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On ideals of pseudo-BCH-algebras

Abstract. In this paper we introduce the notion of a disjoint union of pseudo-BCH-algebras and describe ideals in such algebras. We also investigate ideals of direct products of pseudo-BCH-algebras. Moreover, we establish conditions for the set of all minimal elements of a pseudo-BCH-algebra $\mathfrak{X}$ to be an ideal of $\mathfrak{X}$.

1. Introduction. In 1966, Y. Imai and K. Iséki ([11], [12]) introduced BCK- and BCI-algebras. In 1983, Q. P. Hu and X. Li ([10]) introduced BCH-algebras. It is known that BCK- and BCI-algebras are contained in the class of BCH-algebras.

In 2001, G. Georgescu and A. Iorgulescu ([9]) introduced pseudo-BCK-algebras as an extension of BCK-algebras. In 2008, W. A. Dudek and Y. B. Jun ([3]) introduced pseudo-BCI-algebras as a natural generalization of BCI-algebras and of pseudo-BCK-algebras. These algebras have also connections with other algebras of logic such as pseudo-MV-algebras and pseudo-BL-algebras defined by G. Georgescu and A. Iorgulescu (see [13]). Those algebras were investigated by several authors in [7], [8], [15] and [16]. Recently, A. Walendziak ([18]) introduced pseudo-BCH-algebras as an extension of BCH-algebras and studied the set Cen$\mathfrak{X}$ of all minimal elements of a pseudo-BCH-algebra $\mathfrak{X}$, the so-called centre of $\mathfrak{X}$. He also considered ideals in pseudo-BCH-algebras and established a relationship between the ideals of a pseudo-BCH-algebra and the ideals of its centre.

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In this paper we introduce the notion of a disjoint union of pseudo-BCH-algebras and describe ideals in such algebras. We also investigate ideals of direct products of pseudo-BCH-algebras. Moreover, we establish conditions for the set Cen$\mathfrak{X}$ to be an ideal of a pseudo-BCH-algebra $\mathfrak{X}$.

2. Pseudo-BCH-algebras. We recall that an algebra $\mathfrak{X} = (X;*,0)$ of type $(2,0)$ is called a *BCH-algebra* if it satisfies the following axioms:

(BCH-1) $x * x = 0$;
(BCH-2) $(x * y) * z = (x * z) * y$;
(BCH-3) $x * y = y * x = 0 \Longrightarrow x = y$.

A BCH-algebra $\mathfrak{X}$ is said to be a *BCI-algebra* if it satisfies the identity

(BCI) $((x * y) * (x * z)) * (z * y) = 0$.

A *BCK-algebra* is a BCI-algebra $\mathfrak{X}$ satisfying the law $0 * x = 0$.

Definition 2.1 ([3]). A *pseudo-BCI-algebra* is a structure $\mathfrak{X} = (X;\leq,*,\diamond,0)$, where "$\leq$" is a binary relation on the set X, "$*$" and "$\diamond$" are binary operations on X and "0" is an element of X, satisfying the axioms:

(pBCI-1) $(x * y) \diamond (x * z) \leq z * y, \quad (x \diamond y) * (x \diamond z) \leq z \diamond y$;
(pBCI-2) $x * (x \diamond y) \leq y, \quad x \diamond (x * y) \leq y$;
(pBCI-3) $x \leq x$;
(pBCI-4) $x \leq y,\ y \leq x \Longrightarrow x = y$;
(pBCI-5) $x \leq y \Longleftrightarrow x * y = 0 \Longleftrightarrow x \diamond y = 0$.

A pseudo-BCI-algebra $\mathfrak{X}$ is called a *pseudo-BCK-algebra* if it satisfies the identities

(pBCK) $0 * x = 0 \diamond x = 0$.

Definition 2.2 ([18]). A *pseudo-BCH-algebra* is an algebra $\mathfrak{X} = (X;*,\diamond,0)$ of type $(2,2,0)$ satisfying the axioms:

(pBCH-1) $x * x = x \diamond x = 0$;
(pBCH-2) $(x * y) \diamond z = (x \diamond z) * y$;
(pBCH-3) $x * y = y \diamond x = 0 \Longrightarrow x = y$;
(pBCH-4) $x * y = 0 \Longleftrightarrow x \diamond y = 0$.

