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Properties of the determinant of a rectangular matrix

Abstract. In this paper we present new identities for the Radić's determinant of a rectangular matrix. The results include representations of the determinant of a rectangular matrix as a sum of determinants of square matrices and description how the determinant is affected by operations on columns such as interchanging columns, reversing columns or decomposing a single column.

1. Introduction. In [2] Radić introduced the following definition of the determinant of a rectangular matrix.

Definition 1.1. Let $A = [A_1, A_2, \ldots, A_n]$ be a $m \times n$ matrix with n columns $A_1, \ldots, A_n$ and $m \leq n$. The determinant of A is defined as

$$|A| = \sum_{1 \leq j_1 < \ldots < j_m \leq n} (-1)^{r+j_1+j_2+\ldots+j_m} |A_{j_1}, A_{j_2}, \ldots, A_{j_m}|, \tag{1}$$

where $r = 1 + 2 + \ldots + m$.

The determinant of a square matrix and the determinant (1) of a $m \times n$ matrix, where $m \leq n$, have several common standard properties, including the following (see [2]):

(1) If a row of A is identical to some other row or is a linear combination of other rows then $|A| = 0$.

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(2) If a row of A is multiplied by a number k, then the determinant of the resulting matrix is equal to $k|A|$.
(3) Interchanging two rows of A results in changing the sign of the determinant.
(4) The determinant $|A|$ can be calculated using the Laplace expansion.

The properties of the determinant (1) were investigated by Radić [3, 4, 5] and also by Radić and Sušanj [6]. In the papers cited, the results concerning $2 \times n$ matrices were applied in planar geometry.

Another approach was presented by Amiri, Fathy and Bayat in [1], where the authors proved determinant identities such as Dodgson Condensation Formula and Trahan Formula for rectangular matrices, as well as Cauchy–Binet Formula for non-square products of two matrices.

In this paper we present new identities for determinants of rectangular matrices. The results include representation of the determinant of a rectangular matrix as a sum of determinants of square matrices and description how the determinant is affected by operations on columns such as interchanging two columns, reversing columns or decomposing a single column.

2. Properties of the determinant.

2.1. Representation of the determinant of a rectangular matrix as a sum of determinants of square matrices. For $2 \times n$ matrices, where $n \geq 2$, Radić [3] proved the following theorem.

Theorem 2.1. *Let $A = [A_1, A_2, \ldots, A_n]$ be a $2 \times n$ matrix with $n \geq 2$. Then*

$$\begin{aligned} |A| = {} & |A_1, A_2 - A_3 + A_4 - \ldots + (-1)^n A_n| \\ & + |A_2, A_3 - A_4 + \ldots + (-1)^{n-1} A_n| \\ & + \ldots \\ & + |A_{n-1}, A_n|. \end{aligned}$$

This theorem gives a representation of the determinant of a $2 \times n$ matrix, where $n \geq 2$, as a sum of determinants of square matrices other than the representation (1). We generalize this result to $m \times n$ matrices in the following way.

Theorem 2.2. *Let $A = [A_1, A_2, \ldots, A_n]$ be a $m \times n$ matrix, where m is a number of rows and n is a number of columns, $m \leq n$. Then we have*

$$|A| = \sum_{1 \leq j_1 < \ldots < j_{m-1} < n} (-1)^{r + j_1 + j_2 + \ldots + j_{m-1}} \times \left| A_{j_1}, A_{j_2}, \ldots, A_{j_{m-1}}, \sum_{k = j_{m-1}+1}^{n} (-1)^k A_k \right|.$$

Proof. Applying (1), we have

$$
\begin{aligned}
|A| &= \sum_{1\le j_1<\ldots<j_m\le n} (-1)^{r+j_1+j_2+\ldots+j_m} |A_{j_1}, A_{j_2}, \ldots, A_{j_m}| \\
&= \sum_{1\le j_1<j_2<\ldots<j_{m-1}<n} \sum_{k=j_{m-1}+1}^{n} (-1)^{r+j_1+j_2+\ldots+j_{m-1}+k} \\
&\qquad \times |A_{j_1}, A_{j_2}, \ldots, A_{j_{m-1}}, A_k| \\
&= \sum_{1\le j_1<\ldots<j_{m-1}<n} (-1)^{r+j_1+j_2+\ldots+j_{m-1}} \\
&\qquad \times \left| A_{j_1}, A_{j_2}, \ldots, A_{j_{m-1}}, \sum_{k=j_{m-1}+1}^{n} (-1)^k A_k \right|. \qquad \square
\end{aligned}
$$

Using the same method, one can easily prove the following two theorems.

