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Location of the critical points of certain polynomials

Abstract. Let $\mathbb{D}$ denote the unit disk $\{z : |z| < 1\}$ in the complex plane $\mathbb{C}$. In this paper, we study a family of polynomials P with only one zero lying outside $\overline{\mathbb{D}}$. We establish criteria for P to satisfy implying that each of P and P' has exactly one critical point outside $\overline{\mathbb{D}}$.

1. Introduction. Let P be a polynomial in the complex plane $\mathbb{C}$. We denote the degree of P by $\deg P$. We say that α is a critical point of P if $P'(\alpha) = 0$. Throughout this paper, if not otherwise stated, when we talk about the number of zeros of a polynomial in a domain, we mean the number of zeros counting multiplicities. As the critical points of P are the zeros of P', this applies also to the number of critical points. There are several known results involving the critical points of polynomials. The most classical one is the *Gauss–Lucas Theorem*, [8, p. 25].

Gauss–Lucas Theorem. *Let P be a polynomial of degree n with zeros $z_1, z_2, \ldots, z_n$, not necessarily distinct. The zeros of the derivative P' lie in the convex hull of the set $\{z_1, z_2, \ldots, z_n\}$.*

Another classical theorem concerning the location of the critical points is the *Walsh's Two-Circle Theorem*, [9].

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Walsh's Two-Circle Theorem. *Let P be a polynomial of degree $n \geq 2$. Let n_1 and n_2 be positive integers with $n_1 + n_2 = n$, let α_1 and α_2 be two distinct complex numbers, and let r_1, r_2 be positive real numbers. Let $C_1 = \{z : |z - \alpha_1| \leq r_1\}$, $C_2 = \{z : |z - \alpha_2| \leq r_2\}$, and let $C = \{z : |z - \alpha_0| \leq r\}$, where*

$$\alpha_0 = \frac{\alpha_2 n_1 + \alpha_1 n_2}{n} \qquad \text{and} \qquad r = \frac{n_1 r_2 + n_2 r_1}{n}.$$

Assume that P has n_1 and n_2 zeros in C_1 and C_2 respectively. Then all critical points of P lie in $C_1 \cup C_2 \cup C$.

In this paper we are interested in the location of the critical points of a certain type of polynomials. If P has a zero lying outside the closed unit disk $\overline{\mathbb{D}} = \{z \in \mathbb{C} : |z| \leq 1\}$, by the Gauss–Lucas Theorem, it follows that the zeros of its derivative are in the convex hull of the zeros of P, which includes a region outside $\overline{\mathbb{D}}$. But we do not know how many zeros of P' are outside $\overline{\mathbb{D}}$. We may ask the question of *under what conditions does P have only one critical point outside the closed unit disk*? A consequence of Walsh's theorem gives a partial answer to the question. That is,

Theorem ([5, see (4.1.1) on p. 117]). *If $S \in \{C_1, C_2, C\}$ is a disjoint component of $C_1 \cup C_2 \cup C$, then S contains exactly*

$$n(S) = \begin{cases} n_j - 1 & \text{if} \quad S = C_j \\ 1 & \text{if} \quad S = C \end{cases}$$

critical points of P.

Let P be a polynomial of degree $n \geq 2$ that has only one zero, say α_n, that lies outside the closed unit disk $\overline{\mathbb{D}}$. Let $C_1 = \overline{\mathbb{D}}$ and $C_2 = \{z : |z - \alpha_n| \leq r_2\}$. By taking $r_2 \to 0^+$ we see by the above theorem that if $|\alpha_n| > \frac{n+1}{n-1}$, then P has exactly one critical point α in $C = \left\{|z - \left(\frac{n-1}{n}\right)\alpha_n| \leq \frac{1}{n}\right\}$ while C does not intersect $\overline{\mathbb{D}}$. Hence P has exactly one critical point outside $\overline{\mathbb{D}}$ whenever $|\alpha_n| > \frac{n+1}{n-1}$.

Here we give a general criterion for determining the number of critical points outside $\overline{\mathbb{D}}$.

