doi: 10.2478/umcsmath-2014-0008

ANNALES
UNIVERSITATIS MARIAE CURIE-SKŁODOWSKA
LUBLIN – POLONIA

VOL. LXVIII, NO. 1, 2014 SECTIO A 67–89

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The constructions of general connections on second jet prolongation

Abstract. We determine all natural operators D transforming general connections Γ on fibred manifolds $Y \to M$ and torsion free classical linear connections ∇ on M into general connections $D(\Gamma, \nabla)$ on the second order jet prolongation $J^2Y \to M$ of $Y \to M$.

1. Introduction. The concept of r-th order connections was firstly introduced on groupoids by C. Ehresmann in [2] and next by I. Kolář in [3] for arbitrary fibred manifolds.

Let us recall that an r-th order connection on a fibred manifold $p\colon Y \to M$ is a section $\Theta\colon Y \to J^rY$ of the r-jet prolongation $\beta\colon J^rY \to Y$ of $p\colon Y \to M$. A general connection on $p\colon Y \to M$ is a first order connection $\Gamma\colon Y \to J^1Y$ or (equivalently) a lifting map

$$\Gamma\colon Y \times_M TM \to TY.$$

By $\mathrm{Con}(Y \to M)$ we denote the set of all general connections on a fibred manifold $p\colon Y \to M$.

If $p\colon Y \to M$ is a vector bundle and an r-th order connection $\Theta\colon Y \to J^rY$ is a vector bundle morphism, then Θ is called an r-th order linear connection on $p\colon Y \to M$.

2000 *Mathematics Subject Classification.* 58A05, 58A20, 58A32.

Key words and phrases. General connection, classical linear connection, higher order jet prolongation, bundle functor, natural operator.

An r-th order linear connection on M is an r-th order linear connection $\Lambda\colon TM \to J^rTM$ on the tangent bundle $\pi_M\colon TM \to M$ of M. By $Q^r(M)$ we denote the set of all r-th order linear connections on M.

A classical linear connection on M is a first order linear connection $\nabla\colon TM \to J^1TM$ or (equivalently) a covariant derivative $\nabla\colon \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$. A classical linear connection ∇ on M is called torsion free if its torsion tensor $T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$ is equal to zero. By $Q_\tau(M)$ we denote the set of all torsion free classical linear connections on M.

Let $\mathcal{FM}$ denote the category of fibred manifolds and their fibred maps and let $\mathcal{FM}_{m,n} \subset \mathcal{FM}$ be the (sub)category of fibred manifolds with m-dimensional bases and n-dimensional fibres and their local fibred diffeomorphisms. Let $\mathcal{M}f_m$ denote the category of m-dimensional manifolds and their local diffeomorphisms. Let $F\colon \mathcal{FM}_{m,n} \to \mathcal{FM}$ be a bundle functor on $\mathcal{FM}_{m,n}$ of order r in the sense of [4]. Let $\Gamma\colon Y \times_M TM \to TY$ be the lifting map of a general connection on an object $p\colon Y \to M$ of $\mathcal{FM}_{m,n}$. Let $\Lambda\colon TM \to J^rTM$ be an r-th order linear connection on M. The flow operator $\mathcal{F}$ of F transforming projectable vector fields η on $p\colon Y \to M$ into vector fields $\mathcal{F}\eta := \frac{\partial}{\partial t}_{|t=0} F(Fl^\eta_t)$ on FY is of order r. In other words, the value $\mathcal{F}\eta(u)$ at every $u \in F_yY, y \in Y$ depends only on $j^r_y\eta$. Therefore, we have the corresponding flow morphism $\tilde{\mathcal{F}}\colon FY \times_Y J^rTY \to TFY$, which is linear with respect to J^rTY. Moreover, $\tilde{\mathcal{F}}(u, j^r_y\eta) = \mathcal{F}\eta(u)$, where $u \in F_yY, y \in Y$. Let X^Γ be the Γ-lift of a vector field X on M to Y, i.e. X^Γ is a projectable vector field on $p\colon Y \to M$ defined by $X^\Gamma(y) = \Gamma(y, X(x)), y \in Y_x, x = p(y) \in M$. Then the connection Γ can be extended to a morphism $\tilde{\Gamma}\colon Y \times_M J^rTM \to J^rTY$ by the following formula $\tilde{\Gamma}(y, j^r_xX) = j^r_y(X^\Gamma)$. By applying $\mathcal{F}$, we obtain a map $\mathcal{F}(\tilde{\Gamma})\colon FY \times_M J^rTM \to TFY$ defined by $\mathcal{F}(\tilde{\Gamma})(u, j^r_xX) = \tilde{\mathcal{F}}(u, j^r_y(X^\Gamma)) = \mathcal{F}X^\Gamma(u)$. Further the composition

$$\mathcal{F}(\Gamma, \Lambda) := \mathcal{F}(\tilde{\Gamma}) \circ (id_{FY} \times \Lambda)\colon FY \times_M TM \to TFY$$

is the lifting map of a general connection on $FY \to M$. The connection $\mathcal{F}(\Gamma, \Lambda)$ is called F-prolongation of Γ with respect to Λ and was discovered by I. Kolář [5].

Let ∇ be a torsion free classical linear connection on M. For every $x \in M$, the connection ∇ determines the exponential map $exp^\nabla_x : T_xM \to M$ (of ∇ in x), which is diffeomorphism of some neighbourhood of the zero vector at x onto some neighbourhood of x. Every vector $v \in T_xM$ can be extended to a vector field $\tilde{v}$ on a vector space T_xM by $\tilde{v}(w) = \frac{\partial}{\partial t}_{|t=0}[w + tv]$. Then we can construct an r-th order linear connection $E_r(\nabla)\colon TM \to J^rTM$, which is given by $E_r(\nabla)(v) = j^r_x((exp^\nabla_x)_*\tilde{v})$. This connection is called an exponential extension of ∇ and was presented by W. Mikulski in [9]. Another equivalent definition (for corresponding principal connections in the r-frame bundles) of the exponential extension was independently introduced by I. Kolář in [6]. Hence given a general connection Γ on $Y \to M$ and a torsion free classical linear connection ∇ on M, we have the general connection

$$\mathcal{F}(\Gamma, \nabla) := \mathcal{F}(\Gamma, E_r(\nabla))\colon FY \times_M TM \to TFY.$$

The canonical character of construction of this connection can be described by means of the concept of natural operators. The general concept of natural operators can be found in [4]. In particular, we have the following definitions.

Definition 1. Let $F\colon \mathcal{FM}_{m,n} \to \mathcal{FM}$ be a bundle functor of order r on a category $\mathcal{FM}_{m,n}$. An $\mathcal{FM}_{m,n}$-natural operator $D\colon J^1 \times Q_\tau(\mathcal{B}) \rightsquigarrow J^1(F \to \mathcal{B})$ transforming general connections Γ on fibred manifolds $p\colon Y \to M$ and torsion free classical linear connections ∇ on M into general connections $D(\Gamma, \nabla)\colon FY \to J^1FY$ on $FY \to M$ is a system of regular operators $D_Y\colon \mathrm{Con}(Y \to M) \times Q_\tau(M) \to \mathrm{Con}(FY \to M)$, $(p\colon Y \to M) \in Obj(\mathcal{FM}_{m,n})$ satisfying the $\mathcal{FM}_{m,n}$-invariance condition: for any $\Gamma \in \mathrm{Con}(Y \to M)$, $\Gamma_1 \in \mathrm{Con}(Y_1 \to M_1)$, $\nabla \in Q_\tau(M)$ and $\nabla_1 \in Q_\tau(M_1)$ such that if Γ is f-related to Γ_1 by an $\mathcal{FM}_{m,n}$-map $f\colon Y \to Y_1$ covering $\underline{f}\colon M \to M_1$ (i.e. $J^1 f \circ \Gamma = \Gamma_1 \circ f$) and ∇ is $\underline{f}$-related to ∇_1 (i.e. $J^1T\underline{f} \circ \nabla = \nabla_1 \circ T\underline{f}$), then $D_Y(\Gamma, \nabla)$ is Ff-related to $D_{Y_1}(\Gamma_1, \nabla_1)$ (i.e. $J^1Ff \circ D_Y(\Gamma, \nabla) = D_{Y_1}(\Gamma_1, \nabla_1) \circ Ff$). Equivalently the $\mathcal{FM}_{m,n}$-invariance means that for any $\Gamma \in \mathrm{Con}(Y \to M), \Gamma_1 \in \mathrm{Con}(Y_1 \to M_1), \nabla \in Q_\tau(M)$ and $\nabla_1 \in Q_\tau(M_1)$ if diagrams

$$\begin{tikzcd} J^1Y \arrow[r, "J^1f"] & J^1Y_1 \\ Y \arrow[u, "\Gamma"] \arrow[r, "f"] & Y_1 \arrow[u, "\Gamma_1"'] \end{tikzcd} \qquad \begin{tikzcd} J^1TM \arrow[r, "J^1T\underline{f}"] & J^1TM_1 \\ TM \arrow[u, "\nabla"] \arrow[r, "T\underline{f}"] & TM_1 \arrow[u, "\nabla_1"'] \end{tikzcd}$$

commute for a $\mathcal{FM}_{m,n}$-map $f\colon Y \to Y_1$ covering $\underline{f}\colon M \to M_1$, then the diagram

$$\begin{tikzcd} J^1FY \arrow[r, "J^1Ff"] & J^1FY_1 \\ FY \arrow[u, "D_Y(\Gamma,\nabla)"] \arrow[r, "Ff"] & FY_1 \arrow[u, "D_{Y_1}(\Gamma_1,\nabla_1)"'] \end{tikzcd}$$

commutes. We say that the operator D_Y is regular if it transforms smoothly parametrized families of connections into smoothly parametrized ones.

Definition 2. A $\mathcal{M}f_m$-natural operator $A\colon Q_\tau \rightsquigarrow Q^r$ extending torsion free classical linear connections ∇ on m-dimensional manifolds M into r-th order linear connections $A(\nabla)\colon TM \to J^rTM$ on M is a system of regular operators $A_M\colon Q_\tau(M) \to Q^r(M), M \in Obj(\mathcal{M}f_m)$ satisfying the $\mathcal{M}f_m$-invariance condition: if $\nabla \in Q_\tau(M)$ and $\nabla_1 \in Q_\tau(M_1)$ are f-related by a $\mathcal{M}f_m$-map $f\colon M \to M_1$ (i.e. $J^1Tf \circ \nabla = \nabla_1 \circ Tf$), then $A(\nabla)$ and $A(\nabla_1)$ are f-related, too (i.e. $J^rTf \circ A(\nabla) = A(\nabla_1) \circ Tf$). In other words, the $\mathcal{M}f_m$-invariance means that if for any $\nabla \in Q_\tau(M)$, $\nabla_1 \in Q_\tau(M_1)$ the diagram

$$\begin{array}{ccc} J^1TM & \xrightarrow{J^1Tf} & J^1TM_1 \\ \nabla \uparrow & & \uparrow \nabla_1 \\ TM & \xrightarrow{Tf} & TM_1 \end{array}$$

commutes for a $\mathcal{M}f_m$-map $f\colon M \to M_1$, then the following diagram

$$\begin{array}{ccc} J^rTM & \xrightarrow{J^rTf} & J^rTM_1 \\ A(\nabla) \uparrow & & \uparrow A(\nabla_1) \\ TM & \xrightarrow{Tf} & TM_1 \end{array}$$

commutes, too. The regularity means that every A_M transforms smoothly parametrized families of connections into smoothly parametrized ones.

Thus the construction $\mathcal{F}(\Gamma, \Lambda)$ can be considered as the $\mathcal{FM}_{m,n}$-natural operator $\mathcal{F}\colon J^1 \times Q_\tau(\mathcal{B}) \rightsquigarrow J^1(F \to \mathcal{B})$. Similarly, the correspondence $E_r\colon Q_\tau \rightsquigarrow Q^r$ is the $\mathcal{M}f_m$-natural operator.