Theorem 2.3. *Let $A = [A_1, A_2, \ldots, A_n]$ be a $m \times n$ matrix, $m \le n$. Then we have*

$$
|A| = \sum_{1<j_2<\ldots<j_m\le n} (-1)^{r+j_2+j_3+\ldots+j_m} \left| \sum_{k=1}^{j_2-1} (-1)^k A_k, A_{j_2}, \ldots, A_{j_m} \right|,
$$

where $r = 1 + 2 + \ldots + m$.

Theorem 2.4. *Let $A = [A_1, A_2, \ldots, A_n]$ be a $m \times n$ matrix, $m \le n$. Then for each $p \in \{2, 3, \ldots, m-1\}$ we have*

$$
\begin{aligned}
|A| = &\sum_{\substack{1\le j_1<\ldots<j_{p-1} \\ j_{p+1}<\ldots<j_m\le n \\ j_{p+1}-j_{p-1}>1}} (-1)^{r+j_1+j_2+\ldots+j_{p-1}+j_{p+1}+\ldots+j_m} \\
&\times \left| A_{j_1}, \ldots, A_{j_{p-1}}, \sum_{k=j_{p-1}+1}^{j_{p+1}-1} (-1)^k A_k, A_{j_{p+1}}, \ldots, A_{j_m} \right|,
\end{aligned}
$$

where $r = 1 + 2 + \ldots + m$.

Example 1. Let $[A_1, A_2, A_3, A_4]$ be a 3×4 matrix. Then

$$
\begin{aligned}
|A_1, A_2, A_3, A_4| &= |A_1, A_2, A_3 - A_4| + |A_1, A_3, A_4| - |A_2, A_3, A_4| \\
&= |A_1, A_2, A_3| - |A_1, A_2, A_4| + |A_1 - A_2, A_3, A_4| \\
&= |A_1, A_2, A_3| - |A_1, A_2 - A_3, A_4| - |A_2, A_3, A_4|.
\end{aligned}
$$

2.2. Decomposing a column. If a column K in a square matrix A is a sum of two columns (eg. $K = K_1 + K_2$), then the determinant $|A|$ is a sum of two determinants of matrices obtained from A by replacing K by K_1 and K_2 respectively.

For rectangular matrices we have a similar property.

Theorem 2.5. *Let $A = [A_1, A_2, \dots, A_k, \dots, A_n]$ be a $m \times n$ matrix, $m \le n$, and $A_k = B_k + C_k$ for some $k \in \{1, 2, \dots, n\}$. Then*

$$\begin{aligned} |A| &= |A_1, A_2, \dots, A_{k-1}, B_k, A_{k+1}, \dots, A_n| \\ &\quad + |A_1, A_2, \dots, A_{k-1}, C_k, A_{k+1}, \dots, A_n| \\ &\quad + \sum_{\substack{1 \le j_1 < \dots < j_m \le n \\ k \notin \{j_1, \dots, j_m\}}} (-1)^{r+j_1+j_2+\dots+j_m+1} |A_{j_1}, A_{j_2}, \dots, A_{j_m}|, \end{aligned}$$

where $r = 1 + 2 + \dots + m$.