We define a binary relation $\leqslant$ on X by

$$x \leqslant y \Longleftrightarrow x * y = 0 \Longleftrightarrow x \diamond y = 0.$$

Throughout this paper $\mathfrak{X}$ will denote a pseudo-BCH-algebra.

Remark. Observe that if $(X;*,0)$ is a BCH-algebra, then letting $x \diamond y := x * y$, produces a pseudo-BCH-algebra $(X;*,\diamond,0)$. Therefore, every BCH-algebra is a pseudo-BCH-algebra in a natural way. It is easy to see that if $(X;*,\diamond,0)$ is a pseudo-BCH-algebra, then $(X;\diamond,*,0)$ is also a pseudo-BCH-algebra. From Proposition 3.2 of [3] we conclude that if $(X;\leq,*,\diamond,0)$ is a pseudo-BCI-algebra, then $(X;*,\diamond,0)$ is a pseudo-BCH-algebra.

Example 2.3 ([19]). Let $(G; \cdot, e)$ be a group. Define binary operations $*$ and $\diamond$ on G by

$$a * b = ab^{-1} \quad \text{and} \quad a \diamond b = b^{-1}a$$

for all $a, b \in G$. Then $\mathfrak{G} = (G; *, \diamond, e)$ is a pseudo-BCH-algebra.

We say that a pseudo-BCH-algebra $\mathfrak{X}$ is *proper* if $* \neq \diamond$ and it is not a pseudo-BCI-algebra.

Remark. The class of all pseudo-BCH-algebras is a quasi-variety. Therefore, if $(\mathfrak{X}_t)_{t\in T}$ is an indexed family of pseudo-BCH-algebras, then the direct product $\mathfrak{X} = \prod_{t\in T} \mathfrak{X}_t$ is also a pseudo-BCH-algebra. In the case when at least one of $\mathfrak{X}_t$ is proper, then $\mathfrak{X}$ is proper.

Example 2.4. Let $X_1 = \{0, a, b, c\}$. We define the binary operations $*_1$ and $\diamond_1$ on X_1 as follows:

$*_1$	0	a	b	c
0	0	0	0	0
a	a	0	a	0
b	b	b	0	0
c	c	b	c	0

and

$\diamond_1$	0	a	b	c
0	0	0	0	0
a	a	0	a	0
b	b	b	0	0
c	c	c	a	0

On the set $X_2 = \{0, 1, 2, 3\}$ consider the operation $*_2$ given by the following table:

$*_2$	0	1	2	3
0	0	0	0	0
1	1	0	0	1
2	2	2	0	0
3	3	3	3	0

Then $\mathfrak{X}_1 = (X_1; *_1, \diamond_1, 0)$ and $\mathfrak{X}_2 = (X_2; *_2, *_2, 0)$ are pseudo-BCH-algebras (see [18]). Therefore, the direct product $\mathfrak{X} = \mathfrak{X}_1 \times \mathfrak{X}_2$ is a (proper) pseudo-BCH-algebra.

Let $\mathfrak{X} = (X; *, \diamond, 0)$ be a pseudo-BCH-algebra satisfying (pBCK), and let $(G; \cdot, e)$ be a group. Denote $Y = G - \{e\}$ and suppose that $X \cap Y = \emptyset$. Define the binary operations $*$ and $\diamond$ on $X \cup Y$ by

$$x * y = \begin{cases} x * y & \text{if } x, y \in X \\ xy^{-1} & \text{if } x, y \in Y \text{ and } x \neq y \\ 0 & \text{if } x, y \in Y \text{ and } x = y \\ y^{-1} & \text{if } x \in X,\ y \in Y \\ x & \text{if } x \in Y,\ y \in X \end{cases} \tag{1}$$

and

$$(2) \qquad x \diamond y = \begin{cases} x \diamond y & \text{if } x, y \in X \\ y^{-1}x & \text{if } x, y \in Y \text{ and } x \neq y \\ 0 & \text{if } x, y \in Y \text{ and } x = y \\ y^{-1} & \text{if } x \in X,\ y \in Y \\ x & \text{if } x \in Y,\ y \in X. \end{cases}$$

Routine calculations give that $(X \cup Y; *, \diamond, 0)$ is a pseudo-BCH-algebra; it is proper if $\mathfrak{X}$ is proper.