In [4], the authors described all $\mathcal{FM}_{m,n}$-natural operators $D\colon J^1 \times Q_\tau(\mathcal{B}) \rightsquigarrow J^1(F \to \mathcal{B})$ for a bundle functor $F = J^1\colon \mathcal{FM}_{m,n} \to \mathcal{FM}$. They constructed an additional $\mathcal{FM}_{m,n}$-natural operator P and proved that all $\mathcal{FM}_{m,n}$-natural operators $D\colon J^1 \times Q_\tau(\mathcal{B}) \rightsquigarrow J^1(J^1 \to \mathcal{B})$ form the one parameter family $tP + (1-t)\mathcal{J}^1, t \in \mathbf{R}$.

In this paper we determine all $\mathcal{FM}_{m,n}$-natural operators $D\colon J^1 \times Q_\tau(\mathcal{B}) \rightsquigarrow J^1(J^2 \to \mathcal{B})$. We assume that all manifolds and maps are smooth, i.e. of class C^∞.

2. Quasi-normal fibred coordinate systems. Let $\Gamma\colon Y \to J^1Y$ be a general connection on a fibred manifold $p\colon Y \to M$ with $\dim(M) = m$ and $\dim(Y) = m + n, \nabla$ be a torsion free classical linear connection on M and $y_0 \in Y$ be a point with $x_0 = p(y_0) \in M$.

In [8] W. Mikulski presented a concept of (Γ, ∇, y_0, r)-quasi-normal fibred coordinate systems on Y for any r. For $r = 3$ this concept can be equivalently defined in the following way.

Definition 3. A $(\Gamma, \nabla, y_0, 3)$-quasi-normal fibred coordinate system on Y is a fibred chart ψ on Y with $\psi(y_0) = (0, 0) \in \mathbf{R}^{m,n}$ covering a chart $\underline{\psi}$ on M with centre x_0 if the map $id_{\mathbf{R}^m}$ is a $\underline{\psi}_*\nabla$-normal coordinate system with centre $0 \in \mathbf{R}^m$ and an element $j^2_{(0,0)}(\psi_*\Gamma) \in J^2_{(0,0)}(J^1\mathbf{R}^{m,n} \to \mathbf{R}^{m,n})$ is of the form

$$\tag{1} \begin{aligned} j^2_{(0,0)}(\psi_*\Gamma) = j^2_{(0,0)}\Bigg(\Gamma_0 + \sum_{i,j,k=1}^{m}\sum_{p=1}^{n} a^p_{kij}x^k x^i dx^j \otimes \frac{\partial}{\partial y^p} \\ + \sum_{i,j=1}^{m}\sum_{p,q=1}^{n} b^p_{qij}y^q x^i dx^j \otimes \frac{\partial}{\partial y^p} + \sum_{i,j=1}^{m}\sum_{p=1}^{n} c^p_{ij}x^i dx^j \otimes \frac{\partial}{\partial y^p}\Bigg) \end{aligned}$$

for some (uniquely determined) real numbers a^p_{kij}, b^p_{qij} and c^p_{ij} satisfying

$$\tag{2} \begin{aligned} & a^p_{kij} - a^p_{ikj} = 0 \\ & a^p_{kij} + a^p_{kji} + a^p_{ikj} + a^p_{ijk} + a^p_{jik} + a^p_{jki} = 0 \\ & b^p_{qij} + b^p_{qji} = 0 \\ & c^p_{ij} + c^p_{ji} = 0, \end{aligned}$$

where $\Gamma_0 = \sum_{i=1}^m dx^i \otimes \frac{\partial}{\partial x^i}$ is the trivial general connection on $\mathbf{R}^{m,n}$ and $x^1, \dots, x^m, y^1, \dots, y^n$ are the usual fibred coordinates on $\mathbf{R}^{m,n}$.

In [8] W. Mikulski proved an important proposition ([8], Proposition 2.2) concerning (Γ, ∇, y_0, r)-quasi-normal fibred coordinate systems. Below we recall this result for $r = 3$. A fibred-fibred manifold version of Proposition 2.2 from [8] for $r = 1$ is presented in [7].

Proposition 1. *Let $\Gamma\colon Y \to J^1Y$ be a general connection on a fibred manifold $p\colon Y \to M$ with $\dim(M) = m$ and $\dim(Y) = m + n$, ∇ be a torsion free classical linear connection on M and $y_0 \in Y$ be a point with $x_0 = p(y_0) \in M$. Then:*
(i) There exists a $(\Gamma, \nabla, y_0, 3)$-quasi-normal fibred coordinate system ψ on Y.
(ii) If ψ^1 is another $(\Gamma, \nabla, y_0, 3)$-quasi-normal fibred coordinate system, then

$$\tag{3} j^3_{y_0}\psi^1 = j^3_{y_0}((B \times H) \circ \psi)$$

for a linear map $B \in GL(m)$ and diffeomorphism $H\colon \boldsymbol{R}^n \to \boldsymbol{R}^n$ preserving 0.

From the proof of Proposition 2.2 from [8] it follows that $(B \times H) \circ \psi$ is a $(\Gamma, \nabla, y_0, 3)$-quasi-normal fibred coordinate system for any $B \in GL(m)$ and any diffeomorphism $H\colon \mathbf{R}^n \to \mathbf{R}^n$ preserving 0. In other words, the $\mathcal{FM}_{m,n}$-maps of the form $B \times H$ for $B \in GL(m)$ and diffeomorphisms $H\colon \mathbf{R}^n \to \mathbf{R}^n$ preserving $0 \in \mathbf{R}^n$ transform quasi-normal fibred coordinate systems into quasi-normal ones.

From now on we will usually work in $(\Gamma, \nabla, y_0, 3)$-quasi-normal fibred coordinates for considered Γ and ∇. If coordinates are not necessarily quasi-normal, the reader will be informed.

3. Constructions of connections. Let $\Gamma\colon Y \to J^1Y$ be a general connection on an $\mathcal{FM}_{m,n}$-object $p\colon Y \to M$ and let $\nabla\colon TM \to J^1TM$ be a torsion free classical linear connection on M.

Example 1. Let $A\colon Q_\tau \rightsquigarrow Q^2$ be a $\mathcal{M}f_m$-natural operator and let $\Lambda = A(\nabla)\colon TM \to J^2TM$ be a second order linear connection on M canonically depending on ∇. Then from Introduction for a functor $F = J^2$, we have a general connection

$$\mathcal{J}^2_{(A)}(\Gamma,\nabla) := \mathcal{J}^2(\Gamma, A(\nabla))\colon J^2Y \to J^1J^2Y \tag{4}$$

on $J^2Y \to M$ canonically depending on Γ and ∇.

Because of the canonical character of the construction $\mathcal{J}^2_{(A)}(\Gamma,\nabla)$ we obtain the following proposition.

Proposition 2. *The family* $\mathcal{J}^2_{(A)}\colon J^1 \times Q_\tau(\mathcal{B}) \rightsquigarrow J^1(J^2 \to \mathcal{B})$ *of functions*

$$\mathcal{J}^2_{(A)}\colon \operatorname{Con}(Y \to M) \times Q_\tau(M) \to \operatorname{Con}(J^2Y \to M)$$

for all $\mathcal{FM}_{m,n}$*-objects* $Y \to M$ *is an* $\mathcal{FM}_{m,n}$*-natural operator.*

Example 2. For every torsion free classical linear connection ∇ on a manifold M we have a canonical vector bundle isomorphism $\psi_\nabla\colon J^2TM \to \oplus_{k=0}^2 S^kT^*M \otimes TM$ given by a formula

$$\psi_\nabla(\tau) = \oplus_{k=0}^2 S^kT_0^*\varphi^{-1} \otimes T_0\varphi^{-1}(I(J^2T\varphi(\tau))),$$

where $\tau \in J^2_xTM, x \in M, \varphi$ is a ∇-normal coordinate system on M with centre x and $I\colon J^2_0T\mathbf{R}^m \to \oplus_{k=0}^2 S^kT_0^*\mathbf{R}^m \otimes T_0\mathbf{R}^m$ is the usual identification.

In the main result of [9], W. Mikulski showed that $\mathcal{M}f_m$-natural operators $A\colon Q_\tau \rightsquigarrow Q^2$ are in bijection with $\mathcal{M}f_m$-natural operators $A_0 \equiv 0\colon Q_\tau \rightsquigarrow T^* \otimes T, A_1\colon Q_\tau \rightsquigarrow T^* \otimes T^* \otimes T$ and $A_2\colon Q_\tau \rightsquigarrow T^* \otimes S^2T^* \otimes T$. In other words, the second order linear connections $\Lambda = A(\nabla)\colon TM \to J^2TM$ on M canonically depending on ∇ are in bijection with the tensor fields $A_0(\nabla) \equiv 0\colon M \to T^*M \otimes TM, A_1(\nabla)\colon M \to T^*M \otimes T^*M \otimes TM$ and $A_2(\nabla)\colon M \to T^*M \otimes S^2T^*M \otimes TM$ on M canonically depending on ∇.

Now by means of $\psi_\nabla, A_1(\nabla) \equiv 0$ and $A_2(\nabla)$ we can define a second order linear connection $A(\nabla)\colon TM \to J^2TM$ on M by

$$A(\nabla)(v) = \psi_\nabla^{-1}(v, 0, < A_2(\nabla)(x), v >), v \in T_xM, x \in M \tag{5}$$

In particular, for $A_2(\nabla) \equiv 0\colon M \to T^*M \otimes S^2T^*M \otimes TM$ we obtain

$$A_2^{exp}(\nabla)(v) = \psi_\nabla^{-1}(v, 0, 0)\colon TM \to J^2TM, \tag{6}$$

On the other hand, from [9] it follows that

$$A_2^{exp}(\nabla)(v) = E_2(\nabla)(v).$$

It means that $A_2^{exp}(\nabla)$ is the second order exponential extension of ∇.

Finally, in the accordance with Example 1 we have a general connection

$$\mathcal{J}^2_{(A_2^{exp})}(\Gamma,\nabla) := \mathcal{J}^2(\Gamma, A_2^{exp}(\nabla))\colon J^2Y \to J^1J^2Y \tag{7}$$

on $J^2Y \to M$ canonically depending on Γ and ∇.

Example 3. Let $\rho \in (J^2Y)_{y_0}$, $y_0 \in Y_{x_0}$, $x_0 \in M$ and consider a $(\Gamma,\nabla,y_0,3)$-quasi-normal fibred coordinate system ψ on Y. Then

$$j^2_{(0,0)}(\psi_*\Gamma) = j^2_{(0,0)}\Bigg(\Gamma_0 + \sum_{i,j,k=1}^{m}\sum_{p=1}^{n} a^p_{kij}x^kx^idx^j \otimes \frac{\partial}{\partial y^p} + \sum_{i,j=1}^{m}\sum_{p,q=1}^{n} b^p_{qij}y^qx^idx^j \otimes \frac{\partial}{\partial y^p} + \sum_{i,j=1}^{m}\sum_{p=1}^{n} c^p_{ij}x^idx^j \otimes \frac{\partial}{\partial y^p}\Bigg)$$

for unique real numbers a^p_{kij}, b^p_{qij} and c^p_{ij} satisfying (2). Denote

$$\begin{aligned}\Gamma^{[1]} &= \Gamma_0 + \sum_{i,j,k=1}^{m}\sum_{p=1}^{n} a^p_{kij}x^kx^idx^j \otimes \frac{\partial}{\partial y^p},\\ \Gamma^{[2]} &= \Gamma_0 + \sum_{i,j=1}^{m}\sum_{p,q=1}^{n} b^p_{qij}y^qx^idx^j \otimes \frac{\partial}{\partial y^p} + \sum_{i,j=1}^{m}\sum_{p=1}^{n} c^p_{ij}x^idx^j \otimes \frac{\partial}{\partial y^p}.\end{aligned} \tag{8}$$

Now we define general connections $\mathcal{J}^2_{[1]}(\Gamma,\nabla)\colon J^2Y \to J^1J^2Y$ and $\mathcal{J}^2_{[2]}(\Gamma,\nabla)\colon J^2Y \to J^1J^2Y$ on $J^2Y \to M$ by

$$\begin{aligned}\mathcal{J}^2_{[1]}(\Gamma,\nabla)(\rho) &:= J^1J^2(\psi^{-1})(\mathcal{J}^2_{(A^{exp})}(\Gamma^{[1]},\nabla^0)(J^2\psi(\rho))),\\ \mathcal{J}^2_{[2]}(\Gamma,\nabla)(\rho) &:= J^1J^2(\psi^{-1})(\mathcal{J}^2_{(A^{exp})}(\Gamma^{[2]},\nabla^0)(J^2\psi(\rho))),\end{aligned} \tag{9}$$

where ∇^0 is the usual flat classical linear connection on $\mathbf{R}^m$.

