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L U B L I N – P O L O N I A

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A fixed point theorem
for nonexpansive compact self-mapping

ABSTRACT. A mapping T from a topological space X to a topological space
Y is said to be compact if $T(X)$ is contained in a compact subset of Y. The
aim of this paper is to prove the existence of fixed points of a nonexpansive
compact self-mapping defined on a closed subset having a contractive jointly
continuous family when the underlying space is a metric space. The proved
result generalizes and extends several known results on the subject.

Finding sufficient conditions for the existence of fixed points of a self-
mapping f defined on a topological space X is an interesting aspect of the
theory of fixed points. Several results are known in the literature on this
subject. In 1930, J. Schauder [11] proved that if K is a compact convex
subset of a Banach space X and $T : K \to K$ is a continuous map, then T
has at least one fixed point in K. The compactness condition on K is a
strong one. Many problems in analysis do not have a compact setting and
the spaces are not Banach spaces. So, it is natural to prove the result by
relaxing the condition of compactness and considering spaces more general
than Banach spaces. In this direction many results are available in the
literature (see e.g. [1]–[4], [6]–[8] and [10]). In this paper we also prove
a result on the existence of fixed points of a nonexpansive compact self-
mapping defined on a closed subset having a contractive jointly continuous

family when the underlying space is a metric space. The proved result generalizes and extends some of the results proved in [1], [2]–[4], [7]–[9].

To start with, we recall few definitions and some related results.

Let X and Y be topological spaces. A mapping $T : X \to Y$ is called **compact** if $T(X)$ is contained in a compact subset of Y.

If T is compact, then $\overline{T(X)}$ is compact.

Let (X, d) be a metric space. A continuous mapping $W : X \times X \times [0, 1] \to X$ is said to be a **convex structure** on X if for all $x, y \in X$ and $\lambda \in [0, 1]$,

$$d(u, W(x, y, \lambda)) \le \lambda d(u, x) + (1 - \lambda)d(u, y)$$

holds for all $u \in X$. The metric space (X, d) together with a convex structure is called a **convex metric space** [12].

A subset K of a convex metric space (X, d) is said to be a **convex set** [12] if $W(x, y, \lambda) \in K$ for all $x, y \in K$ and $\lambda \in [0, 1]$. The set K is said to be **p-starshaped** [8] if there exists a $p \in K$ such that $W(x, p, \lambda) \in K$ for all $x \in K$ and $\lambda \in [0, 1]$. Clearly, each convex set is starshaped but not conversely.

A convex metric space (X, d) is said to satisfy **property (I)** [8] if for all $x, y, q \in X$ and $\lambda \in [0, 1]$,

$$d(W(x, q, \lambda), W(y, q, \lambda)) \le \lambda d(x, y).$$

A normed linear space and each of its convex subsets are simple examples of convex metric spaces. There are many convex metric spaces which are not normed linear spaces (see e.g. [8], [12]). Property (I) is always satisfied in a normed linear space.

Let K be a subset of a metric space (X, d) and $\mathfrak{F} = \{f_\alpha : \alpha \in K\}$ a family of functions from $[0, 1]$ into K, having the property $f_\alpha(1) = \alpha$ for each $\alpha \in K$. Such a family $\mathfrak{F}$ is said to be

(i) **contractive** if there exists a function $\phi : (0, 1) \to (0, 1)$ such that for all $\alpha, \beta \in K$ and for $t \in (0, 1)$, we have

$$d(f_\alpha(t), f_\beta(t)) \le \phi(t)d(\alpha, \beta),$$

(ii) **jointly continuous** if $t \to t_0$ in $[0, 1]$ and $\alpha \to \alpha_0$ in K imply $f_\alpha(t) \to f_{\alpha_0}(t_0)$.

In normed linear spaces these notions were discussed by Dotson [5].

If K is a starshaped subset with star centre q of a convex metric space (X, d) with property (I), then the family $\mathfrak{F} = \{f_x : x \in K\}$ defined by $f_x(t) = W(x, q, t)$ satisfies

$$d(f_x(t), f_y(t)) = d(W(x, q, t), W(y, q, t)) \le t\, d(x, y).$$

So taking $\phi(t) = t$, $0 < t < 1$, the family $\mathfrak{F}$ is a contractive jointly continuous family and therefore the class of subsets of X with the property of contractiveness and joint continuity contains the class of starshaped sets which in turn contains the class of convex sets.

