doi: 10.1515/umcsmath-2015-0008

ANNALES
UNIVERSITATIS MARIAE CURIE-SKŁODOWSKA
LUBLIN – POLONIA

VOL. LXVIII, NO. 2, 2014 SECTIO A 85–91

ŁUKASZ PIASECKI

Renormings of c_0 and the minimal displacement problem

Abstract. The aim of this paper is to show that for every Banach space $(X, \|\cdot\|)$ containing asymptotically isometric copy of the space c_0 there is a bounded, closed and convex set $C \subset X$ with the Chebyshev radius $r(C) = 1$ such that for every $k \geq 1$ there exists a k-contractive mapping $T : C \to C$ with $\|x - Tx\| > 1 - \frac{1}{k}$ for any $x \in C$.

1. Introduction and Preliminaries. Let C be a nonempty, bounded, closed and convex subset of an infinitely dimensional real Banach space $(X, \|\cdot\|)$. *The Chebyshev radius of* C *relative to itself* is the number

$$r(C) = \inf_{y \in C} \sup_{x \in C} \|x - y\| .$$

We say that a mapping $T : C \to C$ satisfies *the Lipschitz condition with a constant* k or is k*-lipschitzian*, if for all $x, y \in C$,

$$\|Tx - Ty\| \leq k \|x - y\| .$$

The smallest constant k for which the above inequality holds is called *the Lipschitz constant* for T and it is denoted by $k(T)$. By $L(k)$ we denote the class of all k-lipschitzian mappings $T : C \to C$. A mapping $T : C \to C$ is called k*-contractive* if for all $x, y \in C$, $x \neq y$, we have

$$\|Tx - Ty\| < k \|x - y\| .$$

2000 *Mathematics Subject Classification.* 47H09, 47H10.

Key words and phrases. Minimal displacement, asymptotically isometric copies of c_0, lipschitzian mappings, k-contractive mappings, renormings.

The minimal displacement problem has been raised by Goebel in 1973, see [6]. Standard situation is the following. For any k-lipschitzian mapping $T : C \to C$ *the minimal displacement of* T is the number given by

$$d(T) = \inf \{\|x - Tx\| : x \in C\}.$$

It is known that for every $k \geq 1$ (for the proof see for example [8])

$$d(T) \leq \left(1 - \frac{1}{k}\right) r(C).$$

For any set C we define the function $\varphi_C : [1, \infty) \to [0, r(C))$ by

$$\varphi_C(k) = \sup \{d(T) : T \in L(k)\}.$$

Consequently, for every $k \geq 1$,

$$\varphi_C(k) \leq \left(1 - \frac{1}{k}\right) r(C).$$

The function φ_C is called *the characteristic of minimal displacement of* C. If C is the closed unit ball B_X, then we write ψ_X instead of φ_{B_X}. We also define *the characteristic of minimal displacement of the whole space* X as

$$\varphi_X(k) = \sup \{\varphi_C(k) : C \subset X, r(C) = 1\}.$$

Hence, for every $k > 1$,

$$\psi_X(k) \leq \varphi_X(k) \leq 1 - \frac{1}{k}.$$

The minimal displacement problem is the task to find or evaluate functions φ and ψ for concrete sets or spaces. Obviously this problem is matterless in the case of compact set C because, in view of the celebrated Schauder's fixed point theorem, we have $\varphi_C(k) = 0$ for any $k > 1$. If C is noncompact then by the theorem of Sternfeld and Lin [12] we get $\varphi_C(k) > 0$ for all $k > 1$. Hence we additionally assume that C is noncompact and we restrict our attention to the class of lipschitzian mappings with $k(T) \geq 1$.

The set C for which $\varphi_C(k) = \left(1 - \frac{1}{k}\right) r(C)$ for every $k > 1$ is called *extremal* (with respect to the minimal displacement problem). There are examples of spaces having extremal balls. Among them are spaces of continuous functions $C[a, b]$, bounded continuous functions $BC(R)$, sequences converging to zero c_0, all of them endowed with the standard uniform norm (see [7]). Recently the present author [13] proved that also the space c of converging sequences with the sup norm has extremal balls. It is still unknown if the space l_∞ of all bounded sequences with the sup norm has extremal balls. Very recently Bolibok [1] proved that

$$\psi_{l_\infty}(k) \geq \begin{cases} (3 - 2\sqrt{2})(k - 1) & \text{for } 1 \leq k \leq 2 + \sqrt{2}, \\ 1 - \frac{2}{k} & \text{for } k > 2 + \sqrt{2}. \end{cases}$$

The same estimate holds for the space of summable functions (equivalent classes) $L_1(0,1)$ equipped with the standard norm (see [2]) as well as few other spaces (see [9]).

