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Resolvent and spectrum
of a nonselfadjoint differential operator
in a Hilbert space

ABSTRACT. We consider a second order regular differential operator whose coefficients are nonselfadjoint bounded operators acting in a Hilbert space. An estimate for the resolvent and a bound for the spectrum are established. An operator is said to be stable if its spectrum lies in the right half-plane. By the obtained bounds, stability and instability conditions are established.

1. Introduction. The present paper is devoted to the spectrum localization of regular nonselfadjoint differential operators whose coefficients are bounded operators acting in a separable Hilbert space, and the norm estimates for their resolvents.

The literature on the theory of abstract differential operators is rather rich, but mainly is devoted to the coercitivity of operators and maximal regularity of solutions of the relevant equations, cf. the well-known books [11, 12, 14] and references therein. At the same time, the spectral theory of differential operators with operator coefficients is not enough developed. The works [1, 2, 3, 4] should be mentioned. In [3] the sum of subtraction of the eigenvalues of two selfadjoint differential operators with unbounded operator coefficient is investigated. In the monograph [2], the Hardy type inequalities for abstract differential operators are established. In the paper

[1] the authors consider the Sturm–Liouville operator with variable self-adjoint operators in a Hilbert space. That paper extends some classical results of the spectral theory of the Sturm–Liouville operators. In the paper [4], a regularized trace formula for a differential operator of second order with unbounded operator coefficients on a finite interval is established. The monograph [13] considers the interplay between spectral and oscillatory properties of both finite and infinite systems of linear ordinary differential selfadjoint operators. These can be written as single differential equations with matrix-valued and (bounded) operator-valued coefficients, respectively.

The aim of this paper is to generalize the main result from [8] to a non-selfadjoint differential operator with coefficients from a class of bounded operators in a Hilbert space.

Everywhere below H is a separable Hilbert space with a scalar product $(\cdot, \cdot)_H$ and the norm $\|.\|_H = \sqrt{(\cdot, \cdot)_H}$. I is the unit operator in the corresponding space. All the considered operators are linear. For an operator T, $\sigma(T)$ denotes the spectrum; $R_\lambda(T) = (T - \lambda I)^{-1}$ ($\lambda \notin \sigma(T)$) is the resolvent, T^* is the adjoint operator, $\|T\|_H$ is the operator norm in H; $\lambda_j(T)$ ($j = 1, 2, \ldots$) are the eigenvalues of T, counted with their algebraic multiplicities; $\rho(T, \lambda) := \inf_{t \in \sigma(T)} |\lambda - t|$ is the distance between $\sigma(T)$ and a complex point λ.

Let $B(x), C(x)$ be continuous in the operator norm functions defined on a finite real segment $[a, b]$ whose values are bounded operators acting in H. The main object of this paper is the differential operator

$$(1.1) \qquad E = -\frac{d^2}{dx^2} + 2B(x)\frac{d}{dx} + C(x) \quad (a < x < b)$$

with the boundary value conditions

$$y(a) \cos \alpha + y'(a) \sin \alpha = 0; \quad y(b) \cos \beta + y'(b) \sin \beta = 0$$

for some $\alpha, \beta \in [0, 2\pi)$. Or $y(a) = y(b)$; $y'(a) = y'(b)$.

Introduce the Hilbert space $L^2 = L^2([a, b], H)$ of functions defined on $[a, b]$ with values in H, the scalar product

$$(y, v)_{L^2} := \int_a^b (y(x), v(x))_H dx$$

and the norm $\| \cdot \|_{L^2} = \sqrt{(\cdot, \cdot)_{L^2}}$.

The domain of operator (1.1) is defined as

$$\text{Dom}(E) = \left\{ y \in L^2 = L^2([a, b], H) : \frac{d^2 y(x)}{dx^2} \in L^2, \right.$$

$$\left. y(a) \cos \alpha + y'(a) \sin \alpha = 0; \ y(b) \cos \beta + y'(b) \sin \beta = 0 \right\},$$

or

$$\text{Dom}(E) = \left\{ y \in L^2 : \frac{d^2 y(x)}{dx^2} \in L^2; \ y(a) = y(b), \ y'(a) = y'(b) \right\}.$$

