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On certain general integral operators of analytic functions

Abstract. In this paper, we obtain new sufficient conditions for the operators $F_{\alpha_1,\alpha_2,\ldots,\alpha_n,\beta}(z)$ and $G_{\alpha_1,\alpha_2,\ldots,\alpha_n,\beta}(z)$ to be univalent in the open unit disc $\mathcal{U}$, where the functions $f_1, f_2, \ldots, f_n$ belong to the classes $\mathcal{S}^\star(a,b)$ and $\mathcal{K}(a,b)$. The order of convexity for the operators $F_{\alpha_1,\alpha_2,\ldots,\alpha_n,\beta}(z)$ and $G_{\alpha_1,\alpha_2,\ldots,\alpha_n,\beta}(z)$ is also determined. Furthermore, and for $\beta = 1$, we obtain sufficient conditions for the operators $F_n(z)$ and $G_n(z)$ to be in the class $\mathcal{K}(a,b)$. Several corollaries and consequences of the main results are also considered.

1. Introduction and definitions. Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are analytic in the open unit disc $\mathcal{U} = \{z : |z| < 1\}$. Further, by $\mathcal{S}$ we shall denote the class of all functions in $\mathcal{A}$ which are univalent in $\mathcal{U}$. A function $f(z) \in \mathcal{A}$ is said to be starlike of order $\gamma\,(0 \le \gamma < 1)$ if it satisfies

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \gamma \qquad (z \in \mathcal{U}). \tag{1.1}$$

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Also, we say that a function $f(z) \in \mathcal{A}$ is said to be convex of order $\gamma\,(0 \leq \gamma < 1)$ if it satisfies

$$\text{(1.2)} \qquad \operatorname{Re}\left(1+\frac{zf''(z)}{f'(z)}\right) > \gamma \qquad (z \in \mathcal{U}).$$

We denote by $\mathcal{S}^{\star}(\gamma)$ and $\mathcal{K}(\gamma)$, respectively, the usual classes of starlike and convex functions of order $\gamma\,(0 \leq \gamma < 1)$ in $\mathcal{U}$.

A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{S}^{\star}(a,b)$ if

$$\text{(1.3)} \qquad \left|\frac{zf'(z)}{f(z)} - a\right| < b \qquad (z \in \mathcal{U};\ |a-1| < b \leq a)$$

and a function $f \in \mathcal{A}$ is said to be in the class $\mathcal{K}(a,b)$ if

$$\text{(1.4)} \qquad \left|1+\frac{zf''(z)}{f'(z)} - a\right| < b \qquad (z \in \mathcal{U};\ |a-1| < b \leq a).$$

From (1.3) and (1.4), we have

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > a-b \qquad (z \in \mathcal{U};\ |a-1| < b \leq a)$$

and

$$\operatorname{Re}\left(1+\frac{zf''(z)}{f'(z)}\right) > a-b \qquad (z \in \mathcal{U};\ |a-1| < b \leq a).$$

The class $\mathcal{S}^{\star}(a,b)$ was introduced by Jakubowski [12]. It is clear that $a > \frac{1}{2}$, $\mathcal{S}^{\star}(a,b) \subset \mathcal{S}^{\star}(a-b) \subset \mathcal{S}^{\star}(0) \equiv \mathcal{S}^{\star}$ and $\mathcal{K}(a,b) \subset \mathcal{K}(a-b) \subset \mathcal{K}(0) \equiv \mathcal{K}$. Further, applying the Briot-Bouquet differential subordination [9], we can easily see that $\mathcal{K}(a,b) \subset \mathcal{S}^{\star}(a,b)$.