Because of the canonical character of the construction $\mathcal{J}^2_{[i]}(\Gamma,\nabla)$ for $i = 1,2$ we have the following proposition.

Proposition 3. *The family* $\mathcal{J}^2_{[i]}\colon J^1 \times Q_\tau(\mathcal{B}) \rightsquigarrow J^1(J^2 \to \mathcal{B})$ *of functions*

$$\mathcal{J}^2_{[i]}\colon \operatorname{Con}(Y \to M) \times Q_\tau(M) \to \operatorname{Con}(J^2Y \to M)$$

for all $\mathcal{FM}_{m,n}$*-objects* $Y \to M$ *is an* $\mathcal{FM}_{m,n}$*-natural operator.*

4. The main result. We can consider the first jet prolongation functor J^1 as an affine bundle functor on the category $\mathcal{FM}_{m,n}$. The corresponding vector bundle functor is $T^*\mathcal{B} \otimes V$, where $\mathcal{B}\colon \mathcal{FM}_{m,n} \to \mathcal{M}f_m$ is a base functor and V is a vertical tangent functor. For this reason, for any fibred manifold $p\colon Y \to M$ from the category $\mathcal{FM}_{m,n}$, the first jet prolongation $J^1Y \to Y$ is the affine bundle with the corresponding vector bundle $T^*M \otimes VY$. Therefore, $J^1J^2Y \to J^2Y$ is the affine bundle with corresponding vector bundle $T^*M \otimes VJ^2Y$. Thus the set of all $\mathcal{FM}_{m,n}$-natural operators $D\colon J^1 \times Q_\tau(\mathcal{B}) \rightsquigarrow J^1(J^2 \to \mathcal{B})$ possesses the affine space structure.

The following theorem classifies all $\mathcal{FM}_{m,n}$-natural operators $D\colon J^1 \times Q_\tau(\mathcal{B}) \rightsquigarrow J^1(J^2 \to \mathcal{B})$.

Theorem 1. *Let $D\colon J^1 \times Q_\tau(\mathcal{B}) \rightsquigarrow J^1(J^2 \to \mathcal{B})$ be an $\mathcal{FM}_{m,n}$-natural operator transforming general connections $\Gamma\colon Y \to J^1Y$ on $\mathcal{FM}_{m,n}$-objects $Y \to M$ and torsion free classical linear connections ∇ on M into general connections $D(\Gamma,\nabla)\colon J^2Y \to J^1J^2Y$ on $J^2Y \to M$.*

If $m \geq 2$, then there exist uniquely determined real numbers t_0, t_1, t_2 with $t_0 + t_1 + t_2 = 1$ and $\mathcal{M}f_m$-natural operator $A\colon Q_\tau \rightsquigarrow Q^2$ transforming torsion free classical linear connections ∇ on $\mathcal{M}f_m$-objects M into second order linear connections $A(\nabla)\colon TM \to J^2TM$ on M such that

$$D(\Gamma,\nabla) = t_0\mathcal{J}^2_{(A)}(\Gamma,\nabla) + t_1\mathcal{J}^2_{[1]}(\Gamma,\nabla) + t_2\mathcal{J}^2_{[2]}(\Gamma,\nabla) \tag{10}$$

for any $\mathcal{FM}_{m,n}$-object $Y \to M$, any general connection Γ on $Y \to M$ and any torsion free classical linear connection ∇ on M. Besides, if $t_0 \neq 0$, then A is uniquely determined (else A can be arbitrary).

In the case $m = 1$, $D = \mathcal{J}^2$.

In the proof we use methods for finding natural operators presented in [4] and lemmas from [1].

Proof. Let x^i, y^p be the usual fibred coordinates on $\mathbf{R}^{m,n}$,

$$y^p_i = \frac{\partial y^p}{\partial x^i}, \quad y^p_{ij} = y^p_{ji} = \frac{\partial^2 y^p}{\partial x^i \partial x^j}$$

be the additional coordinates on $J^2\mathbf{R}^{m,n}$ and

$$Y^p = dy^p, \quad Y^p_i = dy^p_i, \quad Y^p_{ij} = Y^p_{ji} = dy^p_{ij}$$

be the essential coordinates on the vertical bundle $VJ^2\mathbf{R}^{m,n}$ of $J^2\mathbf{R}^{m,n} \to \mathbf{R}^m$, where $i,j = 1,\dots,m$ and $p = 1,\dots,n$.

On $J^2_0(J^1\mathbf{R}^{m,n})$ we have the coordinates

$$\Gamma^p_i, \quad \Gamma^p_{ij} = \frac{\partial \Gamma^p_i}{\partial x^j}, \quad \Gamma^p_{iq} = \frac{\partial \Gamma^p_i}{\partial y^q}, \quad \Gamma^p_{ijk} = \frac{\partial^2 \Gamma^p_i}{\partial x^j \partial x^k},$$
$$\Gamma^p_{iqr} = \frac{\partial^2 \Gamma^p_i}{\partial y^q \partial y^r}, \quad \Gamma^p_{ijq} = \frac{\partial^2 \Gamma^p_i}{\partial x^j \partial y^q}.$$

The standard coordinates on $J^1_0(Q_\tau(\mathbf{R}^m))$ are $\nabla^i_{jk} = \nabla^i_{kj}$ and $\nabla^i_{jkl} = \nabla^i_{kjl}$, where $i,j,k,l = 1,\dots,m$.

Let ω_k be the usual coordinates on $T^*\mathbf{R}^m$. Then the induced coordinates on the tensor product $(T^*\mathbf{R}^m \otimes VJ^2\mathbf{R}^{m,n})_0$ are

$$Z^p_k = Y^p\omega_k, \quad Z^p_{i;k} = Y^p_i\omega_k, \quad Z^p_{ij;k} = Y^p_{ij}\omega_k.$$

Let $D\colon J^1 \times Q_\tau(\mathcal{B}) \rightsquigarrow J^1(J^2 \to \mathcal{B})$ be an $\mathcal{FM}_{m,n}$-natural operator transforming general connections $\Gamma\colon Y \to J^1Y$ on $\mathcal{FM}_{m,n}$-objects $Y \to M$ and torsion free classical linear connections ∇ on M into general connections $D(\Gamma,\nabla)\colon J^2Y \to J^1J^2Y$ on $J^2Y \to M$.

Since $J^1J^2Y \to J^2Y$ is the affine bundle with the corresponding vector bundle $T^*M \otimes VJ^2Y$, we have the corresponding $\mathcal{FM}_{m,n}$-natural operator

$$\Delta_D\colon J^1 \times Q_\tau(\mathcal{B}) \rightsquigarrow (J^2, T^*\mathcal{B} \otimes VJ^2).$$

It transforms a general connection $\Gamma\colon Y \to J^1Y$ on an $\mathcal{FM}_{m,n}$-object $Y \to M$ and a torsion free classical linear connection ∇ on M into a fibred map

$$\Delta_D(\Gamma,\nabla) := D(\Gamma,\nabla) - \mathcal{J}^2_{(A_2^{exp})}(\Gamma,\nabla)\colon J^2Y \to T^*M \otimes VJ^2Y. \tag{11}$$

Of course, the operator D is fully determined by Δ_D as $D(\Gamma,\nabla) = \Delta_D(\Gamma,\nabla) + \mathcal{J}^2_{(A_2^{exp})}(\Gamma,\nabla)$ for every $\Gamma \in \mathrm{Con}(Y \to M), \nabla \in Q_\tau(M)$. In other words $D = \Delta_D + \mathcal{J}^2_{(A_2^{exp})}$, so it is sufficient to investigate the operator Δ_D.

Using the invariance of Δ_D with respect to the homotheties $\psi_t = tid_{\mathbf{R}^{m,n}}$ covering $\underline{\psi}_t = tid_{\mathbf{R}^m}$ for $t > 0$, we have the homogeneous conditions

$$\begin{aligned}(T^*(tid_{\mathbf{R}^m}) &\otimes VJ^2(tid_{\mathbf{R}^{m,n}}))(\Delta_D(\Gamma,\nabla)(\rho)) \\ &= (\Delta_D((tid_{\mathbf{R}^{m,n}})_*\Gamma, (tid_{\mathbf{R}^m})_*\nabla))(J^2(tid_{\mathbf{R}^{m,n}})(\rho))\end{aligned}$$

for any general connection Γ on $\mathbf{R}^{m,n}$, any torsion free classical linear connection ∇ on $\mathbf{R}^m$ and any $\rho \in (J^2\mathbf{R}^{m,n})_{(0,0)}$. Using the general theory and the above local coordinates, the above condition can be written as the system of homogeneous conditions. Now, by the non-linear Peetre theorem [4] we obtain that the operator Δ_D is of finite order r in Γ and of order s in ∇. Having the natural operator Δ_D of order r in Γ and of finite order s in ∇, we shall deduce that $r = 2$ and $s = 1$.

The operators Δ_D of order 2 in Γ and of order 1 in ∇ are in bijection with $G^3_{m,n}$-invariant maps of standard fibres $f\colon S_1 \times \Lambda \times S_0 \to Z$ over $\underline{f} = id_{S_0}$, where $S_1 = J^2_0(J^1\mathbf{R}^{m,n}), \Lambda = J^1_0(Q_\tau(\mathbf{R}^m)), S_0 = J^2_0\mathbf{R}^{m,n}, Z = (T^*\mathbf{R}^m \otimes VJ^2\mathbf{R}^{m,n})_0$. This map is of the form

$$\begin{aligned}
Z^p_k &= f^p_k(\Gamma^p_i, \Gamma^p_{ij}, \Gamma^p_{iq}, \Gamma^p_{ijk}, \Gamma^p_{iqr}, \Gamma^p_{ijq}, \nabla^i_{jk}, \nabla^i_{jkl}, y^p_i, y^p_{ij})\\
Z^p_{i;k} &= f^p_{i;k}(\Gamma^p_i, \Gamma^p_{ij}, \Gamma^p_{iq}, \Gamma^p_{ijk}, \Gamma^p_{iqr}, \Gamma^p_{ijq}, \nabla^i_{jk}, \nabla^i_{jkl}, y^p_i, y^p_{ij})\\
Z^p_{ij;k} &= f^p_{ij;k}(\Gamma^p_i, \Gamma^p_{ij}, \Gamma^p_{iq}, \Gamma^p_{ijk}, \Gamma^p_{iqr}, \Gamma^p_{ijq}, \nabla^i_{jk}, \nabla^i_{jkl}, y^p_i, y^p_{ij}).
\end{aligned}$$