In the case of space l_1 of all summable sequences with the classical norm we have:

$$\psi_{l_1}(k) \le \begin{cases} \frac{2+\sqrt{3}}{4}\left(1-\frac{1}{k}\right) & \text{for } 1 \le k \le 3+2\sqrt{3}, \\ \frac{k+1}{k+3} & \text{for } k > 3+2\sqrt{3}. \end{cases}$$

Nevertheless, the subset

$$S^+ = \left\{ \{x_n\}_{n=1}^\infty : x_n \ge 0, \sum_{n=1}^\infty x_n = 1 \right\} \subset l_1$$

is extremal and $\varphi_{S^+}(k) = \left(1-\frac{1}{k}\right) r(S^+) = 2\left(1-\frac{1}{k}\right)$ for every $k > 1$ (for the proof see [7]).

In this paper we deal with a problem of existence of extremal sets in spaces containing isomorphic copies of c_0. Obviously, for every such space X we have $\varphi_X(k) = 1 - \frac{1}{k}$ as an immediate consequence of the following theorem by James [10], its stronger version states:

Theorem 1.1 (James's Distortion Theorem, stronger version). *A Banach space X contains an isomorphic copy of c_0 if and only if, for every null sequence $\{\epsilon_n\}_{n=1}^\infty$ in $(0,1)$, there exists a sequence $\{x_n\}_{n=1}^\infty$ in X such that*

$$(1-\epsilon_k) \sup_{n \ge k} |t_n| \le \left\| \sum_{n=k}^\infty t_n x_n \right\| \le (1+\epsilon_k) \sup_{n \ge k} |t_n|$$

holds for all $\{t_n\}_{n=1}^\infty \in c_0$ and for all $k = 1, 2, \ldots$.

However, it is not known if all isomorphic copies of c_0 contain an extremal subset. We shall prove that the answer is affirmative in the case of spaces containing *an asymptotically isometric copies of* c_0. This class of spaces has been introduced and widely studied by Dowling, Lennard and Turett (see [11], Chapter 9). Let us recall that a Banach space X is said to contain *an asymptotically isometric copy of* c_0 if for every null sequence $\{\epsilon_n\}_{n=1}^\infty$ in $(0,1)$, there exists a sequence $\{x_n\}_{n=1}^\infty$ in X such that

$$\sup_n (1-\epsilon_n)|t_n| \le \left\| \sum_{n=1}^\infty t_n x_n \right\| \le \sup_n |t_n|,$$

for all $\{t_n\}_{n=1}^\infty \in c_0$.

Dowling, Lennard and Turett proved the following theorems.

Theorem 1.2 (see [3] or [11]). *If a Banach space X contains an asymptotically isometric copy of c_0, then X fails the fixed point property for nonexpansive (and even contractive) mappings on bounded, closed and convex subsets of X.*

Theorem 1.3 (see [5] or [11]). *If Y is a closed infinite dimensional subspace of $(c_0, \|\cdot\|_\infty)$, then Y contains an asymptotically isometric copy of c_0.*

Theorem 1.4 (see [5]). *Let Γ be an uncountable set. Then every renorming of $c_0(\Gamma)$ contains an asymptotically isometric copy of c_0.*

Let us recall that a mapping $T : C \to C$ is said to be *asymptotically nonexpansive* if

$$\|T^n x - T^n y\| \leq k_n \|x - y\|$$

for all $x, y \in C$ and for all $n = 1, 2, \ldots$, where $\{k_n\}_{n=1}^{\infty}$ is a sequence of real numbers with $\lim_{n\to\infty} k_n = 1$.

Now we are ready to cite the following theorem.

Theorem 1.5 (see [4] or [11]). *If a Banach space X contains an isomorphic copy of c_0, then there exists a bounded, closed, convex subset C of X and an asymptotically nonexpansive mapping $T : C \to C$ without a fixed point. In particular, c_0 cannot be renormed to have the fixed point property for asymptotically nonexpansive mappings.*

Remark 1.6 (see [5] or [11]). There is an isomorphic copy of c_0 which does not contain any asymptotically isometric copy of c_0.

2. Main result.

Theorem 2.1. *If a Banach space X contains an asymptotically isometric copy of c_0, then there exists a bounded, closed and convex subset C of X with $r(C) = 1$ such that for every $k \geq 1$ there exists a k-contractive mapping $T : C \to C$ with*

$$\|x - Tx\| > 1 - \frac{1}{k}$$

for every $x \in C$.