Example 2.5. Consider the set $X = \{0, a, b, c\}$ with the operation $*$ defined by the following table:

$*$	0	a	b	c
0	0	0	0	0
a	a	0	c	c
b	b	0	0	b
c	c	0	0	0

Then $\mathfrak{X} = (X; *, 0)$ is a BCH-algebra (see [10]). Let $\mathfrak{G}$ be the group of all permutations of $\{1, 2, 3\}$. We have $G = \{\imath, d, e, f, g, h\}$, where $\imath = (1), d = (12), e = (13), f = (23), g = (123)$, and $h = (132)$. Applying (1) and (2) we obtain the following tables:

$*$	0	a	b	c	d	e	f	g	h
0	0	0	0	0	d	e	f	h	g
a	a	0	c	c	d	e	f	h	g
b	b	0	0	b	d	e	f	h	g
c	c	0	0	0	d	e	f	h	g
d	d	d	d	d	0	h	g	e	f
e	e	e	e	e	g	0	h	f	d
f	f	f	f	f	h	g	0	d	e
g	g	g	g	g	e	f	d	0	h
h	h	h	h	h	f	d	e	g	0

and

$\diamond$	0	a	b	c	d	e	f	g	h
0	0	0	0	0	d	e	f	h	g
a	a	0	c	c	d	e	f	h	g
b	b	0	0	b	d	e	f	h	g
c	c	0	0	0	d	e	f	h	g
d	d	d	d	d	0	h	g	f	e
e	e	e	e	e	g	0	h	d	f
f	f	f	f	f	h	g	0	e	d
g	g	g	g	g	e	f	d	0	g
h	h	h	h	h	f	d	e	h	0

Then $(\{0, a, b, c, d, e, f, g, h\}; *, \diamond, 0)$ is a pseudo-BCH-algebra. Observe that it is proper. Indeed, $(b * c) \diamond (b * a) = b \diamond 0 = b \nleq c = a * c$.

Let T be any set and, for each $t \in T$, let $\mathfrak{X}_t = (X_t; *_t, \diamond_t, 0)$ be a pseudo-BCH-algebra satisfying (pBCK). Suppose that $X_s \cap X_t = \{0\}$ for $s \neq t$, $s, t \in T$. Set $X = \bigcup_{t \in T} X_t$ and define the binary operations $*$ and $\diamond$ on X via

$$x * y = \begin{cases} x *_t y & \text{if } x, y \in X_t,\ t \in T, \\ x & \text{if } x \in X_s,\ y \in X_t,\ s \neq t,\ s, t \in T, \end{cases}$$

and

$$x \diamond y = \begin{cases} x \diamond_t y & \text{if } x, y \in X_t,\ t \in T, \\ x & \text{if } x \in X_s,\ y \in X_t,\ s \neq t,\ s, t \in T. \end{cases}$$

It is easy to check that $\mathfrak{X} = (X; *, \diamond, 0)$ is a pseudo-BCH-algebra. Following the terminology for BCH-algebras (see [1]), the algebra $\mathfrak{X}$ will be called the *disjoint union* of $(\mathfrak{X}_t)_{t \in T}$. We shall denote it by $\sum_{t \in T} \mathfrak{X}_t$.