Theorem 1.1. *Let $Q(z) = c\prod_{k=1}^{n}(z - \alpha_k)$ be a polynomial of degree $n \geq 2$, where $c \neq 0$. Suppose that $\alpha_k \notin \overline{\mathbb{D}}$ for $1 \leq k \leq m$, and the remaining points α_k are in $\overline{\mathbb{D}}$. If we have*

$$\sum_{k=m+1}^{n} \frac{1}{1 + |\alpha_k|} > \sum_{k=1}^{m} \frac{1}{|\alpha_k| - 1},$$

then Q has exactly m critical points outside $\overline{\mathbb{D}}$, counting multiplicities. If, in addition, all the points α_k lying on the unit circle are simple zeros of Q, then Q' has no zeros on the unit circle.

Note that if Q has only one zero α_n lying outside $\overline{\mathbb{D}}$ with $|\alpha_n| > \frac{n+1}{n-1}$, which is the same condition as discussed previously, then by Theorem 1.1, Q has exactly one critical point outside $\overline{\mathbb{D}}$. From Theorem 1.1, we can deduce that the result still holds even though $|\alpha_n| \leq \frac{n+1}{n-1}$ if Q satisfies an additional condition.

Corollary 1.2. *Let $Q(z) = c\prod_{k=1}^{n}(z-\alpha_k)$ be a polynomial of degree $n \geq 2$, where $c \neq 0$. Suppose that $\alpha_1 = \alpha$, $\alpha_2 = \alpha^{-1}$, where α is real and $|\alpha| > 1$, and all the remaining points α_k, if any, are in $\overline{\mathbb{D}}$. Then Q has exactly one critical point outside $\overline{\mathbb{D}}$, counting multiplicities. If, in addition, all the points α_k that are on the unit circle are simple zeros of Q, then Q has exactly $n-2$ critical points in $\mathbb{D}$, counting multiplicities.*

A polynomial P is said to be *anti-reciprocal* if $P(z) = -z^{\deg P}P(z^{-1})$. If P is anti-reciprocal, then so is cP for any non-zero complex number c. Note that if P is anti-reciprocal, then 1 is a zero of P, we have $P(0) \neq 0$, and for $\alpha \neq 0$, we have $P(\alpha) = 0$ if, and only if, $P(\alpha^{-1}) = 0$. Furthermore, α and α^{-1} have the same multiplicity as zeros of P, as we see (for $\alpha \neq \pm 1$) by writing $P(z) = (z-\alpha)^m(z-1/\alpha)^n g(z)$, where $g(\alpha)g(1/\alpha) \neq 0$ and using $P(z) = -z^{\deg P}P(z^{-1})$. Therefore, if the leading coefficient of P is real and each zero of P is real or has modulus 1, then the coefficients of P are real. If P is an anti-reciprocal polynomial with exactly one zero, counting multiplicities, lying outside $\overline{\mathbb{D}}$, and which furthermore is real, then P satisfies the assumptions of Corollary 1.2, and so P has only one critical point outside $\overline{\mathbb{D}}$. Indeed, if P is anti-reciprocal with exactly one zero, say α, which is furthermore simple, outside $\overline{\mathbb{D}}$, then P has exactly one zero (namely, $1/\alpha$) in $\mathbb{D}$, and all the other zeros of P must lie on $\partial\mathbb{D}$. In Theorem 1.3, we prove that if P satisfies certain additional conditions, then not only does P' have only one zero outside $\overline{\mathbb{D}}$ but the same is also true for P''.

Theorem 1.3. *Let Q be an anti-reciprocal polynomial with real coefficients of degree $n \geq 3$. Suppose that the zeros of Q are simple and that $\alpha > 1$ is the only zero of Q lying outside $\overline{\mathbb{D}}$. Then each of the polynomials Q' and Q'' has exactly one zero outside $\overline{\mathbb{D}}$, counting multiplicities.*

We can construct a family of anti-reciprocal polynomials satisfying Theorem 1.3. Let P be a polynomial with real coefficients, and set $P^*(z) := z^{\deg P}P(z^{-1})$. Suppose that P has a real zero greater than 1, that the remaining zeros of P are in $\mathbb{D}$ (so $P(1) \neq 0$), and that $P^* \neq P$. Boyd [1, p. 320] showed that the polynomial

$$Q(z) = z^n P(z) - P^*(z) \tag{1}$$

satisfies the assumptions of Theorem 1.3 provided that $n > \deg P - 2\frac{P'(1)}{P(1)}$ and that all zeros of P are simple. The polynomial in (1) was originally introduced by R. Salem [6, Theorem IV, p. 166], [7, p. 30]. Therefore, this gives the following corollary.