Proof. Let $\{\lambda_i\}_{i=1}^{\infty}$ be a strictly decreasing sequence in $(1, \frac{3}{2})$ converging to 1. Then there is a null sequence $\{\epsilon_i\}_{i=1}^{\infty}$ in $(0, 1)$ such that

$$\lambda_{i+1} < (1 - \epsilon_i)\lambda_i$$

for $i = 1, 2, \ldots$. By assumption there exists a sequence $\{x_i\}_{i=1}^{\infty}$ in X such that

$$\sup_i (1 - \epsilon_i) |t_i| \leq \left\| \sum_{i=1}^{\infty} t_i x_i \right\| \leq \sup_i |t_i|$$

for every $\{t_i\}_{i=1}^{\infty} \in c_0$.

Define $y_i = \lambda_i x_i$ for $i = 1, 2, \ldots$ and

$$C = \left\{ \sum_{i=1}^{\infty} t_i y_i : \{t_i\}_{i=1}^{\infty} \in c_0, 0 \leq t_i \leq 1 \text{ for } i = 1, 2, \ldots \right\}.$$

It is clear that C is a bounded, closed and convex subset of X.

We claim that $r(C) = 1$. Fix $w = \sum_{i=1}^{\infty} t_i y_i \in C$ and let $\{z_n\}_{n=2}^{\infty}$ be a sequence of elements in C defined by

$$z_n = \sum_{i=1}^{n-1} t_i y_i + y_n + \sum_{i=n+1}^{\infty} t_i y_i.$$

Then

$$\|w - z_n\| = \|(t_n - 1)y_n\| \geq (1 - \epsilon_n)(1 - t_n)\lambda_n \geq \lambda_{n+1}(1 - t_n).$$

Letting $n \to \infty$, we get $r(w, C) := \sup\{\|w - x\| : x \in C\} \geq 1$ for any $w \in C$ and consequently $r(C) \geq 1$.

Now let $\{z_n\}_{n=1}^{\infty}$ be a sequence in C given by

$$z_n = \sum_{i=1}^{n} \frac{1}{2} y_i.$$

Then for every $w = \sum_{i=1}^{\infty} t_i y_i \in C$ we have

$$\begin{aligned}
\|z_n - w\| &= \left\| \sum_{i=1}^{n} \left(\frac{1}{2} - t_i\right) y_i + \sum_{i=n+1}^{\infty} (-t_i y_i) \right\| \\
&= \left\| \sum_{i=1}^{n} \left(\frac{1}{2} - t_i\right) \lambda_i x_i + \sum_{i=n+1}^{\infty} (-t_i \lambda_i x_i) \right\| \\
&\leq \sup\left\{ \left|\frac{1}{2} - t_1\right| \lambda_1, \ldots, \left|\frac{1}{2} - t_n\right| \lambda_n, t_{n+1}\lambda_{n+1}, t_{n+2}\lambda_{n+2}, \ldots \right\} \\
&\leq \sup\left\{ \frac{1}{2}\lambda_1, \ldots, \frac{1}{2}\lambda_n, \lambda_{n+1}, \lambda_{n+2}, \ldots \right\} \\
&\leq \sup\left\{ \frac{1}{2} \cdot \frac{3}{2}, \ldots, \frac{1}{2} \cdot \frac{3}{2}, \lambda_{n+1}, \lambda_{n+2}, \ldots \right\} \\
&= \sup\left\{ \frac{3}{4}, \lambda_{n+1} \right\} \\
&= \lambda_{n+1}.
\end{aligned}$$

Hence $r(z_n, C) \leq \lambda_{n+1}$. Letting $n \to \infty$, we get $r(C) \leq 1$. Finally $r(C) = 1$.

To construct desired mapping T we shall need the function $\alpha : [0, \infty) \to [0, 1]$ defined by

$$\alpha(t) = \begin{cases} t & \text{if } 0 \leq t \leq 1, \\ 1 & \text{if } t > 1. \end{cases}$$

It is clear that the function α satisfies the Lipschitz condition with the constant 1, that is, for all $s, t \in [0, \infty)$ we have

$$|\alpha(t) - \alpha(s)| \leq |t - s|.$$

For arbitrary $k \geq 1$ we define a mapping $T : C \to C$ by

$$T\left(\sum_{i=1}^{\infty} t_i y_i\right) = y_1 + \sum_{i=2}^{\infty} \alpha(kt_{i-1}) y_i.$$

Then for any $w = \sum_{i=1}^{\infty} t_i y_i$ and $z = \sum_{i=1}^{\infty} s_i y_i$ in C such that $w \neq z$ we have

$$\begin{aligned}
\|Tw - Tz\| &= \left\| \sum_{i=2}^{\infty} \left(\alpha(kt_{i-1}) - \alpha(ks_{i-1})\right) y_i \right\| \\
&= \left\| \sum_{i=2}^{\infty} \left(\alpha(kt_{i-1}) - \alpha(ks_{i-1})\right) \lambda_i x_i \right\| \\
&\leq \sup_{i \geq 2} \lambda_i \left|\alpha(kt_{i-1}) - \alpha(ks_{i-1})\right| \\
&\leq \sup_{i \geq 2} \lambda_i \left|kt_{i-1} - ks_{i-1}\right| \\
&< k \sup_{i=1,2,\dots} (1 - \epsilon_i)\lambda_i \left|t_i - s_i\right| \\
&\leq k \left\| \sum_{i=1}^{\infty} (t_i - s_i)\lambda_i x_i \right\| \\
&= k \left\| w - z \right\|.
\end{aligned}$$

Hence the mapping T is k-contractive.