The group $G^3_{m,n}$ acts on the standard fibre S_0 in the form

$$\begin{aligned}
\overline{y}^p_i &= a^p_q y^q_j \tilde{a}^j_i + a^p_j \tilde{a}^j_i\\
\overline{y}^p_{ij} &= a^p_q y^q_{kl} \tilde{a}^k_i \tilde{a}^l_j + a^p_{qr} y^q_k y^r_l \tilde{a}^k_i \tilde{a}^l_j + a^p_{qk} y^q_l \tilde{a}^k_i \tilde{a}^l_j + a^p_{ql} y^q_k \tilde{a}^k_i \tilde{a}^l_j\\
&\quad + a^p_q y^q_k \tilde{a}^k_{ij} + a^p_k \tilde{a}^k_{ij} + a^p_{kl} \tilde{a}^k_i \tilde{a}^l_j
\end{aligned}$$

and on the fibre S_1 by the formula

$$
\begin{aligned}
\overline{\Gamma}^p_i &= a^p_q \Gamma^q_j \tilde{a}^j_i + a^p_j \tilde{a}^j_i \\
\overline{\Gamma}^p_{ij} &= a^p_q \Gamma^q_{kl} \tilde{a}^k_i \tilde{a}^l_j + a^p_{qr} \Gamma^q_k \Gamma^r_l \tilde{a}^k_i \tilde{a}^l_j + a^p_{qk} \Gamma^q_l \tilde{a}^k_i \tilde{a}^l_j + a^p_{ql} \Gamma^q_k \tilde{a}^k_i \tilde{a}^l_j + a^p_q \Gamma^q_k \tilde{a}^k_{ij} \\
&\quad + a^p_k \tilde{a}^k_{ij} + a^p_{kl} \tilde{a}^k_i \tilde{a}^l_j \\
\overline{\Gamma}^p_{iq} &= a^p_r \Gamma^r_{js} \tilde{a}^s_q \tilde{a}^j_i + a^p_{rs} \Gamma^r_j \tilde{a}^s_q \tilde{a}^j_i + a^p_{rj} \tilde{a}^r_q \tilde{a}^j_i \\
\overline{\Gamma}^p_{ijk} &= [(a^p_{qn} + a^p_{qr} \Gamma^r_n) \Gamma^q_{lm} + (a^p_{nqr} + a^p_{qrs} \Gamma^s_n) \Gamma^q_l \Gamma^r_m + a^p_q \Gamma^q_{lmn} \\
&\quad + a^p_{qr} (\Gamma^q_{ln} \Gamma^r_m + \Gamma^q_l \Gamma^r_{mn}) + (a^p_{qln} + a^p_{qrl} \Gamma^r_n) \Gamma^q_m + a^p_{ql} \Gamma^q_{mn} \\
&\quad + (a^p_{qmn} + a^p_{qrm} \Gamma^r_n) \Gamma^q_l + a^p_{qm} \Gamma^q_{ln} + a^p_{lmn} + a^p_{lmq} \Gamma^q_n] \tilde{a}^l_i \tilde{a}^m_j \tilde{a}^n_k \\
&\quad + (a^p_q \Gamma^q_{lm} + a^p_{qr} \Gamma^q_l \Gamma^r_m + a^p_{ql} \Gamma^q_m + a^p_{qm} \Gamma^q_l + a^p_{lm})(\tilde{a}^l_{ik} \tilde{a}^m_j + \tilde{a}^l_i \tilde{a}^m_{jk}) \\
&\quad + [(a^p_{qn} + a^p_{qr} \Gamma^r_n) \Gamma^q_l + a^p_q \Gamma^q_{ln} + a^p_{ln} + a^p_{ql} \Gamma^q_n] \tilde{a}^l_{ij} \tilde{a}^n_k + (a^p_q \Gamma^q_l + a^p_l) \tilde{a}^l_{ijk} \\
\overline{\Gamma}^p_{iqr} &= (a^p_{su} \Gamma^s_{jt} + a^p_s \Gamma^s_{jtu} + a^p_{stu} \Gamma^s_j + a^p_{st} \Gamma^s_{ju} + a^p_{jtu}) \tilde{a}^j_i \tilde{a}^t_q \tilde{a}^u_r \\
\overline{\Gamma}^p_{ijq} &= (a^p_{rt} \Gamma^r_{kl} + a^p_r \Gamma^r_{klt} + a^p_{rst} \Gamma^r_k \Gamma^s_l + a^p_{rs} \Gamma^r_{kt} \Gamma^s_l + a^p_{rs} \Gamma^r_k \Gamma^s_{lt} + a^p_{rkt} \Gamma^r_l \\
&\quad + a^p_{rk} \Gamma^r_{lt} + a^p_{rlt} \Gamma^r_k + a^p_{rl} \Gamma^r_{kt} + a^p_{klt}) \tilde{a}^k_i \tilde{a}^l_j \tilde{a}^t_q \\
&\quad + (a^p_{rt} \Gamma^r_k + a^p_r \Gamma^r_{kt} + a^p_{kt}) \tilde{a}^k_{ij} \tilde{a}^t_q.
\end{aligned}
$$

The action on Λ is

$$
\begin{aligned}
\overline{\nabla}^i_{jk} &= a^i_l \nabla^l_{mn} \tilde{a}^m_j \tilde{a}^n_k + a^i_{lm} \tilde{a}^l_j \tilde{a}^m_k \\
\overline{\nabla}^i_{jkl} &= a^i_p \nabla^p_{mnq} \tilde{a}^q_l \tilde{a}^n_k \tilde{a}^m_j + a^i_p \nabla^p_{sm} \tilde{a}^m_l \tilde{a}^s_{jk} + a^i_{ps} \nabla^p_{mn} \tilde{a}^n_l \tilde{a}^m_j \tilde{a}^s_k + a^i_{ps} \nabla^s_{nm} \tilde{a}^m_l \tilde{a}^p_j \tilde{a}^n_k \\
&\quad + a^i_{mnq} \tilde{a}^q_l \tilde{a}^m_k \tilde{a}^n_j + a^i_{sm} \tilde{a}^s_{kj} \tilde{a}^m_l.
\end{aligned}
$$

Finally, the group $G^3_{m,n}$ acts on Z in the form

$$
\begin{aligned}
\overline{Z}^p_k &= a^p_q Z^q_l \tilde{a}^l_k \\
\overline{Z}^p_{i;k} &= a^p_{qr} Y^r y^q_j \omega_l \tilde{a}^j_i \tilde{a}^l_k + a^p_q Z^q_{j;l} \tilde{a}^j_i \tilde{a}^l_k + a^p_{qj} Z^q_l \tilde{a}^j_i \tilde{a}^l_k \\
\overline{Z}^p_{ij;k} &= a^p_{qr} Y^r y^q_{lm} \omega_n \tilde{a}^l_i \tilde{a}^m_j \tilde{a}^n_k + a^p_q Z^q_{lm;n} \tilde{a}^l_i \tilde{a}^m_j \tilde{a}^n_k + a^p_{qrs} Y^s y^q_l y^r_m \omega_n \tilde{a}^l_i \tilde{a}^m_j \tilde{a}^n_k \\
&\quad + a^p_{qr} (Y^q_l y^r_m + y^q_l Y^r_m) \omega_n \tilde{a}^l_i \tilde{a}^m_j \tilde{a}^n_k + a^p_{qrl} Y^r y^q_m \omega_n \tilde{a}^l_i \tilde{a}^m_j \tilde{a}^n_k \\
&\quad + a^p_{ql} Z^q_{m;n} \tilde{a}^l_i \tilde{a}^m_j \tilde{a}^n_k + a^p_{qrm} Y^r y^q_l \omega_n \tilde{a}^l_i \tilde{a}^m_j \tilde{a}^n_k + a^p_{qm} Z^q_{l;n} \tilde{a}^l_i \tilde{a}^m_j \tilde{a}^n_k \\
&\quad + a^p_{qr} Y^r y^q_l \omega_n \tilde{a}^l_{ij} \tilde{a}^n_k + a^p_q Z^q_{l;n} \tilde{a}^l_{ij} \tilde{a}^n_k + a^p_{ql} Z^q_n \tilde{a}^l_{ij} \tilde{a}^n_k + a^p_{qlm} Z^q_n \tilde{a}^l_i \tilde{a}^m_j \tilde{a}^n_k.
\end{aligned}
$$

Now we want to show that every $\mathcal{FM}_{m,n}$-natural operator $\Delta_D\colon J^1 \times Q_\tau(\mathcal{B}) \rightsquigarrow (J^2, T^*\mathcal{B} \otimes VJ^2)$ is of order 2 in Γ and of order 1 in ∇. Using the general theory, the operators in question are in bijection with $G^q_{m,n}$-invariant maps

$$
f\colon J^r_0(J^1\mathbf{R}^{m,n}) \times J^s_0(Q_\tau(\mathbf{R}^m)) \times J^2_0\mathbf{R}^{m,n} \to (T^*\mathbf{R}^m \otimes VJ^2\mathbf{R}^{m,n})_0,
$$

where $q = \max\{\text{rank}(J^r J^1), \text{rank}(J^s Q_\tau), \text{rank}(J^2), \text{rank}(T^*), \text{rank}(VJ^2)\} = \max\{r+1, s+2, 2, 1, 3\} = \max\{r+1, s+2, 3\} \geq 3$.

We shall investigate these maps. Let α and γ be multi indices in x^i and β be a multi index in y^p. This associated map of our operator has the form

$$Z^p_k = f^p_k((\Gamma^p_i)_{\alpha\beta}, (\nabla^i_{jk})_\gamma, y^p_i, y^p_{ij})$$
$$Z^p_{i;k} = f^p_{i;k}((\Gamma^p_i)_{\alpha\beta}, (\nabla^i_{jk})_\gamma, y^p_i, y^p_{ij})$$
$$Z^p_{ij;k} = f^p_{ij;k}((\Gamma^p_i)_{\alpha\beta}, (\nabla^i_{jk})_\gamma, y^p_i, y^p_{ij}),$$

where $|\alpha| + |\beta| \leq r$ and $|\gamma| \leq s$.

Using the homotheties

$$\tilde{a}^i_j = t\delta^i_j,\ \tilde{a}^p_q = \delta^p_q,\ a^p_i = 0,\ a^p_{qr} = 0, a^p_{qi} = 0,\ \tilde{a}^k_{ij} = 0,\ a^p_{ij} = 0,\ a^p_{qri} = 0,$$
$$a^p_{qrs} = 0,\ a^p_{qij} = 0,\ a^p_{ijk} = 0,\ \tilde{a}^l_{ijk} = 0,$$

we obtain

$$tf^p_k = f^p_k(t^{1+|\alpha|}(\Gamma^p_i)_{\alpha\beta}, t^{1+|\gamma|}(\nabla^i_{jk})_\gamma, ty^p_i, t^2 y^p_{ij}).$$

From the homogeneous function theorem we deduce that f^p_k is linear in $(\Gamma^p_i)_\beta, \nabla^i_{jk}, y^p_i$ and is independent of y^p_{ij} and of the variables with $|\alpha| > 0$ or $|\gamma| > 0$. Therefore,

$$f^p_k = f^p_k((\Gamma^p_i)_\beta, \nabla^i_{jk}, y^p_i). \tag{12}$$

Considering invariance of (12) with respect to the homotheties

$$\tilde{a}^i_j = \delta^i_j,\ a^p_q = t\delta^p_q,\ a^p_i = 0,\ a^p_{qr} = 0,\ a^p_{qi} = 0,\ \tilde{a}^k_{ij} = 0,\ a^p_{ij} = 0,\ a^p_{qri} = 0,$$
$$a^p_{qrs} = 0,\ a^p_{qij} = 0,\ a^p_{ijk} = 0,\ \tilde{a}^l_{ijk} = 0,$$

we get the condition

$$tf^p_k = f^p_k(t^{1-|\beta|}(\Gamma^p_i)_\beta, \nabla^i_{jk}, ty^p_i).$$

Using again the homogeneous function theorem, we see that f^p_k is independent of $(\Gamma^p_i)_\beta$ with $|\beta| > 1$.

For $f^p_{i;k}$, the homotheties

$$\tilde{a}^i_j = t\delta^i_j,\ \tilde{a}^p_q = \delta^p_q,\ a^p_i = 0,\ a^p_{qr} = 0,\ a^p_{qi} = 0,\ \tilde{a}^k_{ij} = 0,\ a^p_{ij} = 0,\ a^p_{qri} = 0,$$
$$a^p_{qrs} = 0,\ a^p_{qij} = 0,\ a^p_{ijk} = 0,\ \tilde{a}^l_{ijk} = 0,$$

yield

$$t^2 f^p_{i;k} = f^p_{i;k}(t^{1+|\alpha|}(\Gamma^p_i)_{\alpha\beta}, t^{1+|\gamma|}(\nabla^i_{jk})_\gamma, ty^p_i, t^2 y^p_{ij})$$

so that $f^p_{i;k}$ is a polynomial independent of the variables with $|\alpha| > 1$ or $|\gamma| > 1$. In other words,

$$f^p_{i;k} = f^p_{i;k}((\Gamma^p_i)_{\alpha\beta}, (\nabla^i_{jk})_\gamma, y^p_i, y^p_{ij}) \tag{13}$$

for $|\alpha| \leq 1$ and $|\gamma| \leq 1$.