Corollary 1.4. *Let P be a polynomial with real coefficients such that $P^* \neq P$. For $n > \deg P - 2\frac{P'(1)}{P(1)}$, let Q be defined as in* (1)*. Suppose that P has a real zero greater than 1, that the remaining zeros of P are in $\mathbb{D}$, and that all zeros of P are simple. Then each of Q, Q', and Q'' has exactly one zero outside $\overline{\mathbb{D}}$, counting multiplicities.*

2. Proof of Theorem 1.1.

Lemma 2.1. *Let $Q(z) = c\prod_{k=1}^{n}(z-\alpha_k)$ be a polynomial of degree $n \geq 2$, where $c \neq 0$. Suppose that $\alpha_k \notin \overline{\mathbb{D}}$ for $1 \leq k \leq m$, and that the remaining points α_k are in $\overline{\mathbb{D}}$. If we have*

$$\sum_{k=1}^{m} \frac{1}{1-|\alpha_k|} + \sum_{k=m+1}^{n} \frac{1}{1+|\alpha_k|} > 0,$$

then there is a positive δ such that for any $r \in (1, 1+\delta)$, we have

$$\operatorname{Re}\left\{\frac{zQ'(z)}{Q(z)}\right\} > 0 \text{ on } |z| = r.$$

Furthermore, we have $\operatorname{Re}\left\{\frac{zQ'(z)}{Q(z)}\right\} > 0$ whenever $|z| = 1$ and $Q(z) \neq 0$.

Proof. By an elementary calculation, we can show that if $|z| > 1$ and $\alpha_k \neq 0$, then $\operatorname{Re}\left\{\frac{z}{z-\alpha_k}\right\} > \frac{1}{1+|\alpha_k|}$ for $m+1 \leq k \leq n$, the two sides being equal if $\alpha_k = 0$. Also, if $|z| = 1$ then $\operatorname{Re}\left\{\frac{z}{z-\alpha_k}\right\} \geq \frac{1}{1-|\alpha_k|}$ for $1 \leq k \leq m$.

Let

$$\varepsilon = \sum_{k=1}^{m} \frac{1}{1-|\alpha_k|} + \sum_{k=m+1}^{n} \frac{1}{1+|\alpha_k|} > 0.$$

Since $\operatorname{Re}\left\{\frac{z}{z-\alpha_k}\right\}$ is a continuous function except at $z = \alpha_k$ and since $|\alpha_k| > 1$ for $1 \leq k \leq m$, there exists a positive constant δ with $1+\delta < \min\{|\alpha_k| : 1 \leq k \leq m\}$ such that

$$\sum_{k=1}^{m} \operatorname{Re}\left\{\frac{z}{z-\alpha_k}\right\} > \sum_{k=1}^{m} \frac{1}{1-|\alpha_k|} - \frac{\varepsilon}{2}$$

on $|z| = r$, for all $r \in (1, 1+\delta)$. Therefore, if $r \in (1, 1+\delta)$ and $|z| = r$, we have

$$\operatorname{Re}\left\{\frac{zQ'(z)}{Q(z)}\right\} = \sum_{k=1}^{n} \operatorname{Re}\left\{\frac{z}{z-\alpha_k}\right\} > \sum_{k=1}^{m} \frac{1}{1-|\alpha_k|} - \frac{\varepsilon}{2} + \sum_{k=m+1}^{n} \frac{1}{1+|\alpha_k|} = \frac{\varepsilon}{2}.$$

This proves Lemma 2.1. □

Now we are ready to present a proof of Theorem 1.1.