Example 2.6. Let $\mathfrak{X}_1 = (\{0, a, b, c\}; *_1, \diamond_1, 0)$ be the pseudo-BCH-algebra from Example 2.4. Consider the set $X_2 = \{0, 1, 2, 3\}$ with the operation $*_2$ defined by the following table:

$*_2$	0	1	2	3
0	0	0	0	0
1	1	0	2	1
2	2	0	0	2
3	3	3	0	0

Routine calculations show that $\mathfrak{X}_2 = (X_2; *_2, *_2, 0)$ is a (pseudo)-BCH-algebra. Let $X = \{0, a, b, c, 1, 2, 3\}$. We define the binary operations $*$ and $\diamond$ on X as follows

$*$	0	a	b	c	1	2	3
0	0	0	0	0	0	0	0
a	a	0	a	0	a	a	a
b	b	b	0	0	b	b	b
c	c	b	c	0	c	c	c
1	1	1	1	1	0	2	1
2	2	2	2	2	0	0	2
3	3	3	3	3	3	0	0

and

$\diamond$	0	a	b	c	1	2	3
0	0	0	0	0	0	0	0
a	a	0	a	0	a	a	a
b	b	b	0	0	b	b	b
c	c	c	a	0	c	c	c
1	1	1	1	1	0	2	1
2	2	2	2	2	0	0	2
3	3	3	3	3	3	0	0

It is clear that $\mathfrak{X} = (X; *, \diamond, 0)$ is the disjoint union of $\mathfrak{X}_1$ and $\mathfrak{X}_2$. We have $(3 * 1) \diamond (3 * 2) = 3 \diamond 0 = 3 \nleqslant 0 = 2 * 1$, and therefore $\mathfrak{X}$ is not a pseudo-BCI-algebra. Thus $\mathfrak{X}$ is a proper pseudo-BCH-algebra.

From [18] it follows that in any pseudo-BCH-algebra $\mathfrak{X}$ for all $x, y \in X$ we have:

(a1) $x * (x \diamond y) \leqslant y$ and $x \diamond (x * y) \leqslant y$;
(a2) $x * 0 = x \diamond 0 = x$;
(a3) $0 * x = 0 \diamond x$;
(a4) $0 * (0 * (0 * x)) = 0 * x$;
(a5) $0 * (x * y) = (0 * x) \diamond (0 * y)$;

(a6) $0 * (x \diamond y) = (0 * x) * (0 * y)$.

Following the terminology of [18], the set $\{a \in X : a = 0 * (0 * a)\}$ will be called the *centre* of $\mathfrak{X}$. W shall denote it by $\operatorname{Cen}\mathfrak{X}$. By Proposition 4.1 of [18], $\operatorname{Cen}\mathfrak{X}$ is the set of all minimal elements of $\mathfrak{X}$, that is,

$$\operatorname{Cen}\mathfrak{X} = \{a \in X : \forall_{x \in X}(x \leqslant a \Longrightarrow x = a)\}.$$

By (a4),

$$0 * x \in \operatorname{Cen}\mathfrak{X} \tag{3}$$

for all $x \in \mathfrak{X}$.

Minimal elements (also called atoms) were investigated in BCI/BCH-algebras ([17], [14]), pseudo-BCI-algebras ([7]), and in other algebras of logic (see for example [2], [4], and [5]).

Proposition 2.7 ([18]). *Let $\mathfrak{X}$ be a pseudo-BCH-algebra, and let $a \in X$. Then the following conditions are equivalent:*

(i) $a \in \operatorname{Cen}\mathfrak{X}$.
(ii) $a * x = 0 * (x * a)$ *for all* $x \in X$.
(iii) $a \diamond x = 0 * (x \diamond a)$ *for all* $x \in X$.

Proposition 2.8 ([18]). $\operatorname{Cen}\mathfrak{X}$ *is a subalgebra of* $\mathfrak{X}$.

3. Ideals in pseudo-BCH-algebras.

Definition 3.1. A subset I of X is called an *ideal* of $\mathfrak{X}$ if it satisfies for all $x, y \in X$,

(I1) $0 \in I$;
(I2) if $x * y \in I$ and $y \in I$, then $x \in I$.

We will denote by $\operatorname{Id}(\mathfrak{X})$ the set of all ideals of $\mathfrak{X}$. Obviously, $\{0\}, X \in \operatorname{Id}(\mathfrak{X})$.