2. Auxiliary results. Let $V(x)$ be the Cauchy operator of the equation

(2.1) $$u'(x) = B(x)u(x),$$

at a point $x_0 \in [a, b]$. That is, $u(x) = V(x)u(x_0)$ for any solution $u(t)$ of (2.3) and $V'(x) = B(x)V(x)$. So $V(x_0) = I$. We have $V^{-1}(x)u(x) = u(x_0)$. Hence,

$$0 = \frac{d}{dx}\left[V^{-1}(x)u(x)\right] = \left[\frac{d}{dx}V^{-1}(x)\right]u(x) + V^{-1}(x)u'(x)$$

$$= \left[\frac{d}{dx}V^{-1}(x)\right]u(x) + V^{-1}(x)B(x)u(x).$$

Thus

(2.2) $$\frac{d}{dx}V^{-1}(x) = -V^{-1}(x)B(x).$$

Put

$$\Phi(x) = V^{-1}(x)C(x)V(x).$$

Then

$$\frac{d}{dx}\Phi(x) = \left[\frac{d}{dx}V^{-1}(x)\right]C(x)V(x) + V^{-1}(x)C'(x)V(x) + V^{-1}(x)C(x)V'(x)$$

$$= -V^{-1}(x)B(x)C(x)V(x) + V^{-1}(x)C'(x)V(x) + V^{-1}(x)C(x)B(x)V(x)$$

$$= V^{-1}(x)([C(x), B(x)]_c + C'(x))V(x).$$

Here $[C(x), B(x)]_c := C(x)B(x) - B(x)C(x)$ is the commutator. But $V(x_0) = I$, $\Phi(x_0) = C(x_0)$, so

$$\Phi(x) - C(x_0) = \int_{x_0}^{x}\frac{d}{ds}\Phi(s)ds.$$

We thus have proved:

Lemma 2.1. *Let $C(x)$ have the integrable first derivative. Then*

$$V^{-1}(x)C(x)V(x) - C(x_0) = \int_{x_0}^{x}V^{-1}(s)([C(s), B(s)]_c + C'(s))V(s)ds.$$

In particular, if C and B are constant, then $V(x) = e^{Bx}$ and

$$e^{-Bx}Ce^{Bx} = C + \int_{0}^{x}e^{-Bs}[C, B]_ce^{Bs}ds.$$

Moreover, if $B(x)$ and its integral commute:

(2.3) $$B(x)\int_{x_0}^{x}B(s)ds = \int_{x_0}^{x}B(s)ds\,B(x),$$

then

$$V(x) = \exp\left[\int_{x_0}^{x}B(s)ds\right].$$

Indeed, put $J(x) := \int_{x_0}^x B(s)ds$, then

$$\frac{d}{dx} \exp\left[\int_{x_0}^x B(s)ds\right] = \frac{d}{dx} \exp\left[J(x)\right] = \frac{d}{dx} \sum_{k=0}^\infty \frac{J^k(x)}{k!}$$

$$= \sum_{k=1}^\infty B(x) \frac{J^{k-1}(x)}{(k-1)!} = B(x) \exp\left[\int_{x_0}^x B(s)ds\right],$$

as claimed. In the case (2.3), by the previous lemma we have

$$\Phi(x) - B(x_0) = \int_{x_0}^x \exp\left[-\int_{x_0}^s B(s_1)ds_1\right]([C(s), B(s)]_c$$

(2.4)

$$+ C'(s)) \exp\left[\int_{x_0}^s B(s_1)ds_1\right] ds.$$

If $B(x)$ is selfcommuting: $B(x)B(s) = B(s)B(x)$, then (2.3) and therefore, (2.4) hold. If $B(x)$ and $C(x)$ commute, then

$$(2.5) \qquad \Phi(x) - C(x_0) = \int_{x_0}^x V^{-1}(s)C'(s)V(s)ds.$$

3. The autonomous problem. We will investigate operator E as a perturbation of the operator

$$E_0 = -\frac{d^2}{dx^2} + M \quad (0 < x < b)$$

with a constant bounded operator M acting in H. Besides the domain of E_0 is $\mathrm{Dom}(E)$. By $a_j = \lambda_j(S)$, $j = 1, 2, \ldots$ we denote the eigenvalues of the operator $S := -d^2/dx^2$ defined on $\mathrm{Dom}(E)$. Let P_j be the orthogonal eigenprojections of S, defined by

$$P_j = -\frac{1}{2\pi i} \int_{|z-a_j|=\epsilon} (S - zI)^{-1} dz$$

for a sufficiently small $\epsilon > 0$. Then

$$S = \sum_{j=1}^\infty a_j P_j; \ MP_j = P_j M \ \text{ and } \ E_0 = \sum_{j=1}^\infty (a_j I + M)P_j.$$