The homotheties

$\tilde{a}^i_j = \delta^i_j,\ a^p_q = t\delta^p_q,\ a^p_i = 0,\ a^p_{qr} = 0,\ a^p_{qi} = 0,\ \tilde{a}^k_{ij} = 0,\ a^p_{ij} = 0,\ a^p_{qri} = 0,$
$a^p_{qrs} = 0,\ a^p_{qij} = 0,\ a^p_{ijk} = 0,\ \tilde{a}^l_{ijk} = 0,$

imply

$$tf^p_{i;k} = f^p_{i;k}(t^{1-|\beta|}(\Gamma^p_i)_{\alpha\beta}, (\nabla^i_{jk})_\gamma, ty^p_i, ty^p_{ij})$$

for $|\alpha| \leq 1$ and $|\gamma| \leq 1$. Therefore we deduce that $f^p_{i;k}$ is independent of $(\Gamma^p_i)_{\alpha\beta}$ for $|\alpha| + |\beta| > 2$ and $(\nabla^i_{jk})_\gamma$ for $|\gamma| > 1$.

Now invariance of $f^p_{ij;k}$ with respect to the homotheties

$\tilde{a}^i_j = t\delta^i_j,\ \tilde{a}^p_q = t\delta^p_q,\ a^p_i = 0,\ a^p_{qr} = 0,\ a^p_{qi} = 0,\ \tilde{a}^k_{ij} = 0,\ a^p_{ij} = 0,\ a^p_{qri} = 0,$
$a^p_{qrs} = 0,\ a^p_{qij} = 0,\ a^p_{ijk} = 0,\ \tilde{a}^l_{ijk} = 0,$

gives

$$t^2 f^p_{ij;k} = f^p_{ij;k}(t^{|\alpha|+|\beta|}(\Gamma^p_i)_{\alpha\beta}, t^{1+|\gamma|}(\nabla^i_{jk})_\gamma, y^p_i, ty^p_{ij}).$$

So $f^p_{ij;k}$ is a polynomial independent of $(\Gamma^p_i)_{\alpha\beta}$ for $|\alpha| + |\beta| > 2$ and $(\nabla^i_{jk})_\gamma$ for $|\gamma| > 1$. Hence the associated map of our operator is independent of $(\Gamma^p_i)_{\alpha\beta}$ for $|\alpha| + |\beta| > 2$ and $(\nabla^i_{jk})_\gamma$ for $|\gamma| > 1$. This completes the proof of the fact that $\mathcal{FM}_{m,n}$-natural operator $\Delta_D \colon J^1 \times Q_\tau(\mathcal{B}) \rightsquigarrow (J^2, T^*\mathcal{B} \otimes VJ^2)$ is of order 2 in Γ and of order 1 in ∇. In other words it means that the value $\Delta_D(\Gamma, \nabla)(\rho)$ is determined by $j^2_{(0,0)}\Gamma$ and $j^1_0(\nabla)$ and ρ for any $\Gamma \in \mathrm{Con}(\mathbf{R}^{m,n})$, $\nabla \in Q_\tau(\mathbf{R}^m)$ and $\rho \in (J^2\mathbf{R}^{m,n})_{(0,0)}$.

In the rest of the proof, we shall use $(\Gamma, \nabla, y_0, 3)$-quasi-normal fibred coordinate systems,only. Consider the case $m \geq 2$.

Since Δ_D is invariant with respect to $(\Gamma, \nabla, y_0, 3)$-quasi-normal fibred coordinate systems, Δ_D is determined by the contractions $\langle \Delta_D(\Gamma, \nabla)(\rho), v\rangle \in V_\rho J^2\mathbf{R}^{m,n}$ for all $\rho \in (J^2\mathbf{R}^{m,n})_{(0,0)}$, all $v \in T_0\mathbf{R}^m$, all general connections Γ on $\mathbf{R}^{m,n}$ and all torsion free classical linear connections ∇ on $\mathbf{R}^m$ such that $\psi = id_{\mathbf{R}^{m,n}}$ is a $(\Gamma, \nabla, (0,0), 3)$-quasi-normal fibred coordinate system on $\mathbf{R}^{m,n}$ over $\underline{\psi} = id_{\mathbf{R}^m}$.

For vector bundles $E \to M$ we have the standard identification $VE = E \times_M E$ which is a vector bundle isomorphism. As $\mathbf{R}^{m,n}$ is a vector bundle and $J^2\mathbf{R}^{m,n}$ is a vector bundle we can write that $V_\rho J^2\mathbf{R}^{m,n} \cong_\rho J^2_0\mathbf{R}^{m,n}$. This identification $\cong_\rho$ is $GL(m)\times GL(n)$-invariant but not $\mathcal{FM}_{m,n}$-invariant.

Next we use the usual $GL(m) \times GL(n)$-invariant identification

$$J^2_0\mathbf{R}^{m,n} \cong \oplus^2_{k=0} S^k\mathbf{R}^{m*} \otimes \mathbf{R}^n$$

(it is not $\mathcal{FM}_{m,n}$-invariant). Therefore, the values $\langle \Delta_D(\Gamma, \nabla)(\rho), v\rangle$ are determined by the values $\psi^k_{\Gamma,\nabla}(\rho, v) \in S^k\mathbf{R}^{m*} \otimes \mathbf{R}^n$ for $k = 0, 1, 2$ obtained by composing the values $\langle \Delta_D(\Gamma, \nabla)(\rho), v\rangle$ with the respective projections. So we can write

$$\langle \Delta_D(\Gamma, \nabla)(\rho), v\rangle \cong \psi^0_{\Gamma,\nabla}(\rho, v) \oplus \psi^1_{\Gamma,\nabla}(\rho, v) \oplus \psi^2_{\Gamma,\nabla}(\rho, v),$$

where $\psi^0_{\Gamma,\nabla}(\rho,v) \in \mathbf{R}^n$, $\psi^1_{\Gamma,\nabla}(\rho,v) \in \mathbf{R}^{m*} \otimes \mathbf{R}^n$, $\psi^2_{\Gamma,\nabla}(\rho,v) \in S^2\mathbf{R}^{m*} \otimes \mathbf{R}^n$.

Now the values $\psi^k_{\Gamma,\nabla}(\rho,v) \in S^k\mathbf{R}^{m*} \otimes \mathbf{R}^n$ for $k = 0, 1$ are determined by the contractions $\langle \psi^0_{\Gamma,\nabla}(\rho,v), u\rangle$, $\langle \psi^1_{\Gamma,\nabla}(\rho,v), w \otimes u\rangle$ for all $v \in T_0\mathbf{R}^m \cong \mathbf{R}^m$, $u \in \mathbf{R}^{n*}$, $w \in \mathbf{R}^m$ and all Γ, ∇ in question.

Using the polarization formula from linear algebra, we have that every symmetric bilinear form on a vector space is uniquely determined by the corresponding quadratic form. Therefore, for $k = 2$ the values $\psi^2_{\Gamma,\nabla}(\rho,v)$ are determined by the contractions $\langle \psi^2_{\Gamma,\nabla}(\rho,v), (w \odot w) \otimes u\rangle$ for all v, u, w, Γ, ∇ as above, where $\odot$ denotes the symmetric tensor product. Then by the density argument and $m \geq 2$, we can assume that v and w are linearly independent and $u \neq 0$.

Using the $GL(m) \times GL(n)$-invariance of Δ_D and Proposition 1, we can assume $v = e_1$, $w = e_2$, $u = E^1$, where (e_i) is the standard basis in $\mathbf{R}^m$, (E_p) is the standard basis in $\mathbf{R}^n$ and (E^p) is the dual basis in $\mathbf{R}^{n*}$. So we get that the operator Δ_D is uniquely determined by the values $\langle \psi^0_{\Gamma,\nabla}(\rho, \frac{\partial}{\partial x^1}|_0), E^1\rangle$, $\langle \psi^1_{\Gamma,\nabla}(\rho, \frac{\partial}{\partial x^1}|_0), e_2 \otimes E^1\rangle$ and $\langle \psi^2_{\Gamma,\nabla}(\rho, \frac{\partial}{\partial x^1}|_0), (e_2 \odot e_2) \otimes E^1\rangle$. In other words, Δ_D is uniquely determined by the values

$$\begin{gathered}
\Big\langle Y^1_{|\rho}, \Big\langle \Delta_D(\Gamma,\nabla)(\rho), \frac{\partial}{\partial x^1}|_0\Big\rangle\Big\rangle \in \mathbf{R} \\
\Big\langle Y^1_{2|\rho}, \Big\langle \Delta_D(\Gamma,\nabla)(\rho), \frac{\partial}{\partial x^1}|_0\Big\rangle\Big\rangle \in \mathbf{R} \\
\Big\langle Y^1_{22|\rho}, \Big\langle \Delta_D(\Gamma,\nabla)(\rho), \frac{\partial}{\partial x^1}|_0\Big\rangle\Big\rangle \in \mathbf{R}
\end{gathered} \tag{14}$$

for all $\rho \in (J^2\mathbf{R}^{m,n})_{(0,0)}$, all general connections Γ on $\mathbf{R}^{m,n}$ and all torsion free classical linear connections ∇ on $\mathbf{R}^m$ such that $\psi = id_{\mathbf{R}^{m,n}}$ is a $(\Gamma, \nabla, (0,0), 3)$-quasi-normal fibred coordinate system on $\mathbf{R}^{m,n}$ over $\underline{\psi} = id_{\mathbf{R}^m}$.

Consider locally defined $\mathcal{FM}_{m,n}$-maps $\psi_2 \colon \mathbf{R}^{m,n} \to \mathbf{R}^{m,n}$, $\psi_3 \colon \mathbf{R}^{m,n} \to \mathbf{R}^{m,n}$ given by

$$\begin{aligned}
\psi_2(x,y) &= \big(x, y_1 + (y_1)^2, y_2, \ldots, y_n\big) \\
\psi_3(x,y) &= \big(x, y_1 + (y_1)^3, y_2, \ldots, y_n\big)
\end{aligned}$$

for $x \in \mathbf{R}^n$ and $y = (y_1, y_2, \ldots, y_n) \in \mathbf{R}^n$. They preserve $\frac{\partial}{\partial x^1}|_0$ and can be written in the form $\psi_a(x,y) = (id_{\mathbf{R}^m}(x), H_a(y))$, where $H_a(y) = (y_1 + (y_1)^a, y_2, \ldots, y_n)$ and $a = 2, 3$. So $\psi_a = id_{\mathbf{R}^m} \times H_a$ for $H_a \colon \mathbf{R}^n \to \mathbf{R}^n$ being a diffeomorphism preserving 0. Hence by Proposition 1 these $\mathcal{FM}_{m,n}$-maps $\psi_a \colon \mathbf{R}^{m,n} \to \mathbf{R}^{m,n}$ for $a = 2, 3$ transform quasi-normal fibred coordinate systems into quasi-normal ones. Using the invariance of Δ_D with respect to $\psi_a \colon \mathbf{R}^{m,n} \to \mathbf{R}^{m,n}$ for $a = 2, 3$ and the density argument, we show that the values $\langle Y^1_{2|\rho}, \langle \Delta_D(\Gamma,\nabla)(\rho), \frac{\partial}{\partial x^1}|_0\rangle\rangle$ and $\langle Y^1_{|\rho}, \langle \Delta_D(\Gamma,\nabla)(\rho), \frac{\partial}{\partial x^1}|_0\rangle\rangle$

for all $\Gamma \in \mathrm{Con}(\mathbf{R}^{m,n})$, $\nabla \in Q_\tau(\mathbf{R}^m)$, $\rho \in (J^2\mathbf{R}^{m,n})_{(0,0)}$ are determined by the values $\langle Y^1_{22|\rho}, \langle \Delta_D(\Gamma,\nabla)(\rho), \frac{\partial}{\partial x^1}|_0\rangle\rangle$ for all Γ, ∇, ρ as above.

Using the action of the group $G^3_{m,n}$ on S_0 for $a = 2$, we obtain $\overline{y}^1_{22} = y^1_{22} + 2y^1y^1_{22} + 2(y^1_2)^2$ and then

$$\overline{Y}^1_{22} = d\overline{y}^1_{22} = Y^1_{22} + 4y^1_2Y^1_2 + 2y^1_{22}Y^1 + 2y^1Y^1_{22} = Y^1_{22} + 4y^1_2Y^1_2 + 2y^1_{22}Y^1 \tag{15}$$

over $(0,0) \in \mathbf{R}^{m,n}$ (i.e. for $y^1 = 0$). Similarly, for $a = 3$ we get $\tilde{y}^1_{22} = y^1_{22} + 3(y^1)^2y^1_{22} + 6y^1(y^1_2)^2$ and then

$$\begin{aligned}\tilde{Y}^1_{22} = d\tilde{y}^1_{22} &= Y^1_{22} + 6(y^1_2)^2Y^1 + 6y^1y^1_{22}Y^1 + 3(y^1)^2Y^1_{22} + 12y^1y^1_2Y^1_2 \\ &= Y^1_{22} + 6(y^1_2)^2Y^1\end{aligned} \tag{16}$$

over $(0,0) \in \mathbf{R}^{m,n}$.