Proposition 3.2 ([18]). *Let I be an ideal of $\mathfrak{X}$. For any $x, y \in X$, if $y \in I$ and $x \leqslant y$, then $x \in I$.*

Proposition 3.3 ([18]). *Let $\mathfrak{X}$ be a pseudo-BCH-algebra and I be a subset of X satisfying* (I1). *Then I is an ideal of $\mathfrak{X}$ if and only if for all $x, y \in X$,*

(I2') *if $x \diamond y \in I$ and $y \in I$, then $x \in I$.*

Example 3.4. Consider the pseudo-BCH-algebra $\mathfrak{G}$ given in Example 2.3. Let a be an element of G. It is clear that $\{a^n : n \in \mathbb{N} \cup \{0\}\}$ is an ideal of $\mathfrak{G}$.

Example 3.5. Let $\mathfrak{X}_1 = (\{0, a, b, c\}; *_1, \diamond_1, 0)$ be the pseudo-BCH-algebra from Example 2.4. It is easy to check that $I_1 = \{0\}$, $I_2 = \{0, a\}$, $I_3 = \{0, b\}$, and $I_4 = \{0, a, b, c\}$ are ideals of $\mathfrak{X}_1$. Let I be an ideal of $\mathfrak{X}_1$ and suppose that $c \in I$. Since $a *_1 c = b *_1 c = 0 \in I$, (I2) shows that $a, b \in I$, and therefore $I = X_1$. Similarly, if $a, b \in I$, then $I = X_1$. Thus $\operatorname{Id}(\mathfrak{X}_1) = \{I_1, I_2, I_3, I_4\}$.

Theorem 3.6. *Let $\mathfrak{X}$ be a pseudo-BCH-algebra and I be a subset of X containing* 0*. The following statements are equivalent:*

(i) *I is an ideal of $\mathfrak{X}$.*
(ii) $x \in I,\ y \in X - I \Longrightarrow y * x \in X - I$.
(iii) $x \in I,\ y \in X - I \Longrightarrow y \diamond x \in X - I$.

Proof. (i) $\Longrightarrow$ (ii): Assume that I is an ideal of $\mathfrak{X}$, let $x \in I$ and $y \in X - I$. If $y * x \in I$, then $y \in I$ by definition. Therefore $y * x \in X - I$.

(ii) $\Longrightarrow$ (i): To prove that $I \in \mathrm{Id}(\mathfrak{X})$, let $y * x \in I$ and $x \in I$. If $y \notin I$, then (ii) implies $y * x \in X - I$, a contradiction. Hence $y \in I$, which gives that I is an ideal of $\mathfrak{X}$.

Thus we have (i) $\Longleftrightarrow$ (ii). The proof of the equivalence of (i) and (iii) is similar. □

For any pseudo-BCH-algebra $\mathfrak{X}$, we set

$$\mathrm{K}(\mathfrak{X}) = \{x \in X : 0 \leqslant x\}.$$

Proposition 3.7 ([18]). *Let $\mathfrak{X}_1$ and $\mathfrak{X}_2$ be pseudo-BCH-algebras. Then*

$$\mathrm{K}(\mathfrak{X}_1 \times \mathfrak{X}_2) = \mathrm{K}(\mathfrak{X}_1) \times \mathrm{K}(\mathfrak{X}_2).$$

Observe that

$$\operatorname{Cen}\mathfrak{X} \cap \mathrm{K}(\mathfrak{X}) = \{0\}. \tag{4}$$

Indeed, $0 \in \operatorname{Cen}\mathfrak{X} \cap \mathrm{K}(\mathfrak{X})$ and if $x \in \operatorname{Cen}\mathfrak{X} \cap \mathrm{K}(\mathfrak{X})$, then $x = 0 * (0 * x) = 0 * 0 = 0$.

Theorem 3.8.