Proof. After applying (1)

$$|A| = \sum_{1 \le j_1 < \dots < j_m \le n} (-1)^{r+j_1+j_2+\dots+j_m} |A_{j_1}, A_{j_2}, \dots, A_{j_m}|$$

we separate the sum of determinants into two sums: the first one consisting of the determinants of matrices which contain the column $A_k = B_k + C_k$ and the second one consisting of other determinants.

$$\begin{aligned} |A| &= \sum_{\substack{1 \le j_1 < \dots < j_m \le n \\ k \in \{j_1, \dots, j_m\}}} (-1)^{r+j_1+j_2+\dots+j_m} |A_{j_1}, \dots, A_k, \dots, A_{j_m}| \\ &\quad + \sum_{\substack{1 \le j_1 < \dots < j_m \le n \\ k \notin \{j_1, \dots, j_m\}}} (-1)^{r+j_1+j_2+\dots+j_m} |A_{j_1}, A_{j_2}, \dots, A_{j_m}| \\ &= \sum_{\substack{1 \le j_1 < \dots < j_m \le n \\ k \in \{j_1, \dots, j_m\}}} (-1)^{r+j_1+j_2+\dots+j_m} |A_{j_1}, \dots, B_k, \dots, A_{j_m}| \\ &\quad + \sum_{\substack{1 \le j_1 < \dots < j_m \le n \\ k \in \{j_1, \dots, j_m\}}} (-1)^{r+j_1+j_2+\dots+j_m} |A_{j_1}, \dots, C_k, \dots, A_{j_m}| \\ &\quad + \sum_{\substack{1 \le j_1 < \dots < j_m \le n \\ k \notin \{j_1, \dots, j_m\}}} (-1)^{r+j_1+j_2+\dots+j_m} |A_{j_1}, A_{j_2}, \dots, A_{j_m}|. \end{aligned}$$

Now the third sum is added and subtracted so that it can be included into both the first and the second sum:

$$\begin{aligned} |A| &= |A_1, A_2, \dots, A_{k-1}, B_k, A_{k+1}, \dots, A_n| \\ &\quad + |A_1, A_2, \dots, A_{k-1}, C_k, A_{k+1}, \dots, A_n| \\ &\quad - \sum_{\substack{1 \le j_1 < \dots < j_m \le n \\ k \notin \{j_1, \dots, j_m\}}} (-1)^{r+j_1+j_2+\dots+j_m} |A_{j_1}, A_{j_2}, \dots, A_{j_m}| \end{aligned}$$

$$
\begin{aligned}
&= |A_1, A_2, \ldots, A_{k-1}, B_k, A_{k+1}, \ldots, A_n| \\
&\quad + |A_1, A_2, \ldots, A_{k-1}, C_k, A_{k+1}, \ldots, A_n| \\
&\quad + \sum_{\substack{1 \le j_1 < \ldots < j_m \le n \\ k \notin \{j_1, \ldots, j_m\}}} (-1)^{r+j_1+j_2+\ldots+j_m+1} |A_{j_1}, A_{j_2}, \ldots, A_{j_m}|. \qquad \square
\end{aligned}
$$

Example 2. Let $[A_1, A_2, A_3]$ be a 2×3 matrix and $A_1 = B_1 + C_1$. Then according to Theorem 2.5 we have

$$
\begin{aligned}
|B_1 + C_1, A_2, A_3| &= |B_1, A_2, A_3| + |C_1, A_2, A_3| \\
&\quad + \sum_{\substack{1 \le j_1 < j_2 \le 3 \\ 1 \notin \{j_1, j_2\}}} (-1)^{(1+2)+j_1+j_2+1} |A_{j_1}, A_{j_2}| \\
&= |B_1, A_2, A_3| + |C_1, A_2, A_3| + (-1)^{3+2+3+1} |A_2, A_3| \\
&= |B_1, A_2, A_3| + |C_1, A_2, A_3| - |A_2, A_3|.
\end{aligned}
$$

2.3. Interchanging columns. Interchanging columns in a square matrix results in changing the sign of the determinant. Rectangular matrices in which the number of columns is equal to the number of rows increased by one have the same property.