By formula (16) for $y^1_2(\rho) \neq 0$, we have

$$Y^1 = \frac{\tilde{Y}^1_{22} - Y^1_{22}}{6(y^1_2)^2} \tag{17}$$

and consequently the values $\langle Y^1_{|\rho}, \langle \Delta_D(\Gamma,\nabla)(\rho), \frac{\partial}{\partial x^1}|_0\rangle\rangle$ for all $\Gamma \in \mathrm{Con}(\mathbf{R}^{m,n})$, $\nabla \in Q_\tau(\mathbf{R}^m)$, $\rho \in (J^2\mathbf{R}^{m,n})_{(0,0)}$ are determined by the values $\langle Y^1_{22|\rho}, \langle \Delta_D(\Gamma,\nabla)(\rho), \frac{\partial}{\partial x^1}|_0\rangle\rangle$ for all Γ, ∇, ρ as above.

Then analogously from (15) and (17), we see that

$$Y^1_2 = \frac{(\overline{Y}^1_{22} - Y^1_{22}) \cdot 3(y^1_2)^2 - y^1_{22}(\tilde{Y}^1_{22} - Y^1_{22})}{12(y^1_2)^3}$$

and therefore, the values $\langle Y^1_{2|\rho}, \langle \Delta_D(\Gamma,\nabla)(\rho), \frac{\partial}{\partial x^1}|_0\rangle\rangle$ for all $\Gamma \in \mathrm{Con}(\mathbf{R}^{m,n})$, $\nabla \in Q_\tau(\mathbf{R}^m)$, $\rho \in (J^2\mathbf{R}^{m,n})_{(0,0)}$ are determined by the values $\langle Y^1_{22|\rho}, \langle \Delta_D(\Gamma,\nabla)(\rho), \frac{\partial}{\partial x^1}|_0\rangle\rangle$ for all Γ, ∇, ρ as above.

Summing up, we obtain that the operator Δ_D is uniquely determined by the values

$$\left\langle Y^1_{22|\rho}, \left\langle \Delta_D(\Gamma,\nabla)(\rho), \frac{\partial}{\partial x^1}|_0 \right\rangle\right\rangle \in \mathbf{R} \tag{18}$$

for all general connections Γ on $\mathbf{R}^{m,n}$ such that

$$\begin{aligned} j^2_{(0,0)}\Gamma = j^2_{(0,0)}\Bigg(&\Gamma_0 + \sum_{i,j,k=1}^{m}\sum_{p=1}^{n} a^p_{kij}x^kx^idx^j \otimes \frac{\partial}{\partial y^p} \\ &+ \sum_{i,j=1}^{m}\sum_{p,q=1}^{n} b^p_{qij}y^qx^idx^j \otimes \frac{\partial}{\partial y^p} + \sum_{i,j=1}^{m}\sum_{p=1}^{n} c^p_{ij}x^idx^j \otimes \frac{\partial}{\partial y^p}\Bigg)\end{aligned} \tag{19}$$

for unique real numbers a^p_{kij}, b^p_{qij} and c^p_{ij} satisfying (2) and all torsion free classical linear connections ∇ such that the identity map $id_{\mathbf{R}^m}$ is a ∇-normal coordinate system with center zero (then $j^1_0(\nabla) = j^1_0((\sum_{k=1}^m \nabla^l_{ij;k}x^k)^m_{i,j,l=1})$

for some $\nabla^l_{ij;k} = \nabla^l_{ji;k} \in \mathbf{R}$ satisfying some "classical" conditions) and all $\rho \in (J^2\mathbf{R}^{m,n})_{(0,0)}$ of the form

$$\rho = j^2_0\Big(\Big(\sum_{i=1}^{m} g^p_i x^i + \sum_{i,j=1}^{m} h^p_{ij} x^i x^j\Big)_{p=1}^{n}\Big) \tag{20}$$

for real numbers $g^p_i, h^p_{ij} = h^p_{ji}$. So, it is sufficient to study the values (18) for Γ, ∇, ρ as above.

Equivalently, in terms of $G^3_{m,n}$-invariant maps between the standard fibres we obtain that values of functions f^1_1 and $f^1_{2;1}$ are determined by values of functions $f^1_{22;1}$. So we will study the values

$$\begin{aligned} f^1_{22;1}(&\Gamma^p_{kij} = a^p_{kij},\ \Gamma^p_{qij} = b^p_{qij},\ \Gamma^p_{ij} = c^p_{ij},\ \nabla^l_{ijk} = \nabla^l_{ij;k},\\ &y^p_i = g^p_i,\ y^p_{ij} = h^p_{ij}). \end{aligned} \tag{21}$$

The invariance of $f^p_{ij;k}$ with respect to the homotheties

$$\begin{aligned} &\tilde a^i_j = t\delta^i_j,\ \tilde a^p_q = t\delta^p_q,\ a^p_i = 0,\ a^p_{qr} = 0,\ a^p_{qi} = 0,\ \tilde a^k_{ij} = 0,\ a^p_{ij} = 0,\\ &a^p_{qri} = 0,\ a^p_{qrs} = 0,\ a^p_{qij} = 0,\ a^p_{ijk} = 0,\ \tilde a^l_{ijk} = 0, \end{aligned}$$

yields

$$t^2 f^p_{ij;k} = f^p_{ij;k}(t^2 a^p_{kij},\ t^2 b^p_{qij},\ t c^p_{ij},\ t^2\nabla^l_{ij;k},\ g^p_i, t h^p_{ij}).$$

Then the homogeneous function theorem implies that $f^p_{ij;k}$ is linear in a^p_{kij}, b^p_{qij}, $\nabla^l_{ij;k}$, bilinear in c^p_{ij}, h^p_{ij}, quadratic in c^p_{ij} and h^p_{ij}. In other words $f^p_{ij;k}$ is the linear combination of monomials

$$a^p_{kij}, b^p_{qij}, \nabla^l_{ij;k}, c^p_{ij} h^{p_1}_{i_1j_1}, c^p_{ij} c^{p_1}_{i_1j_1}, h^p_{ij} h^{p_1}_{i_1j_1} \tag{22}$$

with the coefficients being smooth functions in the coefficients g^p_i of ρ.

Then using the invariance of $f^p_{ij;k}$ with respect to the homotheties

$$\begin{aligned} &\tilde a^i_j = \delta^i_j,\ a^p_q = t\delta^p_q,\ a^p_i = 0,\ a^p_{qr} = 0,\ a^p_{qi} = 0,\ \tilde a^k_{ij} = 0,\ a^p_{ij} = 0,\\ &a^p_{qri} = 0,\ a^p_{qrs} = 0,\ a^p_{qij} = 0,\ a^p_{ijk} = 0,\ \tilde a^l_{ijk} = 0, \end{aligned}$$

for $t > 0$ and the homogeneous function theorem, we observe that the coefficients on a^p_{kij} are constant, the coefficients on b^p_{qij} and $\nabla^l_{ij;k}$ are linear and the coefficients on other terms from (22) are zero.

Then using the invariance of $f^p_{ij;k}$ with respect to the $\mathcal{FM}_{m,n}$-maps $\psi_{t,\tau} \colon \mathbf{R}^{m,n} \to \mathbf{R}^{m,n}$ given by $\psi_{t,\tau}(x, y) = (t^1x^1, \dots, t^mx^m, \tau^1y^1, \dots, \tau^ny^n)$

for $t^i > 0$, $i = 1, \ldots, m$ and $\tau^p > 0$, $p = 1, \ldots, n$ we deduce that

$$\begin{aligned} f^1_{22;1} &= (\alpha_1 a^1_{122} + \alpha_2 a^1_{212} + \alpha_3 a^1_{221}) \\ &\quad + \left(\sum_{q=1}^n \beta_{q12} b^1_{q12} g^q_2 + \sum_{q=1}^n \beta_{q21} b^1_{q21} g^q_2 + \sum_{q=1}^n \beta_{q22} b^1_{q22} g^q_1 \right) \\ &\quad + \left(\sum_{q=1}^n \gamma_{q12} b^q_{q12} g^1_2 + \sum_{q=1}^n \gamma_{q21} b^q_{q21} g^1_2 + \sum_{q=1}^n \gamma_{q22} b^q_{q22} g^1_1 \right) + g((g^1_l), (\nabla^l_{ij;k})) \end{aligned}$$

for some uniquely determined real numbers α_1, α_2, α_3, β_{q12}, β_{q21}, β_{q22}, γ_{q12}, γ_{q21}, γ_{q22} and some uniquely determined bilinear function g.

Now because of conditions (2) we have

$$\begin{aligned} f^1_{22;1} &= a^1_{122}(\alpha_1 + \alpha_2 - 2\alpha_3) + \sum_{q=1}^n (\beta_{q12} - \beta_{q21}) b^1_{q12} g^q_2 \\ &\qquad + \sum_{q=1}^n (\gamma_{q12} - \gamma_{q21}) b^q_{q12} g^1_2 + g((g^1_l), (\nabla^l_{ij;k})) \\ &= \alpha a^1_{122} + \sum_{q=1}^n \beta_q b^1_{q12} g^q_2 + \sum_{q=1}^n \gamma_q b^q_{q12} g^1_2 + g((g^1_l), (\nabla^l_{ij;k})), \end{aligned}$$

where $\alpha = \alpha_1 + \alpha_2 - 2\alpha_3$, $\beta_q = \beta_{q12} - \beta_{q21}$, $\gamma_q = \gamma_{q12} - \gamma_{q21}$ for $q = 1, \ldots, n$. Further evaluations give

$$\begin{aligned} f^1_{22;1} &= \alpha a^1_{122} + (\beta_1 + \gamma_1) b^1_{112} g^1_2 + \sum_{q=2}^n \beta_q b^1_{q12} g^q_2 \\ &\qquad + \sum_{q=2}^n \gamma_q b^q_{q12} g^1_2 + g((g^1_l), (\nabla^l_{ij;k})) \\ &= \alpha a^1_{122} + \beta b^1_{112} g^1_2 + \sum_{q=2}^n \beta_q b^1_{q12} g^q_2 + \sum_{q=2}^n \gamma_q b^q_{q12} g^1_2 + g((g^1_l), (\nabla^l_{ij;k})), \end{aligned}$$

where $\beta = \beta_1 + \gamma_1$. In other words,

$$\begin{aligned} \left\langle Y^1_{22|\rho}, \left\langle \Delta_D(\Gamma, \nabla)(\rho), \frac{\partial}{\partial x^1}|_0 \right\rangle \right\rangle &= \alpha a^1_{122} + \beta b^1_{112} g^1_2 + \sum_{q=2}^n \beta_q b^1_{q12} g^q_2 \\ &\quad + \sum_{q=2}^n \gamma_q b^q_{q12} g^1_2 + g((g^1_l), (\nabla^l_{ij;k})), \end{aligned} \tag{23}$$

for some uniquely determined real numbers $\alpha, \beta, \beta_q, \gamma_q$ and some uniquely determined bilinear function g, where $j^2_{(0,0)}\Gamma$ is of the form (19) with the coefficients a^p_{kij}, b^p_{qij} and c^p_{ij} satisfying (2), $j^1_0(\nabla) = j^1_0((\sum_{k=1}^m \nabla^l_{ij;k} x^k)^m_{i,j,l=1})$

for some $\nabla^l_{ij;k} = \nabla^l_{ji;k} \in \mathbf{R}$ satisfying some "classical" conditions and ρ is of the form (20) with $g^p_i, h^p_{ij} = h^p_{ji}$.