Proof of Theorem 1.1. We are to show that $zQ'(z)$ and $Q(z)$ have the same number of zeros lying in $\overline{\mathbb{D}}$. By Lemma 2.1, there is $\delta > 0$ such that, for all $r \in (1, 1+\delta)$, we have $\operatorname{Re}\left\{\frac{zQ'(z)}{Q(z)}\right\} > 0$ on $|z| = r$. So, for each fixed $r \in (1, 1+\delta)$, we have

$$\left|1 - \frac{zQ'(z)}{Q(z)}\right| < 1 + \left|\frac{zQ'(z)}{Q(z)}\right|,$$

hence $|zQ'(z) - Q(z)| < |Q(z)| + |zQ'(z)|$, on $|z| = r$. Then, by Rouché's theorem [4, Theorem 3.6, p. 341], $zQ'(z)$ and $Q(z)$ must have the same number of zeros lying in $\{z : |z| \le r\}$ for all $r \in (1, 1+\delta)$. This proves the first part of the theorem.

Next suppose that all the zeros α_k that are on the unit circle, if any, are simple. If Q' has a zero γ on the unit circle, then $\operatorname{Re}\left\{\frac{\gamma Q'(\gamma)}{Q(\gamma)}\right\} = 0$, which contradicts the fact that $\operatorname{Re}\left\{\frac{zQ'(z)}{Q(z)}\right\} > 0$ on $|z| = 1$ outside the zeros of Q. Hence Q' has no zeros on $\partial\mathbb{D}$. The proof of Theorem 1.1 is now complete. □

For a proof of Corollary 1.2, we note that it follows from the fact that $\operatorname{Re}\left\{\frac{z}{z-\alpha} + \frac{z}{z-\alpha^{-1}}\right\} = 1$ for all z with $|z| = 1$ and the argument in the proof of Lemma 2.1.

3. Preliminaries for Theorem 1.3.

To prove Theorem 1.3, we need the following lemmas.

Lemma 3.1. *If $x > 1$ and $y \in [-1, 1)$, then*

$$\frac{1 + x^4 - 2x(1+x^2)y + 2x^2(2y^2-1)}{(x^2 - 2xy + 1)^2} - \frac{y}{2(1-y)} < 2.$$

Proof. This can be proved by using only elementary calculus (see [3, Lemma 5.10, p. 54]). □

Lemma 3.2. *If Q is an anti-reciprocal polynomial of degree $n \ge 2$ with real coefficients, then*

$$\operatorname{Re}\left\{\frac{zQ'(z)}{nQ(z)}\right\} = \frac{1}{2} \quad \text{and} \quad \operatorname{Im}\left\{\frac{z^2Q''(z)}{(n-1)Q(z)}\right\} = \operatorname{Im}\left\{\frac{zQ'(z)}{Q(z)}\right\} \tag{2}$$

whenever $|z| = 1$ and $Q(z) \neq 0$.

Proof. We give a proof that yields the entire statement of this lemma, but we note that the first equality in (2) has been proved in [8, (7.5), p. 229] for reciprocal polynomials Q.

Now, since Q is anti-reciprocal, we have $Q(z) = -z^n Q\left(\frac{1}{z}\right)$. Taking the derivative and multiplying both sides by z, we get

$$zQ'(z) = -nz^n Q\left(\frac{1}{z}\right) + z^{n-1}Q'\left(\frac{1}{z}\right) = nQ(z) + z^{n-1}Q'\left(\frac{1}{z}\right).$$

So, we have

$$z^{n-1}Q'\left(\frac{1}{z}\right) = zQ'(z) - nQ(z). \tag{3}$$

After taking the derivative of both sides of this equation, and then multiplying both sides by z and applying the identity (3), we obtain

$$-z^{n-2}Q''\left(\frac{1}{z}\right) = z^2Q''(z) + 2(1-n)zQ'(z) + n(n-1)Q(z). \tag{4}$$

Let $z \in \partial\mathbb{D}$ with $Q(z) \neq 0$. Next dividing both sides of (4) by $n(n-1)Q(z)$, we get

$$-\frac{z^{n-2}Q''\left(\frac{1}{z}\right)}{n(n-1)Q(z)} = \frac{z^2Q''(z)}{n(n-1)Q(z)} - \frac{2zQ'(z)}{nQ(z)} + 1. \tag{5}$$

By replacing $Q(z)$ on the left side of (5) by $-z^n Q\left(\frac{1}{z}\right)$, the left-hand side becomes

$$\frac{z^{n-2}Q''\left(\frac{1}{z}\right)}{n(n-1)z^nQ\left(\frac{1}{z}\right)} = \frac{z^{-2}Q''\left(\frac{1}{z}\right)}{n(n-1)Q\left(\frac{1}{z}\right)} = \overline{\left(\frac{z^2Q''(z)}{n(n-1)Q(z)}\right)}.$$