Hence,

$$(3.1) \qquad R_\lambda(E_0) = \sum_{j=1}^\infty (M + (a_j - \lambda)I)^{-1} P_j$$

and

$$(3.2) \qquad \|R_\lambda(E_0)\|_{L^2} \le \sup_j \|(M + (a_j - \lambda)I)^{-1}\|_H.$$

It is assumed that there is a continuous monotonically increasing function F_M, satisfying $F_M(0) = 0$, $F_M(\infty) = \infty$, such that

$$(3.3) \qquad \|(M - \lambda I)^{-1}\|_H \leq F_M(1/\rho(M, \lambda)).$$

Then $\|(M + (a_j - \lambda)I)^{-1}\|_H \leq F_M(1/\rho(M, \lambda - a_j))$. But $\inf_j \rho(M, \lambda - a_j) = \rho(E_0, \lambda)$. Here $\rho(E_0, \lambda) = \inf_{t \in \sigma(E_0)} |\lambda - t| = \inf_{s \in \sigma(M), j} |s + a_j - \lambda|$. Consequently, by (3.2) we obtain:

Lemma 3.1. *Let the conditions* (3.3) *and*

$$(3.4) \qquad \lambda \neq a_j + s \quad (j = 1, 2, \ldots; \; s \in \sigma(S))$$

hold. Then $\|R_\lambda(E_0)\|_{L^2} \leq F_M(1/\rho(E_0, \lambda))$.

4. The basic lemma. Let $V(x)$ be the Cauchy operator of equation (2.1) at a point x_0, again. Put

$$Z(x) = -B'(x) + B^2(x) + C(x)$$

and on $\mathrm{Dom}(E)$ introduce the operator T by

$$(Tf)(x) := f''(x) + V^{-1}(x)Z(x)V(x)f(x) \quad (f \in \mathrm{Dom}(E)).$$

Lemma 4.1. *Let* $B(x)$ *have a bounded measurable derivative. Then operator* E *is similar to* T. *Namely,*

$$(4.1) \qquad E = V(\cdot)TV^{-1}(\cdot),$$

where $(V(\cdot)f)(x) = V(x)f(x)$.

Proof. Putting $\lambda I = \lambda$, consider the equation

$$(4.2) \quad (E - \lambda)y = -\frac{d^2}{dx^2}y + 2B(x)\frac{dy(x)}{dx} + (C(x) - \lambda)y = f(x) \quad (a < x < b)$$

with $f \in L^2(H)$. Substitute $y = V(x)v$. Then

$$-[V''(x)v + 2V'(x)v' + V(x)v''] + 2B(x)[V'(x)v + V(x)v']$$
$$+ (C(x) - \lambda)V(x)v = f(x).$$

Note that

$$V''(x) = (B(x)V(x))' = B'(x)V(x) + B(x)V'(x) = B'(x)V(x) + B^2(x)V(x).$$

Hence,

$$-[B'(x)V(x)v + B^2(x)V(x)v + 2B(x)V(x)v' + V(x)v'']$$
$$+ 2B(x)[B(x)v + V(x)v'] + C(x)V(x)v - \lambda V(x)v = f(x).$$

Thus

$$-V(x)v'' + (-B'(x) + B^2(x) + C(x) - \lambda)V(x)v$$
$$= -V(x)v'' + (Z(x) - \lambda)V(x)v = f(x).$$

Therefore,

$$-v'' + V^{-1}(\cdot)Z(\cdot)V(\cdot)v - \lambda v = (T - \lambda)v = V^{-1}(\cdot)f.$$

Hence, $v = R_\lambda(T)V^{-1}(\cdot)f$ and consequently,

$$R_\lambda(E)f = y = V(\cdot)v = V(\cdot)R_\lambda(T)V^{-1}(\cdot).$$

This proves the lemma. $\qquad\square$

By the previous lemma the spectrum of E and T coincide. Moreover,

(4.3) $$\|R_\lambda(E)\|_{L^2} \le m_0 \|R_\lambda(T)\|_{L^2},$$

where $m_0 := \sup_{x\in[a,b]} \|V(x)\|_H \ \sup_{x\in[a,b]} \|V^{-1}(x)\|_H.$

5. The main result. Recall that $Z(x) = -B'(x) + B^2(x) + C(x)$ and for a fixed $x_0 \in [a,b]$, put

(5.1) $$M = Z(x_0) = -B'(x_0) + B^2(x_0) + C(x_0)$$

and

$$q := \sup_x \left\| \int_{x_0}^x V^{-1}(s)([Z(s), B(s)]_c + Z'(s))V(s)ds \right\|_H.$$

Note that

$$[Z(x), B(x)]_c = -[B'(x), B(x)]_c + [C(x), B(x)]_c$$

and

$$\frac{dZ(x)}{dx} = -B''(x) + B'(x)B(x) + B(x)B'(x) + C'(x).$$

Now we are in a position to formulate the main result of the paper.