(i) *For any $t \in T$, let I_t be an ideal of a pseudo-BCH-algebra $(X_t; *_t, \circ_t, 0_t)$. Then $I \coloneqq \prod_{t\in T} I_t$ is an ideal of $\mathfrak{X} \coloneqq \prod_{t\in T} \mathfrak{X}_t$.*
(ii) *If I is an ideal of $\mathfrak{X}$ such that $I \subseteq \mathrm{K}(\mathfrak{X})$, then $I_t \coloneqq \pi_t(I)$, where π_t is the t-th projection of $\mathfrak{X}$ onto $\mathfrak{X}_t$, is an ideal of $\mathfrak{X}_t$, and $I \subseteq \prod_{t\in T} I_t$.*

Proof. (i) The first part of the assertion is obvious.

(ii) The proof of this is similar to that of Theorem 5.1.35 [6]. □

Proposition 3.9. *Let $\mathfrak{X}_1$ and $\mathfrak{X}_2$ be pseudo-BCH-algebras satisfying the condition* (pBCK)*. Then*

$$\mathrm{Id}(\mathfrak{X}_1 \times \mathfrak{X}_2) = \mathrm{Id}(\mathfrak{X}_1) \times \mathrm{Id}(\mathfrak{X}_2).$$

Proof. Let $\mathfrak{X} = \mathfrak{X}_1 \times \mathfrak{X}_2$ and $I \in \mathrm{Id}(\mathfrak{X})$. By Proposition 3.7, $\mathrm{K}(\mathfrak{X}) = \mathrm{K}(\mathfrak{X}_1) \times \mathrm{K}(\mathfrak{X}_2) = X_1 \times X_2 = X$, and therefore $I \subseteq \mathrm{K}(\mathfrak{X})$. From Theorem 3.8 (ii) it follows that $I \subseteq I_1 \times I_2$, where $I_1 = \pi_1(I)$, $I_2 = \pi_2(I)$. Let $a \in I_1$ and $b \in I_2$. There are $c \in X_2$ and $d \in X_1$ such that $(a, c), (d, b) \in I$. Since $(a, 0) \leqslant (a, c)$ and $(0, b) \leqslant (d, b)$, we conclude that $(a, 0), (0, b) \in I$. Observe that $(a, b) \in I$. Indeed, we have $(a, b) * (0, b) = (a, 0)$ and $(a, 0), (0, b) \in I$. From this $(a, b) \in I$. Therefore $I = I_1 \times I_2 \in \mathrm{Id}(\mathfrak{X}_1) \times \mathrm{Id}(\mathfrak{X}_2)$.

Conversely, let $I = I_1 \times I_2$, where $I_1 \in \mathrm{Id}(\mathfrak{X}_1)$ and $I_2 \in \mathrm{Id}(\mathfrak{X}_2)$. By Theorem 3.8 (i), I is an ideal of $\mathfrak{X}$. $\square$

Example 3.10. Let $\mathfrak{X} = \mathfrak{X}_1 \times \mathfrak{X}_2$ be the pseudo-BCH-algebra given in Example 2.4. We know that $\mathrm{Id}(\mathfrak{X}_1) = \{I_1, I_2, I_3, I_4\}$ where $I_1 = \{0\}$, $I_2 = \{0, a\}$, $I_3 = \{0, b\}$, and $I_4 = X_1$ (see Example 3.5). It is easily seen that the only ideals of $\mathfrak{X}_2$ are the following subsets of X_2: $J_1 = \{0\}$, $J_2 = \{0, 1\}$, $J_3 = \{0, 1, 2\}$, and $J_4 = X_2$. Then, by Proposition 3.9, $\mathrm{Id}(\mathfrak{X}) = \{I_m \times J_n : m, n = 1, 2, 3, 4\}$.

Theorem 3.11. *Let* $(\mathfrak{X}_t)_{t\in T}$ *be an indexed family of pseudo-BCH-algebras satisfying* (pBCK) *and* $\mathfrak{X} = \sum_{t\in T} \mathfrak{X}_t$. *Let* I_t *be an ideal of* $\mathfrak{X}_t$ *for* $t \in T$. *Then* $\bigcup_{t\in T} I_t$ *is an ideal of* $\mathfrak{X}$. *Conversely, every ideal of* $\mathfrak{X}$ *is of this form.*

Proof. Let $I = \bigcup_{t\in T} I_t$. Of course, $0 \in I$. Let $x*y \in I$ and $y \in I$. If $x \in X_t$ and $y \in X_u$, where $t \neq u$, then $x = x * y \in I$. Suppose that $x, y \in X_t$. Then $x*y, y \in I_t$. Since I_t is an ideal of $\mathfrak{X}_t$, we conclude that $x \in I_t$. Hence $x \in I$, and consequently, $I \in \mathrm{Id}(\mathfrak{X})$.