Theorem 2.6. *Let $A = [A_1, A_2, \ldots, A_m, A_{m+1}]$ be a $m \times (m+1)$ matrix. Then for each $i, j \in \{1, 2, \ldots, m+1\}$ such that $i < j$, we have*

$$
|A| = -|A_1, A_2, \ldots, A_{i-1}, A_j, A_{i+1}, \ldots, A_{j-1}, A_i, A_{j+1}, \ldots, A_m, A_{m+1}|.
$$

Proof. Let $r = 1 + 2 + \ldots + m$. Fix $i, j \in \{1, 2, \ldots, m+1\}$ such that $i < j$. From all the determinants in the right-hand side of

$$
|A| = \sum_{1 \le j_1 < \ldots < j_m \le n} (-1)^{r+j_1+j_2+\ldots+j_m} |A_{j_1}, A_{j_2}, \ldots, A_{j_m}|,
$$

we distinguish determinants of two matrices which contain either A_i or A_j but not both of them. Thus we have

$$
\begin{aligned}
|A| &= (-1)^{\left[r + \frac{(m+1)(m+2)}{2} - i\right]} \\
&\qquad \times |A_1, A_2, \ldots, A_{i-1}, A_{i+1}, \ldots, A_{j-1}, A_j, A_{j+1}, \ldots, A_{m+1}| \\
&\quad + (-1)^{\left[r + \frac{(m+1)(m+2)}{2} - j\right]} \\
&\qquad \times |A_1, A_2, \ldots, A_{i-1}, A_i, A_{i+1}, \ldots, A_{j-1}, A_{j+1}, \ldots, A_{m+1}| \\
&\quad + \sum_{\substack{1 \le j_1 < \ldots < j_m \le n \\ i, j \in \{j_1, \ldots, j_m\}}} (-1)^{r+j_1+j_2+\ldots+j_m} |A_{j_1}, \ldots, A_i, \ldots, A_j, \ldots A_{j_m}|.
\end{aligned}
$$

Notice that exactly $j-i-1$ inversions are needed to move the column A_j to the position between A_{i-1} and A_{i+1} in the first summand. Similarly, in the second summand, also $j-i-1$ inversions are needed to move the column A_i to the position between A_{j-1} and A_{j+1}.

In other summands we can simply interchange columns A_i and A_j with the sign change. Thus we have

$$\begin{aligned}
|A| = (-1)&^{\left[r+\frac{(m+1)(m+2)}{2}-i+(j-i+1)\right]} \\
&\times |A_1, A_2, \ldots, A_{i-1}, A_j, A_{i+1}, \ldots, A_{j-1}, A_{j+1}, \ldots, A_{m+1}| \\
&+ (-1)^{\left[r+\frac{(m+1)(m+2)}{2}-j+(j-i+1)\right]} \\
&\times |A_1, A_2, \ldots, A_{i-1}, A_{i+1}, \ldots, A_{j-1}, A_i, A_{j+1}, \ldots, A_{m+1}| \\
&- \sum_{\substack{1\le j_1<\ldots<j_m\le n \\ i,j\in\{j_1,\ldots,j_m\}}} (-1)^{r+j_1+j_2+\ldots+j_m} |A_{j_1}, \ldots, A_j, \ldots, A_i, \ldots A_{j_m}| \\
= &- (-1)^{\left[r+\frac{(m+1)(m+2)}{2}-j\right]} \\
&\times |A_1, A_2, \ldots, A_{i-1}, A_j, A_{i+1}, \ldots, A_{j-1}, A_{j+1}, \ldots, A_{m+1}| \\
&- (-1)^{\left[r+\frac{(m+1)(m+2)}{2}-i\right]} \\
&\times |A_1, A_2, \ldots, A_{i-1}, A_{i+1}, \ldots, A_{j-1}, A_i, A_{j+1}, \ldots, A_{m+1}| \\
&- \sum_{\substack{1\le j_1<\ldots<j_m\le n \\ i,j\in\{j_1,\ldots,j_m\}}} (-1)^{r+j_1+j_2+\ldots+j_m} |A_{j_1}, \ldots, A_j, \ldots, A_i, \ldots A_{j_m}| \\
= &- |A_1, A_2, \ldots, A_{i-1}, A_j, A_{i+1}, \ldots, A_{j-1}, A_i, A_{j+1}, \ldots, A_m, A_{m+1}|. \square
\end{aligned}$$

Consider a $m \times n$ matrix A with m rows and n columns, $m \le n$. Let A' be a matrix obtained from A by interchanging two columns. Theorem 2.6 tells us that $|A| + |A'| = 0$ when $n - m = 1$. However, in general, if $n - m > 1$ the sum $|A| + |A'|$ is not zero.