Theorem 5.1. *Let $B(x)$ have an integrable second derivative, $C(x)$ have an integrable first derivative and M be defined by (5.1). Assume that the conditions (3.3) and*

(5.2) $$qF_M(1/\rho(E_0, \lambda)) < 1$$

hold. Then λ is regular for E, and

(5.3) $$\|(E - \lambda I)^{-1}\|_{L^2} \le \frac{m_0 F_M(1/\rho(E_0, \lambda))}{1 - qF_M(1/\rho(E_0, \lambda))}.$$

Proof. Thanks to Lemma 2.1,

(5.4)
$$\|V^{-1}(x)Z(x)V(x) - Z(x_0)\|_H$$
$$= \left\| \int_{x_0}^x V^{-1}(s)([Z(s), B(s)]_c + Z'(s))V(s)ds \right\|_H \le q$$

for all $x \in [a,b]$. So

$$\|T - E_0\|_{L^2} = \sup_x \|V^{-1}(x)Z(x)V^{-1}(x) - Z(x_0)\|_H = q.$$

By the Hilbert identity for resolvents,

$$(T - \lambda I)^{-1} - (E_0 - \lambda I)^{-1} = (T - \lambda I)^{-1}(T - E_0)(E_0 - \lambda I)^{-1}.$$

Thus, if the inequality

(5.5) $$q\|(\lambda I - E_0)^{-1}\|_{L^2} < 1$$

holds, then λ is regular for T, and

$$\|(T - \lambda I)^{-1}\|_{L^2} \leq \|(E_0 - \lambda I)^{-1}\|_{L^2}(1 - q\|(I\lambda - E_0)^{-1}\|_{L^2})^{-1}.$$

Now Lemma 3.1 and (4.3) imply the required result. □

There exist various estimates for the Cauchy operator. For instance, recall the Wintner inequalities

(5.6) $$e^{\int_{x_0}^{x} \beta_R(v)dv} \leq \frac{\|V(x)h\|_H}{\|h\|_H} \leq e^{\int_{x_0}^{x} \alpha_R(v)dv}$$

where $\alpha_R(x)$ and $\beta_R(x)$ are the largest and the smallest eigenvalues, respectively, of the operator $B_R(x) = (B(x) + B^*(x))/2$, e.g. [5, Theorem III.4.7]. Thus

(5.7) $$m_0 \leq \hat{m}_R \quad \text{where} \quad \hat{m}_R := \sup_{x \in [a,b]} e^{-\int_{x_0}^{x} \beta_R(v)dv} \sup_{x \in [a,b]} e^{\int_{x_0}^{x} \alpha_R(v)dv}.$$

Hence,

(5.8) $$q \leq \hat{q} \quad \text{where} \quad \hat{q} := \hat{m}_R \int_a^b \|[Z(s), B(s)]_c + Z'(s)\|_H ds.$$

6. Bounds for the spectrum. Thanks to Theorem 5.1, any $\mu \in \sigma(E)$ satisfies the inequality $qF_M(E_0, \mu) \geq 1$. In other words, for any $\mu \in \sigma(E)$, there is an $s \in \sigma(E_0)$, such that

$$qF_M(1/|\mu - s|) \geq 1.$$

Taking into account the monotonicity of the left-hand part of this inequality we get $|\mu - \lambda(E_0)| \leq r(q)$, where $r(q)$ is the unique positive root of the equation

(6.1) $$qF_M(1/x) = 1.$$

We thus have proved the following:

Theorem 6.1. *Let $B(x)$ and $C(x)$ satisfy the hypothesis of Theorem 5.1. Let operator M be defined by (5.1). Then for any $\mu \in \sigma(E)$, there are integers $j \geq 1$ and $s \in \sigma(M)$, such that $|s + a_j - \mu| \leq r(q)$, where $r(q)$ is the unique positive root of equation (6.1).*

Let A and $\tilde{A}$ be two linear operators. Then

$$sv_A(\tilde{A}) := \sup_{s \in \sigma(\tilde{A})} \inf_{\mu \in \sigma(A)} |s - \mu|$$

will be called *the spectral variation of $\tilde{A}$ with respect to A*.