We claim that for every $x \in C$

$$\|x - Tx\| > 1 - \frac{1}{k}.$$

Indeed, suppose that there exists $w = \sum_{i=1}^{\infty} t_i y_i \in C$ such that $\|w - Tw\| \leq 1 - \frac{1}{k}$, that is,

$$\begin{aligned}
\|w - Tw\| &= \left\| (t_1 - 1) y_1 + \sum_{i=2}^{\infty} (t_i - \alpha(kt_{i-1})) y_i \right\| \\
&= \left\| (t_1 - 1)\lambda_1 x_1 + \sum_{i=2}^{\infty} (t_i - \alpha(kt_{i-1}))\lambda_i x_i \right\| \\
&\leq 1 - \frac{1}{k}.
\end{aligned}$$

This implies that

$$(1 - \epsilon_1)\lambda_1(1 - t_1) \leq 1 - \frac{1}{k}$$

and

$$(1 - \epsilon_i)\lambda_i \left|t_i - \alpha(kt_{i-1})\right| \leq 1 - \frac{1}{k} \quad \text{for } i \geq 2.$$

Hence $t_i \geq \frac{1}{k}$ for $i = 1, 2, \dots$. But $\{t_i\}_{i=1}^{\infty} \in c_0$, a contradiction. □

Corollary 2.2. *If a Banach space X contains an asymptotically isometric copy of c_0, then X contains an extremal subset.*

Corollary 2.3. *If Y is a closed infinite dimensional subspace of $(c_0, \|\cdot\|_\infty)$, then Y contains an extremal subset.*

Corollary 2.4. *Let Γ be uncountable set. Then every renorming of $c_0(\Gamma)$ contains an extremal subset.*

Acknowledgement. The author has been partially supported by MNiSW, grant NN201393737.

References

[1] Bolibok, K., *The minimal displacement problem in the space l^∞*, Cent. Eur. J. Math. **10** (2012), 2211–2214.
[2] Bolibok, K., *Constructions of lipschitzian mappings with non zero minimal displacement in spaces $L^1(0,1)$ and $L^2(0,1)$*, Ann. Univ. Mariae Curie-Skłodowska Sec. A **50** (1996), 25–31.
[3] Dowling, P. N., Lennard C. J., Turett, B., *Reflexivity and the fixed point property for nonexpansive maps*, J. Math. Anal. Appl. **200** (1996), 653–662.
[4] Dowling, P. N., Lennard, C. J., Turett, B., *Some fixed point results in l_1 and c_0*, Nonlinear Anal. **39** (2000), 929–936.
[5] Dowling, P. N., Lennard C. J., Turett , B., *Asymptotically isometric copies of c_0 in Banach spaces*, J. Math. Anal. Appl. **219** (1998), 377–391.
[6] Goebel, K., *On the minimal displacement of points under lipschitzian mappings*, Pacific J. Math. **45** (1973), 151–163.
[7] Goebel, K., *Concise Course on Fixed Point Theorems*, Yokohama Publishers, Yokohama, 2002.
[8] Goebel, K., Kirk, W. A., *Topics in metric fixed point theory*, Cambridge University Press, Cambridge, 1990.
[9] Goebel, K., Marino, G., Muglia, L., Volpe, R., *The retraction constant and minimal displacement characteristic of some Banach spaces*, Nonlinear Anal. **67** (2007), 735–744.
[10] James, R. C., *Uniformly non-square Banach spaces*, Ann. of Math. **80** (1964), 542–550.
[11] Kirk, W. A., Sims, B. (Eds.), *Handbook of Metric Fixed Point Theory*, Kluwer Academic Publishers, Dordrecht, 2001.
[12] Lin, P. K., Sternfeld, Y., *Convex sets with the Lipschitz fixed point property are compact*, Proc. Amer. Math. Soc. **93** (1985), 633–639.
[13] Piasecki, Ł., *Retracting a ball onto a sphere in some Banach spaces*, Nonlinear Anal. **74** (2011), 396–399.

Łukasz Piasecki
Institute of Mathematics
Maria Curie-Skłodowska University
pl. M. Curie-Skłodowskiej 1
20-031 Lublin
Poland
e-mail: `piasecki@hektor.umcs.lublin.pl`

Received October 1, 2012