Now let I be an ideal of $\mathfrak{X}$. It is easy to see that $I_t := I \cap X_t \in \mathrm{Id}(\mathfrak{X}_t)$ for $t \in T$. We have $I = I \cap \bigcup_{t\in T} X_t = \bigcup_{t\in T} I \cap X_t = \bigcup_{t\in T} I_t$. $\square$

Example 3.12. Consider the pseudo-BCH-algebras $\mathfrak{X}_1$, $\mathfrak{X}_2$, and $\mathfrak{X}$, which are described in Example 2.6. We know that $\mathrm{Id}(\mathfrak{X}_1) = \{\{0\}, \{0, a\}, \{0, b\}, X_1\}$ (by Example 3.5). It is easy to check that $\mathrm{Id}(\mathfrak{X}_2) = \{\{0\}, \{0, 3\}, X_2\}$. Applying Theorem 3.11, we get $\mathrm{Id}(\mathfrak{X}) = \{\{0\}, \{0, a\}, \{0, b\}, X_1, \{0, 3\}, \{0, 3, a\}, \{0, 3, b\}, X_1 \cup \{3\}, X_2, X_2 \cup \{a\}, X_2 \cup \{b\}, X\}$.

$\mathrm{Cen}\,\mathfrak{X}$ is a subalgebra of $\mathfrak{X}$ (see Proposition 2.8) but it may not be an ideal. For example, let $Y = \{0, a, b, c, d, e, f, g, h\}$ and $\mathfrak{Y} = (Y; *, \diamond, 0)$ be the pseudo-BCH-algebra given in Example 2.5. Then $\mathrm{Cen}\,\mathfrak{Y} = \{0, d, e, f, g, h\}$. It is easy to see that $\mathrm{Cen}\,\mathfrak{Y}$ is not an ideal of $\mathfrak{Y}$. Now we establish conditions for the set $\mathrm{Cen}\,\mathfrak{X}$ to be an ideal of a pseudo-BCH-algebra $\mathfrak{X}$.

Theorem 3.13. *Let* $\mathfrak{X}$ *be a pseudo-BCH-algebra. The following statements are equivalent:*

(i) $\mathrm{Cen}\,\mathfrak{X}$ *is an ideal of* $\mathfrak{X}$.
(ii) $x = (x * a) * (0 * a)$ *for* $x \in X$, $a \in \mathrm{Cen}\,\mathfrak{X}$.
(iii) *For all* $x \in X$, $a \in \mathrm{Cen}\,\mathfrak{X}$, $x * a = 0 * a$ *implies* $x = 0$.
(iv) *For all* $x \in \mathrm{K}(\mathfrak{X})$, $a \in \mathrm{Cen}\,\mathfrak{X}$, $x * a = 0 * a$ *implies* $x = 0$.

Proof. (i) $\Longrightarrow$ (ii): Write $I = \mathrm{Cen}\,\mathfrak{X}$, and suppose that I is an ideal of $\mathfrak{X}$. Let $x \in X$ and $a \in I$. By (pBCH-2) and (pBCH-1),

$$((x*a)*(0*a))\diamond x = ((x*a)\diamond x)*(0*a) = ((x\diamond x)*a))*(0*a) = (0*a)*(0*a) = 0,$$

and hence

$$(x * a) * (0 * a) \leqslant x. \tag{5}$$

Using (pBCH-2) and (a1), we obtain

(6) $\quad (x \diamond ((x * a) * (0 * a))) * a = (x * a) \diamond ((x * a) * (0 * a)) \leqslant 0 * a.$