For a $m \times n$ matrix $M = [M_1, M_2, \ldots, M_n]$ and each $i, j \in \{1, 2, \ldots, m\}$, such that $i < j$, denote

$$S_1(M, i, j) = \sum_{\substack{1\le j_1<\ldots<j_m\le n \\ i,j\notin\{j_1,\ldots,j_m\}}} (-1)^{r+j_1+j_2+\ldots+j_m} |M_{j_1}, M_{j_2}, \ldots, M_{j_m}|,$$

$$S_2(M,i,j) = \sum_{\substack{1\le j_1<\ldots<j_m\le n\\ i,j\in\{j_1,\ldots,j_m\}}} (-1)^{r+j_1+j_2+\ldots+j_m}|M_{j_1},M_{j_2},\ldots,M_{j_m}|,$$

$$S_3(M,i,j) = \sum_{\substack{1\le j_1<\ldots<j_m\le n\\ (i\in J,\, j\notin J \text{ or } i\notin J,\, j\in J)\\ J=\{i,\ldots,j\}\setminus\{j_1,\ldots,j_m\}\\ \mathrm{card}(J)\equiv 1 \pmod 2}} (-1)^{r+j_1+j_2+\ldots+j_m}|M_{j_1},M_{j_2},\ldots,M_{j_m}|,$$

$$S_4(M,i,j) = \sum_{\substack{1\le j_1<\ldots<j_m\le n\\ (i\in J,\, j\notin J \text{ or } i\notin J,\, j\in J)\\ J=\{i,\ldots,j\}\setminus\{j_1,\ldots,j_m\}\\ \mathrm{card}(J)\equiv 0 \pmod 2}} (-1)^{r+j_1+j_2+\ldots+j_m}|M_{j_1},M_{j_2},\ldots,M_{j_m}|,$$

where $r = 1+2+\ldots+m$ and $\mathrm{card}(X)$ stands for the cardinality of X.

Theorem 2.7. *Let $A = [A_1, A_2, \ldots, A_n]$ be a $m \times n$ matrix with m rows and n columns, $m \le n$. For $i, j \in \{1, 2, \ldots, n\}$ such that $i \ne j$ denote by $A_{A_i\leftrightarrow A_j}$ the matrix obtained from A by interchanging columns A_i and A_j. Then*

$$\begin{aligned} |A| + |A_{A_i\leftrightarrow A_j}| &= 2S_1(A,i,j) + 2S_4(A,i,j)\\ &= 2S_1(A_{A_i\leftrightarrow A_j},i,j) + 2S_4(A_{A_i\leftrightarrow A_j},i,j). \end{aligned}$$

Proof. Fix $i, j \in \{1, 2, \ldots, m\}$ such that $i < j$. (If $i > j$ we can proceed analogously). We have

$$|A| = S_1(A,i,j) + S_2(A,i,j) + S_3(A,i,j) + S_4(A,i,j).$$

It is easy to verify that

$$\begin{aligned} S_1(A_{A_i\leftrightarrow A_j},i,j) &= S_1(A,i,j),\\ S_2(A_{A_i\leftrightarrow A_j},i,j) &= -S_2(A,i,j). \end{aligned}$$

Notice that each of the matrices in $S_3(A,i,j) + S_4(A,i,j)$ needs exactly $(j - i - \mathrm{card}(J))$ column inversions to move the column A_i to the position where A_j would be, and also $(j-i-\mathrm{card}(J))$ inversions are needed to move the column A_j to the position where A_i would be.