Here we have used the fact that since $|z| = 1$ and Q has real coefficients, we have $Q(1/z) = Q(\overline{z}) = \overline{Q(z)}$, and similarly for Q'' instead of Q. Then from (5) we derive

$$\overline{\left(\frac{z^2Q''(z)}{n(n-1)Q(z)}\right)} - \frac{z^2Q''(z)}{n(n-1)Q(z)} = 1 - \frac{2zQ'(z)}{nQ(z)},$$

which gives $2i\,\mathrm{Im}\left\{\frac{z^2Q''(z)}{n(n-1)Q(z)}\right\} = \frac{2zQ'(z)}{nQ(z)} - 1$. This implies that $\mathrm{Re}\left\{\frac{zQ'(z)}{nQ(z)}\right\} = \frac{1}{2}$ and $\mathrm{Im}\left\{\frac{z^2Q''(z)}{(n-1)Q(z)}\right\} = \mathrm{Im}\left\{\frac{zQ'(z)}{Q(z)}\right\}$, as desired. □

Lemma 3.3. *Let $Q(z) = \prod_{k=1}^{n}(z - \alpha_k)$ be an anti-reciprocal polynomial of degree $n \geq 3$. Suppose that $\alpha_1 = \tau > 1$, $\alpha_2 = \tau^{-1}$, $\alpha_3 = 1$, and $|\alpha_k| = 1$ for $k > 3$. For $|z| = 1$ with $Q(z) \neq 0$, if $\frac{z^2Q''(z)}{Q(z)}$ is a real number, then it is positive. In particular, then $Q''(z) \neq 0$.*

Proof. Since Q is monic and each zero of Q is real or has modulus 1, Q has real coefficients. Let z be a point on the unit circle with $Q(z) \neq 0$. We have

$$\frac{z^2Q''(z)}{Q(z)} = z^2\left(\left(\frac{Q'}{Q}\right)'(z) + \left(\left(\frac{Q'}{Q}\right)(z)\right)^2\right) = \left(\frac{zQ'(z)}{Q(z)}\right)^2 - \sum_{k=1}^{n}\frac{z^2}{(z-\alpha_k)^2}.$$

Suppose that $\frac{z^2Q''(z)}{Q(z)}$ is a real number. Thus, by Lemma 3.2, $\frac{zQ'(z)}{nQ(z)}$ is real as well, and so is also $\sum_{k=1}^{n}\frac{z^2}{(z-\alpha_k)^2}$. Since $\mathrm{Re}\left\{\frac{zQ'(z)}{nQ(z)}\right\} = \frac{1}{2}$ on $|z| = 1$ when $Q(z) \neq 0$, we have

$$\frac{z^2Q''(z)}{Q(z)} = \frac{n^2}{4} - \sum_{k=1}^{n}\frac{z^2}{(z-\alpha_k)^2}. \tag{6}$$

Next we want to find an upper bound for the real part of $\sum_{k=1}^{n}\frac{z^2}{(z-\alpha_k)^2}$ on the unit circle. Let $z = e^{i\theta}$, where $\theta \in (0, 2\pi)$ (note that $z \neq 1$ since $Q(1) = 0$). If α is real, we have

$$\mathrm{Re}\left\{\frac{z^2}{(z-\alpha)^2}\right\} = \frac{1 - 2\alpha\cos\theta + \alpha^2(2\cos^2\theta - 1)}{(1 + \alpha^2 - 2\alpha\cos\theta)^2}.$$

For $k \geq 3$, by letting $\alpha_k = e^{i\theta_k}$, $\theta_k \in [0, 2\pi)$, we have $\mathrm{Re}\left\{\frac{z^2}{(z-\alpha_k)^2}\right\} = \frac{-\cos\beta_k}{2-2\cos\beta_k}$, where $\beta_k = \theta - \theta_k$. Therefore,

$$\begin{aligned}\mathrm{Re}&\left\{\sum_{k=1}^{n}\frac{z^2}{(z-\alpha_k)^2}\right\}\\ &= \frac{1 + \tau^4 - 2\tau(1+\tau^2)\cos\theta + 2\tau^2(2\cos^2\theta - 1)}{(1 + \tau^2 - 2\tau\cos\theta)^2} - \sum_{k=3}^{n}\frac{\cos\beta_k}{2 - 2\cos\beta_k}.\end{aligned}$$