By (3), $0 * a \in I$. From (6) and Proposition 3.2 we conclude that

$$(x \diamond ((x * a) * (0 * a))) * a \in I.$$

Since $a \in I$, by the definition of ideal we deduce that

(7) $$x \diamond ((x * a) * (0 * a)) \in I.$$

Applying (a6) and Proposition 2.7, we get

$0*((x*a)*(0*a)) = (0*(x*a))\diamond(0*(0*a)) = (a*x)\diamond a = (a\diamond a)*x = 0*x.$

Then $0 * (x \diamond ((x * a) * (0 * a))) = (0 * x) * (0 * x) = 0$, and hence

$$x \diamond ((x * a) * (0 * a)) \in \mathrm{K}(\mathfrak{X}).$$

From this and (7) we have $x \diamond ((x * a) * (0 * a)) \in I \cap \mathrm{K}(\mathfrak{X}) = \{0\}$ (see (4)), that is, $x \diamond ((x * a) * (0 * a)) = 0$. Therefore

(8) $$x \leqslant (x * a) * (0 * a).$$

By (5) and (8) we obtain $x = (x * a) * (0 * a)$.

(ii) $\Longrightarrow$ (iii): Let $x \in X$, $a \in \mathrm{Cen}\,\mathfrak{X}$, and $x * a = 0 * a$. Then $x = (x * a) * (0 * a) = (x * a) * (x * a) = 0$.

(iii) $\Longrightarrow$ (iv) is obvious.

(iv) $\Longrightarrow$ (i): To prove that $\mathrm{Cen}\,\mathfrak{X}$ is an ideal, let a, $x*a \in \mathrm{Cen}\,\mathfrak{X}$. Observe that $x \diamond (0 * (0 * x)) \in \mathrm{K}(\mathfrak{X})$. By (a6) and (a4), $0 * [x \diamond (0 * (0 * x))] = (0 * x) * (0 * (0 * (0 * x))) = (0 * x) * (0 * x) = 0$, and hence

(9) $$x \diamond (0 * (0 * x)) \in \mathrm{K}(\mathfrak{X}).$$

We have

$$\begin{aligned} x * a &= 0 * (0 * (x * a)) && [\text{since } x * a \in \mathrm{Cen}\,\mathfrak{X}] \\ &= (0 * (0 * x)) * (0 * (0 * a)) && [\text{by (a5) and (a6)}] \\ &= (0 * (0 * x)) * a. && [\text{since } a \in \mathrm{Cen}\,\mathfrak{X}] \end{aligned}$$

Then by (pBCH-2) and (pBCH-1),

$[x\diamond(0*(0*x))]*a = (x*a)\diamond(0*(0*x)) = [(0*(0*x))*a]\diamond(0*(0*x)) = 0*a,$

that is,

$$[x \diamond (0 * (0 * x))] * a = 0 * a.$$

Applying (iv) we get $x \diamond (0 * (0 * x)) = 0$. Hence $x \leqslant 0 * (0 * x)$. By (a3) and (a1), $0 * (0 * x) = 0 * (0 \diamond x) \leqslant x$, and therefore $x = 0 * (0 * x)$. From this $x \in \mathrm{Cen}\,\mathfrak{X}$. Thus $\mathrm{Cen}\,\mathfrak{X}$ is an ideal of $\mathfrak{X}$. □

We also have theorem analogous to Theorem 3.13.

Theorem 3.14. *Let $\mathfrak{X}$ be a pseudo-BCH-algebra. The following statements are equivalent:*

(i) $\mathrm{Cen}\mathfrak{X}$ *is an ideal of* $\mathfrak{X}$.
(ii) $x = (x \diamond a) \diamond (0 \diamond a)$ *for* $x \in X$, $a \in \mathrm{Cen}\,\mathfrak{X}$.
(iii) *For all* $x \in X$, $a \in \mathrm{Cen}\,\mathfrak{X}$, $x \diamond a = 0 \diamond a$ *implies* $x = 0$.
(iv) *For all* $x \in \mathrm{K}(\mathfrak{X})$, $a \in \mathrm{Cen}\,\mathfrak{X}$, $x \diamond a = 0 \diamond a$ *implies* $x = 0$.

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