Therefore,

$$
\begin{aligned}
S_3(A,i,j) = & \sum_{\substack{1\le j_1<\ldots<j_m\le n\\ i\in J,\, j\notin J\\ J=\{i,\ldots,j\}\setminus\{j_1,\ldots,j_m\}\\ \operatorname{card}(J)\equiv 1 \pmod 2}} (-1)^{r+\left(\sum_{k=1}^{m} j_k+j\right)-j+(j-i-\operatorname{card}(J))} \\
&\qquad \times |A_{j_1},\ldots,A_{j_p},A_i,A_{j_q},\ldots,A_{j_m}| \\
&+ \sum_{\substack{1\le j_1<\ldots<j_m\le n\\ i\notin J,\, j\in J\\ J=\{i,\ldots,j\}\setminus\{j_1,\ldots,j_m\}\\ \operatorname{card}(J)\equiv 1 \pmod 2}} (-1)^{r+\left(\sum_{k=1}^{m} j_k+i\right)-i+(j-i-\operatorname{card}(J))} \\
&\qquad \times |A_{j_1},\ldots,A_{j_u},A_j,A_{j_v},\ldots,A_{j_m}| \\
= & - \sum_{\substack{1\le j_1<\ldots<j_m\le n\\ i\in J,\, j\notin J\\ J=\{i,\ldots,j\}\setminus\{j_1,\ldots,j_m\}\\ \operatorname{card}(J)\equiv 1 \pmod 2}} (-1)^{r+\left(\sum_{k=1}^{m} j_k+j\right)-i} \\
&\qquad \times |A_{j_1},\ldots,A_{j_p},A_i,A_{j_q},\ldots,A_{j_m}| \\
&- \sum_{\substack{1\le j_1<\ldots<j_m\le n\\ i\notin J,\, j\in J\\ J=\{i,\ldots,j\}\setminus\{j_1,\ldots,j_m\}\\ \operatorname{card}(J)\equiv 1 \pmod 2}} (-1)^{r+\left(\sum_{k=1}^{m} j_k+i\right)-j} \\
&\qquad \times |A_{j_1},\ldots,A_{j_u},A_j,A_{j_v},\ldots,A_{j_m}| \\
= & - S_3(A_{A_i\leftrightarrow A_j},i,j),
\end{aligned}
$$

where $r = 1+2+\ldots+m$ and $j_p < j < j_q$, $j_u < i < j_v$ for some p,q,u,v. Similarly, we have

$$S_4(A_{A_i\leftrightarrow A_j},i,j) = S_4(A,i,j),$$

and finally,

$$
\begin{aligned}
|A| + |A_{A_i\leftrightarrow A_j}| &= S_1(A,i,j)+S_2(A,i,j)+S_3(A,i,j)+S_4(A,i,j) \\
&\quad + S_1(A_{A_i\leftrightarrow A_j},i,j)+S_2(A_{A_i\leftrightarrow A_j},i,j) \\
&\quad + S_3(A_{A_i\leftrightarrow A_j},i,j)+S_4(A_{A_i\leftrightarrow A_j},i,j) \\
&= 2S_1(A,i,j)+2S_4(A,i,j).
\end{aligned}
$$

$\square$

Corollary 2.8. *Let A be a $m \times n$ matrix, $m \leq n$. If $i, j \in \{1, 2, \ldots, n\}$ satisfy $|i - j| = 1$, then*

$$|A| + |A_{A_i \leftrightarrow A_j}| = 2S_1(A, i, j) = 2S_1(A_{A_i \leftrightarrow A_j}, i, j).$$

Example 3. Below we present a few identities obtained from Theorem 2.6, Theorem 2.7 and Corollary 2.8.

(a) Let $[A_1, A_2, A_3, A_4, A_5]$ be a 4×5 matrix. Then

$$|A_1, A_2, A_3, A_4, A_5| = -|A_5, A_2, A_3, A_4, A_1| = |A_5, A_4, A_3, A_2, A_1|.$$

(b) Let $[A_1, A_2, A_3, A_4]$ be a 2×4 matrix. Then

$$\begin{aligned}
&|A_1, A_2, A_3, A_4| + |A_2, A_1, A_3, A_4| = 2|A_3, A_4|,\\
&|A_1, A_2, A_3, A_4| + |A_1, A_4, A_3, A_2| = 2(|A_1, A_2| - |A_1, A_3| + |A_1, A_4|),\\
&|A_1, A_2, A_3, A_4| + |A_4, A_2, A_3, A_1| = 2(|A_1, A_2| - |A_1, A_3| + |A_2, A_3|\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad - |A_2, A_4| + |A_3, A_4|).
\end{aligned}$$

2.4. Reversing columns. Reversing columns in a $n \times n$ square matrix results in changing the sign of its determinant if and only if n is congruent to 2 or 3 (mod 4). Surprisingly, the determinant of a rectangular matrix also either changes or does not change the sign after column reversing, depending on the number of rows and the number of columns of the matrix.