Several authors (e.g., see [4, 5, 6, 8, 10, 11, 15, 16]), obtained many sufficient conditions for the univalency of the integral operators

$$\text{(1.5)} \qquad F_{\alpha_1,\alpha_2,\ldots,\alpha_n,\beta}(z) = \left\{\beta\int_0^z t^{\beta-1}\prod_{i=1}^{n}\left(\frac{f_i(t)}{t}\right)^{\alpha_i} dt\right\}^{\frac{1}{\beta}}$$

and

$$\text{(1.6)} \qquad G_{\alpha_1,\alpha_2,\ldots,\alpha_n,\beta}(z) = \left\{\beta\int_0^z t^{\beta-1}\prod_{i=1}^{n}\left(f_i'(t)\right)^{\alpha_i} dt\right\}^{\frac{1}{\beta}},$$

where the functions $f_1, f_2, \ldots, f_n$ belong to the class $\mathcal{A}$ and the parameters $\alpha_1, \alpha_2, \ldots, \alpha_n$, and β are complex numbers such that the integrals in (1.5) and (1.6) exist. Here and throughout in the sequel every many-valued function is taken with the principal branch.

For $\beta = 1$, we obtain the integral operators

$$F_n(z) = \int_0^z \left(\frac{f_1(t)}{t}\right)^{\alpha_1} \dots \left(\frac{f_n(t)}{t}\right)^{\alpha_n} dt \tag{1.7}$$

and

$$G_n(z) = \int_0^z \left(f_1'(t)\right)^{\alpha_1} \dots \left(f_n'(t)\right)^{\alpha_n} dt \tag{1.8}$$

introduced and studied by Breaz and Breaz [5] and Breaz et al. [7], respectively.

In this paper, we obtain new sufficient conditions for the operators $F_{\alpha_1,\alpha_2,\dots,\alpha_n,\beta}(z)$ and $G_{\alpha_1,\alpha_2,\dots,\alpha_n,\beta}(z)$ defined by (1.5) and (1.6) to be univalent in the open unit disc $\mathcal{U}$, where the functions $f_1, f_2, \dots, f_n$ belong to the above classes $\mathcal{S}^\star(a,b)$ and $\mathcal{K}(a,b)$. The order of convexity for the operators $F_{\alpha_1,\alpha_2,\dots,\alpha_n,\beta}(z)$ and $G_{\alpha_1,\alpha_2,\dots,\alpha_n,\beta}(z)$ is also determined. Furthermore, we obtain sufficient conditions for the operators $F_n(z)$ and $G_n(z)$ defined by (1.5) and (1.6) to be in the class $\mathcal{K}(a,b)$.

In the proofs of our main results we need the following univalence criteria. The first result, i.e. Lemma 1.1 is a generalization of the well-known univalence criterion of Becker [2] (which in fact corresponds to the case $\beta = \delta = 1$), while the second, i.e. Lemma 1.2 is a generalization of Ahlfors' and Becker's univalence criterion [1, 3] (which corresponds to the case $\beta = 1$).

Lemma 1.1 ([13]). *Let δ be a complex number with $\mathrm{Re}(\delta) > 0$. If $f \in \mathcal{A}$ satisfies*

$$\frac{1 - |z|^{2\,\mathrm{Re}(\delta)}}{\mathrm{Re}(\delta)} \left|\frac{zf''(z)}{f'(z)}\right| \leq 1,$$

for all $z \in \mathcal{U}$, then, for any complex number β with $\mathrm{Re}(\beta) \geq \mathrm{Re}(\delta)$, the integral operator

$$F_\beta(z) = \left\{\beta \int_0^z t^{\beta-1} f'(t) dt\right\}^{\frac{1}{\beta}}$$

is in the class $\mathcal{S}$.

Lemma 1.2 ([14]). *Let β be a complex number with $\mathrm{Re}(\beta) > 0$ and c be a complex number with $|c| \leq 1$, $c \neq -1$. If $f \in \mathcal{A}$ satisfies*

$$\left|c\,|z|^{2\beta} + \left(1 - |z|^{2\beta}\right) \frac{zf''(z)}{\beta f'(z)}\right| \leq 1$$

for all $z \in \mathcal{U}$, *then the integral operator*

$$F_\beta(z) = \left\{ \beta \int_0^z t^{\beta-1} f'(t) dt \right\}^{\frac{1}{\beta}}$$

is in the class $\mathcal{S}$.