So Theorem 6.1 means that $sv_{E_0}(E) \leq r(q)$.

Corollary 6.2. *Under the hypothesis of Theorem 5.1, the spectrum of E lies in the union of the sets*

$$\{z \in \mathbb{C} : |z - s(M) - a_j| \leq r(q)\},$$

$s \in \sigma(M); \; j = 1, 2, \dots$.

Operator E is said to be *stable* if $\operatorname{Re} \sigma(A) > 0$. E is *unstable* if $\inf \operatorname{Re} \sigma(A) < 0$.

Now Theorem 6.1 implies

Corollary 6.3. *Under the hypothesis of Theorem 5.1, let*

$$\inf_{j=1,2,\dots} a_j + \inf \operatorname{Re} \sigma(M) > r(q).$$

Then E is stable. Conversely, if $\inf_{j=1,2,\dots} a_j + \inf \operatorname{Re} \sigma(M) < -r(q)$, then E is unstable.

7. Operators admitting triangular representations. For a compact operator K in H, $s_k(A)$, $k = 1, 2, \dots$, are the singular numbers taken with their multiplicities and enumerated in the decreasing order. By SN_r ($1 \leq r < \infty$), the Schatten–von Neumann ideal is denoted. That is, the ideal of compact operators K in H with the finite norm $N_r(K) :=$ $[\operatorname{Trace} (K^*K)^{r/2}]^{1/r}$.

Let c_0 be the space of all scalar sequences tending to zero, $\hat{c}$ be the set of all sequences from c_0 having a finite number of nonzero entries. A real function $F(h) = F(h_1, h_2, \dots)$ defined on c_0 is called a norming function, if it satisfies the following conditions: $F(h) > 0$, $h \neq 0$; $F(ah) = |a|F(h)$ ($a \in \mathbb{C}$) and $F(h + w) \leq F(h) + F(w)$ for all $h = \{h_k\}, w = \{w_k\} \in \hat{c}$. In addition, $F(1, 0, 0, \dots) = 1$. A norming function F is called a symmetrically norming function, if $F(h_1, h_2, \dots, h_n, 0, 0, \dots) = F(|h_1|, |h_2|, \dots, |h_n|, 0, 0, \dots)$. The following relations are well-known [9, Section 3.3]: if for $h = \{h_k\}, w = \{w_k\} \in \hat{c}$, the inequalities $|h_k| \leq |w_k|$ hold, then $F(h) \leq F(w)$. Moreover, if

$$w_1 \geq w_2 \geq \dots \geq 0; h_1 \geq h_2 \geq \dots \geq 0 \text{ and } \sum_{k=1}^{j} w_k \leq \sum_{k=1}^{j} h_k,$$

then $F(w) \leq F(h)$ for any symmetrically norming function F. Furthermore, on the set of all compact operators K acting in H, introduce the norm $N_W(K)$ by

$$N_W(K) = F(s_1(K), s_2(K), \dots).$$

Then the set $W = W(F)$ of operators with the finite norm N_W is a symmetrically normed ideal, e.g. [9, Section 3.4].

Let $E(t)$ be a left-continuous orthogonal resolution of the identity in H, defined on a real segment $[a, b]$. E is called a maximal resolution of the identity (m.r.i.), if its every gap $E(t_0 + 0) - E(t_0)$ (if it exists) is one-dimensional, cf. [6].

A compact quasinilpotent operator will be called *a Volterra operator*. We will say that an m.r.i. $E(t)$ belongs to A (or A has an m.r.i. $E(t)$), if $E(t)AE(t) = AE(t)$ $(t \in [a, b])$. We will say that A is *a E-triangular operator* if it has an m.r.i. E defined on $[a, b]$ and admits the representation

$$(7.1) \qquad\qquad A = D + V$$

where D is a normal operator and V is a Volterra one, having the following properties:

$$(7.2) \qquad E(t)VE(t) = VE(t) \text{ and } DE(t) = E(t)D \quad (t \in [a, b]).$$

An E-triangular operator A has the property

$$(7.3) \qquad\qquad \sigma(A) = \sigma(D),$$

cf. [6, Lemma 7.5.1]. Each compact operator is E-triangular [9] and each operator having the Schatten-von Neumann Hermitian component is E-triangular; for more details see [6, Chapter 7]. We will call D and V *the diagonal and nilpotent part of A*, respectively.