From (23) it follows that Δ_D is determined by the real number α, the bilinear map g and the values

$$\begin{aligned}(24)\qquad &\Delta_D\Bigg(\Gamma_0 + (x^1dx^2 - x^2dx^1)\otimes\frac{\partial}{\partial y^1}\\ &+\sum_{p,q=1}^n b^p_{q12}y^q(x^1dx^2 - x^2dx^1)\otimes\frac{\partial}{\partial y^p}, \nabla^0\Bigg)(\rho)\\ &=\Delta_D\Bigg(\Gamma_0 + (x^1dx^2 - x^2dx^1)\Bigg(\frac{\partial}{\partial y^1} + \sum_{p,q=1}^n b^p_{q12}y^q\frac{\partial}{\partial y^p}\Bigg), \nabla^0\Bigg)(\rho)\end{aligned}$$

for all $b^p_{q12} \in \mathbf{R}$ and all $\rho \in (J^2\mathbf{R}^{m,n})_{(0,0)}$, where ∇^0 is the usual flat torsion free classical linear connection on $\mathbf{R}^m$.

Considering the invariance of Δ_D with respect to the maps $id_{\mathbf{R}^m} \times H$ for diffeomorphisms $H\colon \mathbf{R}^n \to \mathbf{R}^n$ preserving 0, we get that $\sum_{p,q=1}^n b^p_{q12}y^q\frac{\partial}{\partial y^p}$ is near 0 equal to zero modulo some diffeomorphism $H\colon \mathbf{R}^n \to \mathbf{R}^n$ preserving 0. Hence we have that Δ_D is determined by the real number α, the bilinear map g and the values

$$(25)\qquad \Delta_D\Bigg(\Gamma_0 + a(x^1dx^2 - x^2dx^1)\otimes\frac{\partial}{\partial y^1}, \nabla^0\Bigg)(\rho)$$

for all $a \in \mathbf{R}$ and all $\rho \in (J^2\mathbf{R}^{m,n})_{(0,0)}$.

Next using the invariance of Δ_D with respect to the homotheties

$$\begin{aligned}&\tilde{a}^i_j = \delta^i_j,\ a^p_q = t\delta^p_q,\ a^p_i = 0,\ a^p_{qr} = 0,\ a^p_{qi} = 0,\ \tilde{a}^k_{ij} = 0,\ a^p_{ij} = 0,\\ &a^p_{qri} = 0,\ a^p_{qrs} = 0,\ a^p_{qij} = 0,\ a^p_{ijk} = 0,\ \tilde{a}^l_{ijk} = 0,\end{aligned}$$

from the homogeneous function theorem, it follows that (25) depends linearly in (a, ρ). This implies that Δ_D is determined by the real number α, the bilinear map g and the values

$$\Delta_D\Bigg(\Gamma_0 + (x^1dx^2 - x^2dx^1)\otimes\frac{\partial}{\partial y^1}, \nabla^0\Bigg)(j^2_0 0) \quad\text{and}\quad \Delta_D(\Gamma_0, \nabla^0)(\rho)$$

for all $\rho \in (J^2\mathbf{R}^{m,n})_{(0,0)}$.

Now the values $\Delta_D(\Gamma_0, \nabla^0)(\rho)$ are determined by the values $\langle\Delta_D(\Gamma_0, \nabla^0)(\rho), v\rangle \in V_\rho J^2\mathbf{R}^{m,n} \cong_\rho J^2_0\mathbf{R}^{m,n} \cong \oplus_{k=0}^2 S^k\mathbf{R}^{m*}\otimes\mathbf{R}^n$ for all $\rho \in (J^2\mathbf{R}^{m,n})_{(0,0)}$, $v \in T_0\mathbf{R}^m$ such that $\psi = id_{\mathbf{R}^{m,n}}$ is a $(\Gamma_0, \nabla^0, (0,0), 3)$-quasi-normal fibred coordinate system on $\mathbf{R}^{m,n}$ over $\underline{\psi} = id_{\mathbf{R}^m}$. Since the $\mathcal{FM}_{m,n}$-maps of the form $B \times H$ (in question) preserve the trivial general connection Γ_0 and the flat torsion free classical linear connection ∇^0 then we deduce that the values $\Delta_D(\Gamma_0, \nabla^0)(\rho)$ are determined by the values

$\langle Y^1_{22|\rho}, \langle \Delta_D(\Gamma_0, \nabla^0)(\rho), \frac{\partial}{\partial x^1}|_0\rangle\rangle$. But using the formula (23), we see that the last values are equal to zero. Therefore,

$$\Delta_D(\Gamma_0, \nabla^0)(\rho) = 0 \tag{26}$$

for any $\rho \in (J^2\mathbf{R}^{m,n})_{(0,0)}$. This gives that Δ_D is determined by the real number α, the bilinear map g and the values

$$\Delta_D\Big(\Gamma_0 + (x^1dx^2 - x^2dx^1) \otimes \frac{\partial}{\partial y^1}, \nabla^0\Big)(j^2_0 0). \tag{27}$$

The value (27) is determined by the evaluations

$$\begin{aligned} &\Big\langle Y^p_{|j^2_0 0}, \Big\langle \Delta_D\Big(\Gamma_0 + (x^1dx^2 - x^2dx^1) \otimes \frac{\partial}{\partial y^1}, \nabla^0\Big)(j^2_0 0), \frac{\partial}{\partial x^k}|_0\Big\rangle\Big\rangle \\ &\Big\langle Y^p_{i|j^2_0 0}, \Big\langle \Delta_D\Big(\Gamma_0 + (x^1dx^2 - x^2dx^1) \otimes \frac{\partial}{\partial y^1}, \nabla^0\Big)(j^2_0 0), \frac{\partial}{\partial x^k}|_0\Big\rangle\Big\rangle \\ &\Big\langle Y^p_{ij|j^2_0 0}, \Big\langle \Delta_D\Big(\Gamma_0 + (x^1dx^2 - x^2dx^1) \otimes \frac{\partial}{\partial y^1}, \nabla^0\Big)(j^2_0 0), \frac{\partial}{\partial x^k}|_0\Big\rangle\Big\rangle \end{aligned} \tag{28}$$

for all $p = 1, \dots, n$ and all $i, j, k = 1, \dots, m$.

Since (25) depends linearly on a, using the invariance of Δ_D with respect to the homotheties

$$\begin{aligned} &\tilde a^i_j = \delta^i_j,\ \tilde a^p_q = t\delta^p_q, a^p_i = 0,\ a^p_{qr} = 0, a^p_{qi} = 0,\ \tilde a^k_{ij} = 0,\ a^p_{ij} = 0, \\ &a^p_{qri} = 0,\ a^p_{qrs} = 0,\ a^p_{qij} = 0,\ a^p_{ijk} = 0,\ \tilde a^l_{ijk} = 0, \end{aligned}$$

we see that

$$\begin{aligned} \Big\langle Y^p_{|j^2_0 0}, \Big\langle \Delta_D\Big(\Gamma_0 + (x^1dx^2 - x^2dx^1) \otimes \frac{\partial}{\partial y^1}, \nabla^0\Big)(j^2_0 0), \frac{\partial}{\partial x^k}|_0\Big\rangle\Big\rangle = 0, \\ \Big\langle Y^p_{ij|j^2_0 0}, \Big\langle \Delta_D\Big(\Gamma_0 + (x^1dx^2 - x^2dx^1) \otimes \frac{\partial}{\partial y^1}, \nabla^0\Big)(j^2_0 0), \frac{\partial}{\partial x^k}|_0\Big\rangle\Big\rangle = 0. \end{aligned}$$

Therefore, Δ_D is determined by the evaluations

$$\Big\langle Y^p_{i|j^2_0 0}, \Big\langle \Delta_D\Big(\Gamma_0 + (x^1dx^2 - x^2dx^1) \otimes \frac{\partial}{\partial y^1}, \nabla^0\Big)(j^2_0 0), \frac{\partial}{\partial x^k}|_0\Big\rangle\Big\rangle. \tag{29}$$

Then using the invariance of Δ_D with respect to $a_t\colon \mathbf{R}^{m,n} \to \mathbf{R}^{m,n}$ by $a_t(x,y) = (x, ty_1, y_2, \dots, y_n)$ for $t > 0$, we may assume $p = 1$, i.e. Δ_D is determined by the evaluations

$$\Big\langle Y^1_{i|j^2_0 0}, \Big\langle \Delta_D(\Gamma_0 + (x^1dx^2 - x^2dx^1) \otimes \frac{\partial}{\partial y^1}, \nabla^0)(j^2_0 0), \frac{\partial}{\partial x^k}|_0\Big\rangle\Big\rangle. \tag{30}$$

Then using the invariance of Δ_D with respect to $b_t\colon \mathbf{R}^{m,n} \to \mathbf{R}^{m,n}$ by $b_t(x,y) = (t_1x_1, \dots, t_mx_m, y_1, \dots, y_n)$, we see that the values (30) are all zero except the values

$$\Big\langle Y^1_{1|j^2_0 0}, \Big\langle \Delta_D\Big(\Gamma_0 + (x^1dx^2 - x^2dx^1) \otimes \frac{\partial}{\partial y^1}, \nabla^0\Big)(j^2_0 0), \frac{\partial}{\partial x^2}|_0\Big\rangle\Big\rangle \tag{31}$$

and

$$\Big\langle Y^1_{2|j^2_0 0}, \Big\langle \Delta_D\Big(\Gamma_0 + (x^1dx^2 - x^2dx^1)\otimes \frac{\partial}{\partial y^1}, \nabla^0\Big)(j^2_0 0), \frac{\partial}{\partial x^1}|_0\Big\rangle\Big\rangle. \tag{32}$$

Because of the invariance of Δ_D with respect to exchanging x^1 and x^2 (i.e. with respect to the map $c\colon \mathbf{R}^{m,n} \to \mathbf{R}^{m,n}$ given by $c(x^1, x^2, \dots, x_m, y) = (x^2, x^1, \dots, x_m, y)$), we get

$$\begin{aligned}
&\Big\langle Y^1_{1|j^2_0 0}, \Big\langle \Delta_D\Big(\Gamma_0 + (x^1dx^2 - x^2dx^1)\otimes \frac{\partial}{\partial y^1}, \nabla^0\Big)(j^2_0 0), \frac{\partial}{\partial x^2}|_0\Big\rangle\Big\rangle \\
&\quad = -\Big\langle Y^1_{2|j^2_0 0}, \Big\langle \Delta_D\Big(\Gamma_0 + (x^1dx^2 - x^2dx^1)\otimes \frac{\partial}{\partial y^1}, \nabla^0\Big)(j^2_0 0), \frac{\partial}{\partial x^1}|_0\Big\rangle\Big\rangle.
\end{aligned}$$

Consequently, the vector space of all possible values (27) is of dimension ≤ 1. So, the vector space of all possible Δ_D is of dimension $\leq 2 + K$, where K is the dimension of the vector space of all possible g.

If $D = \mathcal{J}^2_{[i]}$ for $i = 1, 2$ is as in Example 3, then we have

$$\begin{aligned}
&\Big\langle \Delta_{\mathcal{J}^2_{[1]}}\Big(\Gamma_0 + (x^1x^2dx^2 - (x^2)^2dx^1)\otimes \frac{\partial}{\partial y^1}, \nabla^0\Big)(j^2_0 0), \frac{\partial}{\partial x^1}|_0\Big\rangle = 0, \\
&\Big\langle \Delta_{\mathcal{J}^2_{[1]}}\Big(\Gamma_0 + (x^1dx^2 - x^2dx^1)\otimes \frac{\partial}{\partial y^1}, \nabla^0\Big)(j^2_0 0), \frac{\partial}{\partial x^1}|_0\Big\rangle \\
&\qquad = \mathcal{J}^2\Big(x^2\frac{\partial}{\partial y^1}\Big)(j^2_0 0), \\
&\Big\langle \Delta_{\mathcal{J}^2_{[2]}}\Big(\Gamma_0 + (x^1x^2dx^2 - (x^2)^2dx^1)\otimes \frac{\partial}{\partial y^1}, \nabla^0\Big)(j^2_0 0), \frac{\partial}{\partial x^1}|_0\Big\rangle \\
&\qquad = \mathcal{J}^2\Big((x^2)^2\frac{\partial}{\partial y^1}\Big)(j^2_0 0), \\
&\Big\langle \Delta_{\mathcal{J}^2_{[2]}}\Big(\Gamma_0 + (x^1dx^2 - x^2dx^1)\otimes \frac{\partial}{\partial y^1}, \nabla^0\Big)(j^2_0 0), \frac{\partial}{\partial x^1}|_0\Big\rangle = 0, \\
&\Delta_{\mathcal{J}^2_{[i]}}(\Gamma_0, \nabla)(\rho) = 0 \quad \text{for} \quad i = 1, 2
\end{aligned}$$

for any $\rho \in (J^2\mathbf{R}^{m,n})_{(0,0)}$ and any torsion free classical linear connection $\nabla \in Q_\tau(\mathbf{R}^m)$ such that $id_{\mathbf{R}^m}$ is a ∇-normal coordinate system with center 0. By the flow argument we see that

$$\begin{aligned}
\mathcal{J}^2\Big((x^2)^2\frac{\partial}{\partial y^1}\Big)(j^2_0 0) &\cong j^2_0((x^2)^2)\mathcal{J}^2\Big(\frac{\partial}{\partial y^1}\Big)(j^2_0 0), \\
\mathcal{J}^2\Big(x^2\frac{\partial}{\partial y^1}\Big)(j^2_0 0) &\cong j^2_0(x^2)\mathcal{J}^2\Big(\frac{\partial}{\partial y^1}\Big)(j^2_0 0),
\end{aligned}$$

and then they are linearly independent.