2. Univalence conditions for $F_{\alpha_1,\alpha_2,\ldots,\alpha_n,\beta}(z)$. We first prove

Theorem 2.1. *Let* $f_i(z) \in \mathcal{S}^\star(a_i, b_i)$; $|a_i - 1| < b_i \leq a_i$, $\alpha_i \in \mathbb{C}$ *for all* $i = 1, \ldots, n$, *and* $\delta \in \mathbb{C}$ *with*

$$\mathrm{Re}(\delta) \geq 2 \sum_{i=1}^{n} |\alpha_i|\, b_i. \tag{2.1}$$

Then for any $\beta \in \mathbb{C}$ *with* $\mathrm{Re}(\beta) \geq \mathrm{Re}(\delta)$, *the integral operator* $F_{\alpha_1,\alpha_2,\ldots,\alpha_n,\beta}(z)$ *defined by* (1.5) *is analytic and univalent in* $\mathcal{U}$.

Proof. Defining

$$h(z) = \int_0^z \prod_{i=1}^{n} \left(\frac{f_i(t)}{t} \right)^{\alpha_i} dt,$$

we observe that $h(0) = h'(0) - 1 = 0$, where

$$h'(z) = \prod_{i=1}^{n} \left(\frac{f_i(z)}{z} \right)^{\alpha_i}. \tag{2.2}$$

Differentiating both sides of (2.2) logarithmically, we obtain

$$\frac{zh''(z)}{h'(z)} = \sum_{i=1}^{n} \alpha_i \left(\frac{zf_i'(z)}{f_i(z)} - 1 \right)$$

which is equivalent to

$$\frac{zh''(z)}{h'(z)} = \sum_{i=1}^{n} \alpha_i \left(\frac{zf_i'(z)}{f_i(z)} - a_i \right) + \sum_{i=1}^{n} \alpha_i a_i - \sum_{i=1}^{n} \alpha_i. \tag{2.3}$$

Since $f_i(z) \in \mathcal{S}^\star(a_i, b_i)$; $|a_i - 1| < b_i \leq a_i$ for all $i = 1, 2, \ldots, n$, it follows from (2.3) that

$$\begin{aligned} \left| \frac{zh''(z)}{h'(z)} \right| &\leq \sum_{i=1}^{n} |\alpha_i| \left| \frac{zf_i'(z)}{f_i(z)} - a_i \right| + \sum_{i=1}^{n} |\alpha_i|\, |a_i - 1| \\ &< 2 \sum_{i=1}^{n} |\alpha_i|\, b_i. \end{aligned} \tag{2.4}$$

Multiplying both sides of (2.4) by $\frac{1-|z|^{2\operatorname{Re}(\delta)}}{\operatorname{Re}(\delta)}$ and making use of (2.1), we obtain

$$\begin{aligned}\frac{1-|z|^{2\operatorname{Re}(\delta)}}{\operatorname{Re}(\delta)}\left|\frac{zh''(z)}{h'(z)}\right| &\leq 2\left(\frac{1-|z|^{2\operatorname{Re}(\delta)}}{\operatorname{Re}(\delta)}\right)\sum_{i=1}^{n}|\alpha_i|\,b_i \\ &< \frac{2}{\operatorname{Re}(\delta)}\sum_{i=1}^{n}|\alpha_i|\,b_i \leq 1.\end{aligned}$$

Applying Lemma 1.1 for the function $h(z)$, we prove that $F_{\alpha_1,\alpha_2,\ldots,\alpha_n,\beta}(z) \in \mathcal{S}$. □

Letting $n = 1$, $\alpha_1 = \alpha$, $a_1 = a$, $b_1 = b$ and $f_1 = f$ in Theorem 2.1, we have

Corollary 2.2. *Let $f(z) \in \mathcal{S}^\star(a,b)$; $|a-1| < b \leq a$, $\alpha \in \mathbb{C}$ and $\delta \in \mathbb{C}$ with $\operatorname{Re}(\delta) > 2\,|\alpha|\,b$. Then for any $\beta \in \mathbb{C}$ with $\operatorname{Re}(\beta) \geq \operatorname{Re}(\delta)$, the integral operator*

$$F_{\alpha,\beta}(z) = \left\{\beta\int_0^z t^{\beta-1}\left(\frac{f(t)}{t}\right)^{\alpha} dt\right\}^{\frac{1}{\beta}} \tag{2.5}$$

is analytic and univalent in $\mathcal{U}$.