Theorem 2.9. *Let $[A_1, A_2, \ldots, A_n]$ be a $m \times n$ matrix, $m \leq n$. Then we have*

$$\begin{aligned}
|A_n, A_{n-1}, \ldots, A_2, A_1| &= |A_1, A_2, \ldots, A_{n-1}, A_n| \cdot (-1)^{\frac{m}{2}(2n+m+1)}\\
&= \begin{cases}
|A_1, A_2, \ldots, A_{n-1}, A_n| & \text{if } m \equiv 0 \pmod 4,\\
|A_1, A_2, \ldots, A_{n-1}, A_n| \cdot (-1)^{n+1} & \text{if } m \equiv 1 \pmod 4,\\
|A_1, A_2, \ldots, A_{n-1}, A_n| \cdot (-1) & \text{if } m \equiv 2 \pmod 4,\\
|A_1, A_2, \ldots, A_{n-1}, A_n| \cdot (-1)^n & \text{if } m \equiv 3 \pmod 4.
\end{cases}
\end{aligned}$$

Proof. Let $r = 1+2+\ldots+m = \frac{m(m+1)}{2}$ and $B_k = A_{n+1-k}$, $k \in \{1, 2, \ldots, n\}$. Since exactly $(m-1) + (m-2) + \ldots + 1 = \frac{(m-1)m}{2}$ inversions of (adjacent) columns are needed to reverse the columns of a $m \times m$ matrix, we have

$$\begin{aligned}
|B_1, B_2, \ldots, B_n| &= \sum_{1 \leq i_1 < \ldots < i_m \leq n} (-1)^{r+i_1+i_2+\ldots+i_m} |B_{i_1}, B_{i_2}, \ldots, B_{i_m}|\\
&= \sum_{1 \leq i_1 < \ldots < i_m \leq n} (-1)^{r+i_1+i_2+\ldots+i_m+\frac{(m-1)m}{2}}\\
&\qquad\qquad \times |B_{i_m}, B_{i_{m-1}}, \ldots, B_{i_1}|\\
&= \sum_{1 \leq i_1 < \ldots < i_m \leq n} (-1)^{r+i_1+i_2+\ldots+i_m+\frac{(m-1)m}{2}}\\
&\qquad\qquad \times |A_{n+1-i_m}, A_{n+1-i_{m-1}}, \ldots, A_{n+1-i_1}|.
\end{aligned}$$

Applying the following change of variables: $j_k = n + 1 - i_{m-k+1}$ for each $k \in \{1, 2, \ldots, m\}$, we get

$$
\begin{aligned}
|A_n, &A_{n-1}, \ldots, A_2, A_1| = |B_1, B_2, \ldots, B_n| \\
&= \sum_{1 \leq j_1 < \ldots < j_m \leq n} (-1)^{r+m(n+1)-(j_1+j_2+\ldots+j_m)+\frac{(m-1)m}{2}} |A_{j_1}, A_{j_2}, \ldots, A_{j_m}| \\
&= |A_1, A_2, \ldots, A_{n-1}, A_n| \cdot (-1)^{m(n+1)+\frac{(m-1)m}{2}} \\
&= |A_1, A_2, \ldots, A_{n-1}, A_n| \cdot (-1)^{\frac{m}{2}(2n+m+1)}.
\end{aligned}
$$

Finally, we state that

$$
(-1)^{\frac{m}{2}(2n+m+1)} = \begin{cases} 1 & \text{if } m \equiv 0 \pmod 4, \\ (-1)^{n+1} & \text{if } m \equiv 1 \pmod 4, \\ (-1) & \text{if } m \equiv 2 \pmod 4, \\ (-1)^n & \text{if } m \equiv 3 \pmod 4, \end{cases}
$$

which is easy to verify. □

Example 4. Let

$$[A_1, A_2, A_3, A_4, A_5, A_6, A_7, A_8, A_9]$$

be a 5×9 matrix. Then

$$|A_1, A_2, A_3, A_4, A_5, A_6, A_7, A_8, A_9| = |A_9, A_8, A_7, A_6, A_5, A_4, A_3, A_2, A_1|.$$

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