Using the dimension argument and the formula (23), we deduce that there exist unique real numbers t_1 and t_2 and an $\mathcal{FM}_{m,n}$-natural operator

D_1 such that

$$D = (1 - t_1 - t_2)D_1 + t_1\mathcal{J}^2_{[1]} + t_2\mathcal{J}^2_{[2]} \tag{33}$$

(the affine combination) and

$$\Delta_{D_1}(\Gamma, \nabla^0)(\rho) = 0 \tag{34}$$

for all $\rho \in (J^2\mathbf{R}^{m,n})_{(0,0)}$ and all general connections Γ on $\mathbf{R}^{m,n}$ such that the identity map $\psi = id_{\mathbf{R}^{m,n}}$ is a $(\Gamma, \nabla^0, (0,0), 3)$-quasi-normal fibred coordinate system on $\mathbf{R}^{m,n}$. The operator D_1 is uniquely determined if $t_1 + t_2 \neq 1$.

It remains to show that D_1 is of the form

$$D_1 = \mathcal{J}^2_{(A)} \tag{35}$$

for a uniquely determined $\mathcal{M}f_m$-natural operator A transforming torsion free classical linear connections ∇ on m-manifolds M into second order linear connections $A(\nabla)\colon TM \to J^2TM$ on M, where $\mathcal{J}^2_{(A)}$ is as in Example 1.

We construct A in the following way. Given a torsion free classical linear connection ∇ on a m-manifold M we define a tensor field $\tilde{A}(\nabla)\colon M \to T^*M \otimes S^2T^*M \otimes TM$ on M by

$$\langle \tilde{A}(\nabla)_{|x}, \omega\rangle = pr_1 \circ \Delta_{D_1}(\Gamma_M, \nabla)(j^2_x(f, 0, \dots, 0)) \in T^*_xM \otimes S^2T^*_xM, \tag{36}$$

where $\omega = d_xf \in T^*_xM$, $f\colon M \to \mathbf{R}$, $f(x) = 0$, Γ_M is the trivial general connection on the trivial bundle $M \times \mathbf{R}^n \to M$ and

$$pr_1\colon T^*M\otimes S^2T^*M\otimes V(M\times\mathbf{R}^n) = T^*M\otimes S^2T^*M\otimes\mathbf{R}^n \to T^*M\otimes S^2T^*M$$

is the projection onto the first factor.

The definition (36) is correct because

$$\begin{aligned}\Delta_{D_1}(\Gamma_M, \nabla)(j^2_x(f, 0, \dots, 0)) &\in T^*M \otimes S^2T^*M \otimes V(M \times \mathbf{R}^n)\\ &\subset T^*M \otimes VJ^2(M \times \mathbf{R}^n)\end{aligned}$$

as $\Delta_{D_1}(\Gamma_M, \nabla)(j^2_x(f, 0, \dots, 0))$ projects onto zero by

$$id_{T^*M} \otimes V\pi^2_1\colon T^*M \otimes VJ^2(M \times \mathbf{R}^n) \to T^*M \otimes VJ^1(M \times \mathbf{R}^n),$$

where $\pi^2_1\colon J^2Y \to J^1Y$ is the jet projection. Indeed, in order to observe that $\Delta_{D_1}(\Gamma_M, \nabla)(j^2_x(f, 0, \dots, 0))$ projects onto zero, we can assume that $M = \mathbf{R}^m$, $x = 0$ and $\psi = id_{\mathbf{R}^{m,n}}$ is a $(\Gamma_0, \nabla, (0,0), 3)$-quasi-normal fibred coordinate system on $\mathbf{R}^{m,n}$ because of the $\mathcal{FM}_{m,n}$-invariance of Δ_{D_1}. From (26) for Δ_{D_1} instead of Δ_D we have $\Delta_{D_1}(\Gamma_0, \nabla^0)(j^2_0(f, 0, \dots, 0)) = 0$. Then using the invariance of Δ_{D_1} with respect to the homotheties and applying the homogeneous function theorem, we complete the observation.

Using the invariance of Δ_{D_1} with respect to the fiber homotheties $id_M \times t id_{\mathbf{R}^n}$ and applying the homogeneous function theorem, we see that the value (36) depends linearly on ω. Hence $\tilde{A}$ is really a tensor field.

Let

$$A(\nabla) := A^{exp}_2(\nabla) + \tilde{A}(\nabla)\colon TM \to J^2TM \tag{37}$$

be the second order connection corresponding to $\tilde{A}$. So, we have constructed an $\mathcal{M}f_m$-natural operator A transforming torsion free classical linear connections ∇ on m-manifolds M into second order linear connections $A(\nabla)\colon TM \to J^2TM$ on M.

We prove (35) as follows. Using the invariance of $A - A_2^{exp}$ with respect to the homotheties and applying the homogeneous function theorem, we see that $A(\nabla^0) - A_2^{exp}(\nabla^0)$ is the zero tensor field of type $T^* \otimes S^2T^* \otimes T$. Therefore, we obtain (34) for $\Delta_{\mathcal{J}_{(A)}}$ instead of Δ_{D_1}. Then using the condition (34), we get

$$\begin{aligned} &\left\langle Y^1_{22|\rho}, \left\langle \Delta_{D_1}(\Gamma,\nabla)(\rho), \frac{\partial}{\partial x^1}|_0 \right\rangle\right\rangle \\ &= \left\langle Y^1_{22|\rho}, \left\langle \Delta_{D_{\mathcal{J}_{(A)}}}(\Gamma,\nabla)(\rho), \frac{\partial}{\partial x^1}|_0 \right\rangle\right\rangle = g((g^1_l),(\nabla^l_{ij;k})) \end{aligned} \tag{38}$$

for any $\rho \in (J^2\mathbf{R}^{m,n})_{(0,0)}$, any general connection Γ on $\mathbf{R}^{m,n}$ and any torsion free classical linear connection ∇ on $\mathbf{R}^m$ such that the identity map $\psi = id_{\mathbf{R}^{m,n}}$ is a $(\Gamma,\nabla,(0,0),3)$-quasi-normal coordinate system on $\mathbf{R}^{m,n}$, where

$$\begin{aligned} j^2_{(0,0)}\Gamma = j^2_{(0,0)}\Bigg(\Gamma_0 &+ \sum_{i,j,k=1}^{m}\sum_{p=1}^{n} a^p_{kij}x^kx^idx^j \otimes \frac{\partial}{\partial y^p} \\ &+ \sum_{i,j=1}^{m}\sum_{p,q=1}^{n} b^p_{qij}y^qx^idx^j \otimes \frac{\partial}{\partial y^p} + \sum_{i,j=1}^{m}\sum_{p=1}^{n} c^p_{ij}x^idx^j \otimes \frac{\partial}{\partial y^p}\Bigg) \end{aligned}$$

with coefficients a^p_{kij}, b^p_{qij} and c^p_{ij} satisfying (2),

$$j^1_0(\nabla) = j^1_0\left(\left(\sum_{k=1}^{m}\nabla^l_{ij;k}x^k\right)_{i,j,l=1}^{m}\right)$$

for $\nabla^l_{ij;k} = \nabla^l_{ji;k} \in \mathbf{R}$ satisfying some "classical" conditions, ρ is of the form

$$\rho = j^2_0\left(\left(\sum_{i=1}^{m} g^p_ix^i + \sum_{i,j=1}^{m} h^p_{ij}x^ix^j\right)_{p=1}^{n}\right)$$

for real numbers $g^p_i, h^p_{ij} = h^p_{ji}$ and g is the bilinear map as in (23). Then we have (35) because any Δ_D (and then any D) is determined by the values (18).

If $D_1 = \mathcal{J}^2_{(A_1)}$ for another $\mathcal{M}f_m$-natural operator A_1 (of the type as the one of A), then

$$\langle \tilde{A}(\nabla)_{|x}, \omega\rangle = \langle \tilde{A}_1(\nabla)_{|x}, \omega\rangle \tag{39}$$

for any torsion free classical linear connection ∇ on M and any $\omega \in T^*_xM, x \in M$, where $\tilde{A}_1(\nabla) = A_1(\nabla) - A_2^{exp}(\nabla)\colon M \to T^*M \otimes S^2T^*M \otimes TM$ is the tensor field corresponding to $A_1(\nabla)\colon TM \to J^2TM$.

Because of $\mathcal{M}f_m$-invariance it is sufficient to show (39) in the case $M = \mathbf{R}^m, x = 0$ and the identity map $\psi = id_{\mathbf{R}_{m,n}}$ is a $(\Gamma, \nabla, (0,0), 3)$-quasi-normal fibred coordinate system on $\mathbf{R}_{m,n}$. It is not difficult. So, $A_1 = A$, i.e. A satisfying (35) is uniquely determined. The proof of Theorem 1 for $m \geq 2$ is complete.

If $m = 1$, we proceed similarly as in the case $m \geq 2$. Therefore, Δ_D is uniquely determined by the values

$$\left\langle Y^1_{|\rho}, \left\langle \Delta_D(\Gamma, \nabla)(\rho), \frac{\partial}{\partial x^1}|_0 \right\rangle \right\rangle \in \mathbf{R}$$

$$\left\langle Y^1_{1|\rho}, \left\langle \Delta_D(\Gamma, \nabla)(\rho), \frac{\partial}{\partial x^1}|_0 \right\rangle \right\rangle \in \mathbf{R}$$

$$\left\langle Y^1_{11|\rho}, \left\langle \Delta_D(\Gamma, \nabla)(\rho), \frac{\partial}{\partial x^1}|_0 \right\rangle \right\rangle \in \mathbf{R}$$

for all $\rho \in (J^2\mathbf{R}^{1,n})_{(0,0)}$, all general connections Γ on $\mathbf{R}^{1,n}$ and all torsion free classical linear connections ∇ on $\mathbf{R}$ such that $\psi = id_{\mathbf{R}^{1,n}}$ is a $(\Gamma, \nabla, (0,0), 3)$-quasi-normal fibred coordinate system on $\mathbf{R}^{1,n}$ over $\underline{\psi} = id_{\mathbf{R}}$. Then the operator Δ_D is uniquely determined by the values

$$\left\langle Y^1_{11|\rho}, \left\langle \Delta_D(\Gamma, \nabla)(\rho), \frac{\partial}{\partial x^1}|_0 \right\rangle \right\rangle \in \mathbf{R}.$$

If the identity map $\psi = id_{\mathbf{R}^{1,n}}$ is a $(\Gamma, \nabla, (0,0), 3)$-quasi-normal fibred coordinate system, then $j^2_{(0,0)}\Gamma = j^2_{(0,0)}(\Gamma_0)$ and $j^1_0(\nabla) = j^1_0(\nabla^0)$ (as the curvature of ∇ is zero). Consequently, Δ_D is determined by the values

$$\left\langle Y^1_{11}, \left\langle \Delta_D(\Gamma_0, \nabla^0)(\rho), \frac{\partial}{\partial x^1}|_0 \right\rangle \right\rangle \in \mathbf{R} \tag{40}$$

for all $\rho \in J^2_0(\mathbf{R}, \mathbf{R}^n)_0$. But the values (40) are zero because of the similar arguments as in the proof of formula (23).

The proof of Theorem 1 is complete. $\square$

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Received June 10, 2013