Making use of Lemma 1.2, we prove the following theorem:

Theorem 2.3. *Let $f_i(z) \in \mathcal{S}^\star(a_i,b_i)$; $|a_i-1| < b_i \leq a_i$, $\alpha_i \in \mathbb{C}$ for all $i = 1,2,\ldots,n$, and $\beta \in \mathbb{C}$ with*

$$\operatorname{Re}(\beta) \geq 2\sum_{i=1}^{n}|\alpha_i|\,b_i$$

and

$$|c| \leq 1 - \frac{2}{\operatorname{Re}(\beta)}\sum_{i=1}^{n}|\alpha_i|\,b_i \qquad (c \in \mathbb{C}). \tag{2.6}$$

Then the integral operator $F_{\alpha_1,\alpha_2,\ldots,\alpha_n,\beta}(z)$ defined by (1.5) is analytic and univalent in $\mathcal{U}$.

Proof. Let $f_i(z) \in \mathcal{S}^\star(a_i, b_i)$; $|a_i - 1| < b_i \leq a_i$ for all $i = 1, 2, \ldots, n$, it follows from (2.4) that

$$\begin{aligned}\left|c\,|z|^{2\beta} + (1 - |z|^{2\beta})\frac{zh''(z)}{\beta h'(z)}\right| &\leq |c| + \left|\frac{1 - |z|^{2\beta}}{\beta}\right| \left|\frac{zh''(z)}{h'(z)}\right| \\ &\leq |c| + 2\left|\frac{1 - |z|^{2\beta}}{\beta}\right| \sum_{i=1}^{n} |\alpha_i|\, b_i \\ &< |c| + \frac{2}{|\beta|} \sum_{i=1}^{n} |\alpha_i|\, b_i \\ &< |c| + \frac{2}{\mathrm{Re}(\beta)} \sum_{i=1}^{n} |\alpha_i|\, b_i\end{aligned}$$

which, in the light of the hypothesis (2.6), yields

$$\left|c\,|z|^{2\beta} + (1 - |z|^{2\beta})\frac{zh''(z)}{\beta h'(z)}\right| \leq 1.$$

Finally, by applying Lemma 1.2, we conclude that $F_{\alpha_1, \alpha_2, \ldots, \alpha_n, \beta}(z) \in \mathcal{S}$. $\square$

Letting $n = 1$, $\alpha_1 = \alpha$, $a_1 = a$, $b_1 = b$ and $f_1 = f$ in Theorem 2.3, we have

Corollary 2.4. *Let $f(z) \in \mathcal{S}^\star(a, b)$; $|a - 1| < b \leq a$, $\alpha \in \mathbb{C}$, and $\beta \in \mathbb{C}$ with*

$$\mathrm{Re}(\beta) \geq 2\,|\alpha|\, b$$

and

$$|c| \leq 1 - \frac{2}{\mathrm{Re}(\beta)}\,|\alpha|\, b \qquad (c \in \mathbb{C}).$$

Then the integral operator $F_{\alpha,\beta}(z)$ defined by (2.5) is analytic and univalent in $\mathcal{U}$.

3. Univalence conditions for $G_{\alpha_1, \alpha_2, \ldots, \alpha_n, \beta}(z)$. Now, we prove

Theorem 3.1. *Let $f_i(z) \in \mathcal{K}(a_i, b_i)$; $|a_i - 1| < b_i \leq a_i$, $\alpha_i \in \mathbb{C}$ for all $i = 1, \ldots, n$, and $\delta \in \mathbb{C}$ with*

$$\mathrm{Re}(\delta) \geq 2 \sum_{i=1}^{n} |\alpha_i|\, b_i.$$

Then for any $\beta \in \mathbb{C}$ with $\mathrm{Re}(\beta) \geq \mathrm{Re}(\delta)$, the integral operator $G_{\alpha_1, \alpha_2, \ldots, \alpha_n, \beta}(z)$ defined by (1.6) is analytic and univalent in $\mathcal{U}$.