Taking $x = \tau$ and $y = \cos\theta$ in Lemma 3.1, we see that

$$\frac{1 + \tau^4 - 2\tau\left(1+\tau^2\right)\cos\theta + 2\tau^2\left(2\cos^2\theta - 1\right)}{(1 + \tau^2 - 2\tau\cos\theta)^2} - \frac{\cos\theta}{2 - 2\cos\theta} < 2.$$

It is easy to see that $\frac{-\cos\omega}{2-2\cos\omega} \leq \frac{1}{4}$ for all $\omega \in (0, 2\pi)$. So, we obtain

$$\mathrm{Re}\left\{\sum_{k=1}^{n}\frac{z^2}{(z-\alpha_k)^2}\right\} < 2 + \frac{1}{4}(n-3) = \frac{n+5}{4}.$$

Hence, from (6), we derive

$$\frac{z^2Q''(z)}{Q(z)} = \frac{n^2}{4} - \sum_{k=1}^{n}\frac{z^2}{(z-\alpha_k)^2} > \frac{n^2}{4} - \frac{n+5}{4} > 0$$

if $n \geq 3$, as desired. This proves Lemma 3.3. □

4. Proof of Theorem 1.3. Let the assumptions of Theorem 1.3 be satisfied. By Corollary 1.2 we know that Q' has only one zero outside $\overline{\mathbb{D}}$ and has no zeros on $\partial\mathbb{D}$. Let $G(z) = -z^{n-2}Q''\left(\frac{1}{z}\right)$ and $T(z) = z^{n-1}Q'\left(\frac{1}{z}\right)$. In order to prove that Q'' has exactly one zero outside $\overline{\mathbb{D}}$, it is equivalent to show that G has only one zero in $\mathbb{D}$. Since Q' has only one zero outside $\overline{\mathbb{D}}$ and has no zeros on $\partial\mathbb{D}$, T has exactly one zero in $\mathbb{D}$ and has no zeros on $\partial\mathbb{D}$. If we have

$$|G(z) + 2(n-1)T(z)| < |G(z)| + 2(n-1)|T(z)| \tag{7}$$

on $\partial\mathbb{D}$, then, by a form of Rouché's Theorem [4, Theorem 3.6, p. 341], both G and T have the same number of zeros inside $\mathbb{D}$. This will prove the theorem. From (3) and (4), we have

$$G(z) + 2(n-1)T(z) = z^2Q''(z) - n(n-1)Q(z). \tag{8}$$

Let $z \in \partial\mathbb{D}$. It is easy to see that if $Q(z) = 0$, then (7) holds. Now, for $Q(z) \neq 0$, write $\frac{z^2Q''(z)}{(n-1)Q(z)} = a+ib$, where $a, b \in \mathbb{R}$. So $G(z)+2(n-1)T(z) = (a-n+ib)(n-1)Q(z)$. Since, by Lemma 3.2, $\operatorname{Im}\left\{\frac{z^2Q''(z)}{(n-1)Q(z)}\right\} = \operatorname{Im}\left\{\frac{zQ'(z)}{Q(z)}\right\}$ and $\operatorname{Re}\left\{\frac{zQ'(z)}{nQ(z)}\right\} = \frac{1}{2}$, we have $zQ'(z) = (\frac{n}{2}+ib)Q(z)$. We also have $|G(z)| = |z^2Q''(z)| = (n-1)|a+ib||Q(z)|$ and, by (3),

$$2|T(z)| = 2|zQ'(z) - nQ(z)| = |-n+2ib||Q(z)|.$$

Thus, the inequality (7) is equivalent to

$$|a-n+ib| < |a+ib| + |-n+2ib|$$

which is clearly true if $b \neq 0$. If $b = 0$, then by Lemma 3.3, we have $a > 0$ and so the inequality above is true. Therefore, the inequality (7) holds on $\partial\mathbb{D}$, as desired. The proof of Theorem 1.3 is now complete.

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