class of multiplier transformations*, Math. Comput. Modelling **37** (1–2) (2003), 39–49.

[9] Cho, N. E., Kim, T. H., *Multiplier transformations and strongly close-to-convex functions*, Bull. Korean Math. Soc. **40** (3) (2003), 399–410.

[10] Duren, P. L., *Univalent Functions*, Springer-Verlag, New York, 1983.

[11] Dziok, J., Srivastava, H. M., *Classes of analytic functions associated with the generalized hypergeometric function*, Appl. Math. Comput. **103** (1999), 1–13.

[12] Dziok, J., Srivastava, H. M., *Certain subclasses of analytic functions associated with the generalized hypergeometric function*, Integral Transforms Spec. Funct. **14** (2003), 7–18.

[13] El-Ashwah, R. M., Aouf, M. K., *Differential subordination and superordination for certain subclasses of p-valent functions*, Math. Comput. Modelling **51** (2010), 349–360.

[14] El-Ashwah, R. M., Aouf, M. K., *Some properties of new integral operator*, Acta Univ. Apulensis **24** (2010), 51–61.

[15] Grenander, U., Szegö, G., *Toeplitz Forms and Their Application*, Univ. of California Press, Berkeley, 1958.

[16] Janteng, A., Halim, S. A., Darus, M., *Hankel determinant for starlike and convex functions*, Int. J. Math. Anal. (Ruse) **1** (13) (2007), 619–625.
[17] Keogh, F. R., Markes, E. P., *A coefficient inequality for certain classes of analytic functions*, Proc. Amer. Math. Soc. **20** (1969), 8–12.
[18] Libera, R. J., Złotkewicz, E. J., *Coefficient bounds for the inverse of a function with derivative in* $\mathcal{P}$, Proc. Amer. Math. Soc. **87** (2) (1983), 251–257.
[19] Miller, S. S., Mocanu, P. T., *Differential Subordination: Theory and Applications*, Marcel Dekker Inc., New York–Basel, 2000.
[20] Mishra, A. K., Gochhayat, P., *Second Hankel determinant for a class of analytic functions defined by fractional derivative*, Int. J. Math. Math. Sci. (2008), Art. ID 153280, 1–10.
[21] Mishra, A. K., Kund, S. N., *Second Hankel determinant for a class of functions defined by the Carlson–Shaffer*, Tamkang J. Math. **44** (1) (2013), 73–82.
[22] Mohammed, A., Darus, M., *Second Hankel determinant for a class of analytic functions defined by a linear operator*, Tamkang J. Math. **43** (3) (2012), 455–462.
[23] Noonan, J. W., Thomas, D. K., *On the second Hankel determinant of areally mean p-valent functions*, Trans. Amer. Math. Soc. **223** (2) (1976), 337–346.
[24] Prajapat, J. K., *Subordination and superordination preserving properties for generalized multiplier transformation operator*, Math. Comput. Modelling **55** (2012), 1456–1465.
[25] Raina, R. K., Bansal, D., *Some properties of a new class of analytic functions defined in tems of a Hadmard product*, J. Inequal. Pure Appl. Math. **9** (2008), Art. 22, 1–9.
[26] Rogosinski, W., *On the coefficients of subordinate functions*, Proc. London Math. Soc. **48** (1943), 48–82.
[27] Sălăgean, G. S., *Subclasses of univalent functions*, in *Complex analysis – fifth Romanian-Finnish seminar*, Part 1 (Bucharest, 1981), Springer-Verlag, Berlin, 1983, 362–372.
[28] Selvaraj, C., Karthikeyan, K. R., *Differential subordinant and superordinations for certain subclasses of analytic functions*, Far East J. Math. Sci. **29** (2) (2008), 419–430.
[29] Srivastava, H. M., Karlsson, P. W., *Multiple Gaussian Hypergeometric Series*, Ellis Horwood Limited, Chichester; Halsted Press, New York, 1985.

S. M. El-Deeb
Department of Mathematics
Faculty of Science
Damietta University
New Damietta 34517
Egypt
e-mail: shezaeldeeb@yahoo.com

M. K. Aouf
Department of Mathematics
Faculty of Science
Mansoura University
Mansoura 35516
Egypt
e-mail: mkaouf127@yahoo.com

Received April 21, 2015