Proof. Defining

$$h(z) = \int_0^z \prod_{i=1}^n \left(f_i'(t)\right)^{\alpha_i} dt,$$

we observe that $h(0) = h'(0) - 1 = 0$. On the other hand, it is easy to see that

$$h'(z) = \prod_{i=1}^n \left(f_i'(z)\right)^{\alpha_i}. \tag{3.1}$$

Differentiating both sides of (3.1) logarithmically, we obtain

$$\frac{zh''(z)}{h'(z)} = \sum_{i=1}^n \alpha_i \left(\frac{zf_i''(z)}{f_i'(z)}\right).$$

Thus, we have

$$\frac{zh''(z)}{h'(z)} = \sum_{i=1}^n \alpha_i \left(1 + \frac{zf_i''(z)}{f_i'(z)} - a_i\right) + \sum_{i=1}^n \alpha_i(a_i - 1). \tag{3.2}$$

Let $f_i(z) \in \mathcal{K}(a_i, b_i)$; $|a_i - 1| < b_i \leq a_i$, for all $i = 1, 2, \ldots, n$, and following the same steps in the proof of Theorem 2.1, we get the required result. □

Letting $n = 1$, $\alpha_1 = \alpha$, $a_1 = a$, $b_1 = b$ and $f_1 = f$ in Theorem 3.1, we have

Corollary 3.2. *Let $f(z) \in \mathcal{K}(a, b)$; $|a - 1| < b \leq a$, α and $\delta \in \mathbb{C}$ with $\mathrm{Re}(\delta) \geq 2|\alpha| b$. Then for any $\beta \in \mathbb{C}$ with $\mathrm{Re}(\beta) \geq \mathrm{Re}(\delta)$, the integral operator*

$$G_{\alpha,\beta}(z) = \left\{\beta \int_0^z t^{\beta-1} \left(f'(t)\right)^{\alpha} dt\right\}^{\frac{1}{\beta}} \tag{3.3}$$

is analytic and univalent in $\mathcal{U}$.

Using (3.1), (1.4) and applying Lemma 1.2, we prove the following theorem:

Theorem 3.3. *Let $f_i(z) \in \mathcal{K}(a_i, b_i)$; $|a_i - 1| < b_i \leq a_i$, $\alpha_i \in \mathbb{C}$ for all $i = 1, \ldots, n$ and $\beta \in \mathbb{C}$ with*

$$\mathrm{Re}(\beta) \geq 2 \sum_{i=1}^n |\alpha_i| b_i$$

and

$$|c| \leq 1 - \frac{2}{\mathrm{Re}(\beta)} \sum_{i=1}^n |\alpha_i| b_i \qquad (c \in \mathbb{C}).$$

Then the integral operator $G_{\alpha_1,\alpha_2,\ldots,\alpha_n,\beta}(z)$ defined by (1.6) is analytic and univalent in $\mathcal{U}$.

Letting $n = 1$, $\alpha_1 = \alpha$, $a_1 = a$, $b_1 = b$ and $f_1 = f$ in Theorem 3.3, we have

Corollary 3.4. *Let $f(z) \in \mathcal{K}(a, b)$; $|a-1| < b \le a$, α and $\beta \in \mathbb{C}$ with*

$$\operatorname{Re}(\beta) \ge 2\,|\alpha|\,b$$

and

$$|c| \le 1 - \frac{2}{\operatorname{Re}(\beta)}\,|\alpha|\,b.$$

Then the integral operator $G_{\alpha,\beta}(z)$ defined by (3.3) is analytic and univalent in $\mathcal{U}$.