Lemma 7.1. *Let A be an E-triangular operator whose nilpotent part V belongs to a symmetrically normed ideal W. In addition, let*

$$(7.4) \qquad\qquad \|(I - V)^{-1}\| \leq \phi(N_W(V)),$$

where $\phi(x)$, $x > 0$ is a nondecreasing continuous function. Then

$$(7.5) \qquad \|R_\lambda(A)\| \leq \frac{1}{\rho(A, \lambda)} \phi\left(\frac{N_W(V)}{\rho(A, \lambda)}\right) \quad (\lambda \notin \sigma(A)).$$

Proof. Let λ be a regular point of the operator D. According to the triangular representation (7.1), we obtain

$$R_\lambda(A) = (D + V - \lambda I)^{-1} = R_\lambda(D)(I + VR_\lambda(D))^{-1}.$$

Operator $VR_\lambda(D)$ for a regular point λ of the operator D is a Volterra one due to Lemma 7.3.4 from [6]. Therefore,

$$\|R_\lambda(A)\| = \|R_\lambda(D)(I - VR_\lambda(D)))^{-1}\| \leq \|R_\lambda(D)\|\phi(N_W(VR_\lambda(D))).$$

But $N_W(VR_\lambda(D)) \leq \|R_\lambda(D)\|N_W(V)$ and $\|R_\lambda(D)\| = 1/\rho(A, \lambda)$. This proves the result. $\qquad\square$

If $\phi(0) = 1$, then the previous lemma is sharp: if A is a normal operator, then $V = 0$ and inequality (7.5) becomes the equality $\|R_\lambda(A)\| = \frac{1}{\rho(A,\lambda)}$. By the Weyl inequalities,

$$N_W(D) = F(\lambda_1(A), \lambda_2(A), \dots) \leq F(s_1(A), s_2(A), \dots) = N_W(A).$$

We thus have

$$(7.6) \qquad N_W(V) \leq N_W(D) + N_W(A) \leq 2N_W(A).$$

Now the previous lemma implies:

Corollary 7.2. *Let* $A \in W$ *and condition* (7.4) *hold. Then*

$$\|R_\lambda(A)\| \leq \frac{1}{\rho(A,\lambda)} \phi\left(\frac{2N_W(A)}{\rho(A,\lambda)}\right) \quad (\lambda \notin \sigma(A)).$$

8. Operators with Schatten–von Neumann components. In this section we illustrate the results of the previous section in the case

$$(8.1) \qquad A_{I,t} := (e^{it}A - e^{-it}A^*)/2i \in SN_p \quad (p = 1, 2, \dots)$$

for a $t \in [0, 2\pi]$. To this end we need the following result.

Lemma 8.1. *Let* $V \in SN_1$ *be a quasinilpotent operator. Then*

$$\|(I - V)^{-1}\|^2 \leq \prod_{k=1}^\infty (1 + s_k(2V_R - V^*V)).$$

Here $s_k(2V_R - V^*V)$ *are the singular values of* $2V_R - V^*V$ *counted with their multiplicities and enumerated in the decreasing way.*

Proof. We have $\det(I - V)(I - V^*) = \det(I - V)\det(I - V^*) = 1$. So

$$1 = \det(I - V)(I - V^*) = \det(I - T), \text{ where } T = 2V_R - VV^*.$$

Let $\lambda_k = \lambda_k(T)$. Then $|\lambda_k| = s_k(T) = s_k(2V_R - VV^*)$. Assume that $\min_k |1 - \lambda_k| = |1 - \lambda_j|$ for an index j. Then $\|(I - V)^{-1}\|^2 = |1 - \lambda_j|^{-1}$. So

$$1 = |\det(I - T)| = \|(I - V)^{-1}\|^{-2} \prod_{k=1, k \neq j}^\infty |1 - \lambda_k|$$

and thus,

$$(8.2) \qquad \|(I - V)^{-1}\|^2 = \prod_{k=1, k \neq j}^\infty |1 - \lambda_k| \leq \prod_{k=1}^\infty (1 + |\lambda_k|),$$

as claimed. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \square$

Introduce the function

$$\Phi(t_1, t_2, \dots, t_j) = \prod_{k=1}^\infty (1 + t_k) \quad (t_1 > t_2 > \cdots > t_j \geq 0).$$

Clearly,

$$\frac{\partial \Phi}{\partial t_k} = \frac{\Phi}{1 + t_k} > \frac{\partial \Phi}{\partial t_{k+1}} \geq 0.$$

But

$$\sum_{k=1}^{j} s_k(2V_R - V^*V) \leq \sum_{k=1}^{j} s_k(2V_R) + s_k(V^*V).$$

Therefore, by Lemma 2.7.5 [9],

$$\prod_{k=1}^{\infty}(1 + s_k(2V_R - V^*V)) \leq \prod_{k=1}^{\infty}(1 + 2s_k(V_R) + s_k^2(V)) = \prod_{k=1}^{\infty}(1 + s_k(V))^2.$$

Now the latter lemma yields our next result.