Considering the existence of fixed points for nonexpansive mappings in convex metric spaces, Beg and Abbas [2] proved that if K is a nonempty compact and convex subset of a complete convex metric space (X, d) with property (I), then any nonexpansive mapping $T : K \to K$ has a fixed point. This result is not true in general for noncompact sets K (see [7], p. 12). Various extensions and generalizations of the results of Schauder [11] and of Beg and Abbas [2] are known in the literature (see e.g. [1], [3], [4], [6]–[10]). In this direction, we also prove the following:

Theorem 1. *Let (X, d) be a metric space and K a non-empty closed subset with a contractive jointly continuous family $\{f_\alpha : \alpha \in K\}$. If $T : K \to K$ is a nonexpansive compact mapping, then T has a fixed point.*

Proof. Let $k_n = \frac{n}{n+1}$, $n = 1, 2, 3, \ldots$. Define $T_n : K \to K$ as

$$T_n x = f_{Tx}(k_n), \ x \in K.$$

Since $T(K) \subset K$ and $k_n < 1$, $T_n : K \to K$ is well defined. Consider

$$
\begin{aligned}
d(T_n x, T_n y) &= d(f_{Tx}(k_n), f_{Ty}(k_n)) \\
&\leq \phi(k_n) d(Tx, Ty) \\
&\leq \phi(k_n) d(x, y), x, y \in K
\end{aligned}
$$

and so each T_n is a contraction mapping on K. Since $cl \ T_n(K)$ is T_n-invariant and also compact for each n and hence complete, by Banach contraction principle, each T_n has a unique fixed point $x_n \in K$ i.e. $T_n x_n = x_n$.

Since $T(K)$ lies in a compact subset of K, (Tx_n) has a subsequence (Tx_{n_i}) such that $(Tx_{n_i}) \to x_0 \in K$.

Now $x_{n_i} = T_{n_i}(x_{n_i}) = f_{Tx_{n_i}}(k_{n_i}) \to f_{x_0}(1) = x_0$. Since T is a continuous, $(Tx_{n_i}) \to Tx_0$ and hence $x_0 = Tx_0$ i.e. x_0 is a fixed point of T. $\square$

If K is a starshaped subset with a star centre p, of a convex metric space (X, d) with property (I), then the family $\{f_\alpha : \alpha \in K\}$ defined by $f_\alpha(t) = W(\alpha, p, t)$ is contractive if we take $\phi(t) = t$ for $0 < t < 1$, and is jointly continuous. So, we have

Corollary 1 ([3]). *Let (X, d) be a convex metric space with property (I) and K a non-empty closed starshaped subset of X. If $T : K \to K$ is a nonexpansive compact map, then T has a fixed point.*

Since every normed linear space is a convex metric with property (I), as an immediate consequence of Corollary 1, we get:

Corollary 2 ([9]). *If $T : K \to K$ is a nonexpansive, where K is a closed and starshaped subset of a normed linear space E, and $cl[T(K)]$ is compact, then T has a fixed point.*

Since every convex set is starshaped, we have

Corollary 3. *Let K be a non-empty closed convex subset of a convex metric space (X, d) with property (I), then any nonexpansive compact mapping $T : K \to K$ has a fixed point.*

Since for a compact set K, any continuous mapping $T : K \to K$ is a compact mapping, we have

Corollary 4 ([2]). *Let K be a compact subset of a complete convex metric space (X, d). Suppose there exists a contractive, jointly continuous family of maps associated with K, then any nonexpansive mapping T of K into itself has a fixed point in K.*

Note. We do not require the completeness of the space in Corollary 4 proved by Beg and Abbas [2].

Corollary 5 ([2]). *Let K be a non-empty compact convex (or starshaped) subset of a convex metric space (X, d) with property (I), then any nonexpansive mapping $T : K \to K$ has a fixed point.*

Corollary 6 ([8]). *Let X be a compact starshaped metric space satisfying property (I). Let $T : X \to X$ be a nonexpansive mapping. Then T has a fixed point in X.*

Since every normed linear space is a convex metric space with property (I), we have

Corollary 7 ([7], Theorem 3.1, p. 28). *If K is a non-empty compact and convex subset of a Banach space, then any nonexpansive mapping $T : K \to K$ has a fixed point.*

Corollary 8 ([1]). *Let K be a closed starshaped or convex subset of a Banach space X and $f : K \to K$ a nonexpansive map with $f(K)$ contained in a compact set K. Then f has a fixed point.*

Corollary 9 ([4]). *Let X be a Banach space and K be a nonempty compact starshaped subset of X. Let $T : K \to K$ be a nonexpansive mapping. Then T has a fixed point in K.*

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