4. Order of convexity. Now, we prove

Theorem 4.1. *Let $f_i(z) \in \mathcal{S}^\star(a_i, b_i)$; $|a_i - 1| < b_i \le a_i$, and $\alpha_i > 0$ for all $i = 1, \ldots, n$, with*

$$0 \le 1 - \sum_{i=1}^{n} \alpha_i \left(b_i + \frac{1}{2}\right) < 1 \qquad \text{and} \qquad \sum_{i=1}^{n} \alpha_i \left(b_i + \frac{1}{2}\right) \le 1.$$

Then the integral operator $F_n(z)$ defined by (1.7) is in the class

$$\mathcal{K}\left(1 - \sum_{i=1}^{n} \alpha_i \left(b_i + \frac{1}{2}\right)\right).$$

Proof. From (1.7), it follows that

$$F_n'(z) = \prod_{i=1}^{n} \left(\frac{f_i(z)}{z}\right)^{\alpha_i}. \tag{4.1}$$

Differentiating both sides of (4.1) logarithmically, we obtain

$$1 + \frac{zF_n''(z)}{F_n'(z)} = \sum_{i=1}^{n} \alpha_i \left(\frac{zf_i'(z)}{f_i(z)}\right) - \sum_{i=1}^{n} \alpha_i + 1.$$

Since $f_i(z) \in \mathcal{S}^\star(a_i, b_i)$; $|a_i - 1| < b_i \le a_i$ and $a_i > \frac{1}{2}$ for all $i = 1, 2, \ldots, n$, we have

$$\begin{aligned} \operatorname{Re}\left(1 + \frac{zF_n''(z)}{F_n'(z)}\right) &= \sum_{i=1}^{n} \alpha_i \operatorname{Re}\left(\frac{zf_i'(z)}{f_i(z)}\right) - \sum_{i=1}^{n} \alpha_i + 1 \\ &\ge \sum_{i=1}^{n} \alpha_i (a_i - b_i - 1) + 1 \qquad (4.2) \\ &> 1 - \sum_{i=1}^{n} \alpha_i \left(b_i + \frac{1}{2}\right). \end{aligned}$$

Therefore, $F_n(z)$ is convex of order $1 - \sum_{i=1}^{n} \alpha_i \left(b_i + \frac{1}{2}\right)$ in $\mathcal{U}$. $\square$

Letting $n = 1$, $\alpha_1 = \alpha$, $a_1 = a$, $b_1 = b$ and $f_1 = f$ in Theorem 4.1, we have

Corollary 4.2. *Let* $f(z) \in \mathcal{S}^\star(a, b)$; $|a - 1| < b \leq a$, *and* $\alpha > 0$ *with* $0 \leq 1 - \alpha\left(b + \frac{1}{2}\right) < 1$ *and* $\alpha\left(b + \frac{1}{2}\right) \leq 1$. *Then* $\int_0^z \left(\frac{f(t)}{t}\right)^\alpha dt \in \mathcal{K}(1 - \alpha(b + \frac{1}{2}))$.

Next, we prove

Theorem 4.3. *Let* $f_i(z) \in \mathcal{K}(a_i, b_i)$; $|a_i - 1| < b_i \leq a_i$, *and* $\alpha_i > 0$ *for all* $i = 1, \ldots, n$, *with*

$$0 \leq 1 - \sum_{i=1}^n \alpha_i \left(b_i + \frac{1}{2}\right) < 1 \qquad \text{and} \qquad \sum_{i=1}^n \alpha_i \left(b_i + \frac{1}{2}\right) \leq 1.$$

Then the integral operator $G_n(z)$ *defined by* (1.8) *is in the class*

$$\mathcal{K}\left(1 - \sum_{i=1}^n \alpha_i \left(b_i + \frac{1}{2}\right)\right).$$

Proof. From (1.8), we have

$$1 + \frac{zG_n''(z)}{G_n'(z)} = \sum_{i=1}^n \alpha_i \left(1 + \frac{zf_i''(z)}{f_i'(z)}\right) - \sum_{i=1}^n \alpha_i + 1. \tag{4.3}$$

Let $f_i(z) \in \mathcal{K}(a_i, b_i)$; $|a_i - 1| < b_i \leq a_i$; $a_i > \frac{1}{2}$ for all $i = 1, 2, \ldots, n$, and following the same steps in the proof of Theorem 4.1, we get the required result. □