Corollary 8.2. *Let $V \in SN_1$ be a quasinilpotent operator. Then*

$$(8.3) \qquad \|(I - V)^{-1}\| \leq \prod_{k=1}^{\infty}(1 + s_k(V)).$$

Note that inequality (8.3) is a particular case of the well-known Theorem V.5.1 from [9], but we suggest a considerably new approach. Besides, the previous lemma is sharper than (8.3). Furthermore, clearly,

$$(8.4) \qquad (I - V)^{-1} = (I + V + \cdots + V^{p-1})(I - V^p)^{-1}$$

for any positive integer p. Now Corollary 8.2 implies:

Lemma 8.3. *Let $V \in SN_p$ $(p = 1, 2, \dots)$ be a quasinilpotent operator. Then*

$$\|(I - V)^{-1}\| \leq \|I + V + \cdots + V^{p-1}\| \prod_{k=1}^{\infty}(1 + s_k^p(V)) \leq \psi_p(N_p(V)),$$

where

$$\psi_p(x) := e^{x^p} \sum_{k=0}^{p-1} x^k \quad (x \geq 0).$$

This result and Lemma 7.1 yield the next result.

Corollary 8.4. *Let A be an E-triangular operator whose nilpotent part $V \in SN_p$ $(p = 1, 2, \dots)$. Then*

$$\|R_\lambda(A)\| \leq \frac{1}{\rho(A, \lambda)} \psi_p\left(\frac{N_p(V)}{\rho(A, \lambda)}\right) \quad (\lambda \notin \sigma(A)).$$

As it is proved in Theorems III.6.2 and III.6.3 [10], if $V_I \in SN_p$, then $N_p(V) \leq \tilde{b}_p N_p(V_I)$, where $\tilde{b}_p$ is a constant dependent only on p, and $\tilde{b}_p \leq 1 + \frac{p}{e^{2/3} \ln 2}$. Besides, $\tilde{b}_2 = \sqrt{2}$. Since $N_p(V) = N_p(e^{it}V)$ for a real t, we have

$$(8.5) \qquad N_p(V) \leq \tilde{b}_p N_p(V_{I,t}).$$

But by the Weyl inequalities [9], $N_p(D_{I,t}) \leq N_p(A_{I,t})$ and therefore,

$$(8.6) \qquad N_p(V) \leq \tilde{b}_p N_p(V_{I,t}) \leq \tilde{b}_p N_p(A_{I,t}) + \tilde{b}_p N_p(D_{I,t}) \leq 2\tilde{b}_p N_p(A_{I,t}).$$

Thanks to Theorem 7.6.1 [6], under condition (8.1), A is an E-triangular operator. Now Corollary 8.4 implies the following result.

Theorem 8.5. *Let condition* (8.1) *hold. Then*

$$(8.7) \qquad \|R(A)\| \leq \frac{1}{\rho(A,\lambda)} \psi_p \left(\frac{b_p N_p(A_{I,t})}{\rho(A,\lambda)} \right) \quad (b_p = 2\tilde{b}_p, \ \lambda \notin \sigma(A)).$$

Remark 8.6. In the case $A \in SN_p$, $p \geq 1$, we have $N_p(D) \leq N_p(A)$ and therefore, $N_p(V) \leq 2N_p(A)$. Thus in (8.7) one can replace $b_p N_p(A_{I,t})$ by $2N_p(A)$.

Note that (8.7) is a generalization of Theorem 7.7.1 [6], which is proved in the case $A - A^* \in SN_{2r}$, $r = 1, 2, \ldots$.

In the case $M_{I,t} = (Me^{it} - e^{-it}M^*)/2i \in SN_p$, $1 < p < \infty$, according to Theorem 8.5, equation (6.1) takes the form

$$(8.8) \qquad \frac{q_0}{x} \psi_p \left(\frac{b_p N_p(M_{I,t})}{x} \right) = 1.$$

To estimate the roots of this equation we need the following result.