Letting $n = 1$, $\alpha_1 = \alpha$, $a_1 = a$, $b_1 = b$ and $f_1 = f$ in Theorem 4.3, we have

Corollary 4.4. *Let* $f(z) \in \mathcal{K}(a, b)$; $|a - 1| < b \leq a$, *and* $\alpha > 0$ *with* $0 \leq 1 - \alpha\left(b + \frac{1}{2}\right) < 1$ *and* $\alpha\left(b + \frac{1}{2}\right) \leq 1$. *Then* $\int_0^z (f'(t))^\alpha dt \in \mathcal{K}(1 - \alpha(b + \frac{1}{2}))$.

5. Sufficient conditions for the operators $F_n(z)$ and $G_n(z)$.

Theorem 5.1. *Let* $f_i(z) \in \mathcal{S}^\star(\gamma_i)$; $0 \leq \gamma_i < 1$, *for all* $i = 1, 2, \ldots, n$. *Then the integral operator* $F_n(z)$ *defined by* (1.7) *is in the class* $\mathcal{K}(a_i, b_i)$, *where* $a_i = \sum_{i=1}^n \alpha_i\gamma_i + 1$, $b_i = \sum_{i=1}^n \alpha_i$ *and* $\sum_{i=1}^n \alpha_i(1 - \gamma_i) \leq 1$ *for all* $i = 1, 2, \ldots, n$.

Proof. Let $f_i(z) \in \mathcal{S}^\star(\gamma_i)$; $0 \leq \gamma_i < 1$, for all $i = 1, 2, \ldots, n$. Then it follows from (4.2) that

$$\begin{aligned} \operatorname{Re}\left(1 + \frac{zF_n''(z)}{F_n'(z)}\right) &= \sum_{i=1}^n \alpha_i \operatorname{Re}\left(\frac{zf_i'(z)}{f_i(z)}\right) + 1 - \sum_{i=1}^n \alpha_i \\ &> \sum_{i=1}^n \alpha_i\gamma_i + 1 - \sum_{i=1}^n \alpha_i \end{aligned}$$

which proves that $F_n(z) \in \mathcal{K}(a_i, b_i)$, where $a_i = \sum_{i=1}^{n} \alpha_i\gamma_i + 1$ and $b_i = \sum_{i=1}^{n} \alpha_i$ for all $i = 1, 2, \ldots, n$. $\square$

Letting $n = 1$, $\alpha_1 = \alpha$, $\gamma_1 = \gamma$, $a_1 = a$, $b_1 = b$ and $f_1 = f$ in Theorem 5.1, we have

Corollary 5.2. *Let $f(z) \in \mathcal{S}^\star(\gamma)$; $0 \le \gamma < 1$. Then $\int_0^z \left(\frac{f(t)}{t}\right)^\alpha dt \in \mathcal{K}(\alpha\gamma + 1, \alpha)$, where $0 < \alpha(1-\gamma) \le 1$.*

Using (4.3), we can prove the following theorem:

Theorem 5.3. *Let $f_i(z) \in \mathcal{K}(\gamma_i)$; $0 \le \gamma_i < 1$, for all $i = 1, 2, \ldots, n$. Then the integral operator $G_n(z)$ defined by (1.8) is in the class $\mathcal{K}(a_i, b_i)$, where $a_i = \sum_{i=1}^{n} \alpha_i\gamma_i + 1$, $b_i = \sum_{i=1}^{n} \alpha_i$ and $\sum_{i=1}^{n} \alpha_i(1-\gamma_i) \le 1$ for all $i = 1, 2, \ldots, n$.*

Letting $n = 1$, $\alpha_1 = \alpha$, $\gamma_1 = \gamma$, $a_1 = a$, $b_1 = b$ and $f_1 = f$ in Theorem 5.3, we have

Corollary 5.4. *Let $f(z) \in \mathcal{K}(\gamma)$; $0 \le \gamma < 1$. Then $\int_0^z (f'(t))^\alpha dt \in \mathcal{K}(\alpha\gamma + 1, \alpha)$, where $0 < \alpha(1-\gamma) \le 1$.*

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