Lemma 8.7. *For any integer* $p \geq 1$, *the unique positive root* z_a *of the equation*

$$(8.9) \qquad \sum_{j=0}^{p-1} \frac{1}{y^{j+1}} \exp\left[\frac{1}{y^p} \right] = a \quad (a = const > 0)$$

satisfies the inequality $z_a \leq \delta_p(a)$, *where*

$$\delta_p(a) := \begin{cases} pe/a & \text{if } a \leq pe, \\ \sqrt[p]{2}[\ln(ae/p)]^{-1/p} & \text{if } a > pe. \end{cases}$$

Proof. Let

$$(8.10) \qquad pe \geq a.$$

Since the function

$$f(y) = \sum_{j=0}^{p-1} \frac{1}{y^{j+1}} \exp\left[\frac{1}{y^p} \right]$$

is nonincreasing and $f(1) = pe$, we have $z_a \geq 1$. But because of (8.9),

$$z_a = \frac{1}{a} \sum_{j=0}^{p-1} \frac{1}{z_a^j} \exp[z_a^{-p}] \leq \frac{pe}{a}.$$

So in the case (8.10), the lemma is proved. Now let $pe < a$. Then $z_a \leq 1$. But

$$\sum_{j=0}^{p-1} x^{j+1} \leq px^p \leq p\exp[x^p - 1] \quad (x \geq 1).$$

So

$$f(y) = \sum_{j=0}^{p-1} \frac{1}{y^{j+1}} \exp\left[\frac{1}{y^p}\right] \leq pe^{-1}\exp\left[\frac{2}{y^p}\right] \quad (y \leq 1).$$

We thus have $a = f(z_a) \leq pe^{-1}\exp\left[\frac{2}{z_a^p}\right]$. This finishes the proof. $\qquad\square$

Put in (8.8) $y = \frac{x}{b_p N_p(M_{I,t})}$. Then we obtain (8.9) with $a = \gamma(q)$ with

$$\gamma(q) := \frac{b_p N_p(M_{I,t})}{q_0}.$$

Hence, a unique positive root $r(q)$ of (8.8) satisfies the inequality

$$(8.11) \quad r(q) \leq \delta(q) \text{ where } \delta(q) := \begin{cases} pe/\gamma(q) & \text{if } \gamma(q) \leq pe, \\ \sqrt[p]{2}[\ln(\gamma(q)e/p)]^{-1/p} & \text{if } \gamma(q) > pe. \end{cases}$$

Now Theorem 6.1 implies:

Corollary 8.8. *Let $B(x)$ and $C(x)$ satisfy the hypothesis of Theorem 5.1 and $M_{I,t} \in SN_p$ ($t \in [0, 2\pi]$; $1 < p < \infty$). Then for any $\mu \in \sigma(E)$, there is an integer $j \geq 1$ and an $s \in \sigma(M)$, such that $|s + a_j - \mu| \leq r(q) \leq \delta(q)$, where $r(q)$ is the unique positive root of equation (8.8) and $\delta(q)$ is defined by (8.11).*

In particular, if $\inf_{j=1,2,...} a_j + \inf \operatorname{Re} \sigma(M) > \delta(q)$, then E is stable. Conversely, if $\inf_{j=1,2,...} a_j + \inf \operatorname{Re} \sigma(M) < -\delta(q)$, then E is unstable.

9. Example. Let $H = L^2([-\pi, \pi], \mathbb{C}^n)$ be a Hilbert space of functions defined on $[-\pi, \pi]$ with values in a complex Euclidean space $\mathbb{C}^n$ and equipped with the scalar product

$$(f, h)_H = \int_{-\pi}^{\pi} (f(s), h(s))_{\mathbb{C}^n} ds,$$

where $(\cdot, \cdot)_{\mathbb{C}^n}$ is the scalar product in $\mathbb{C}^n$. In addition, $\|\cdot\|_{\mathbb{C}^n} = \sqrt{(\cdot, \cdot)_{\mathbb{C}^n}}$. Consider the Dirichlet problem

$$(9.1) \qquad Eu = -\frac{\partial^2 u(x, y)}{\partial x^2} + b(x)\frac{\partial u(x, y)}{\partial x} + \int_{-\pi}^{\pi} K_0(x - s)u(x, s)ds$$

($x \in (0, 1)$, $y \in [-\pi, \pi]$),

$$(9.2) \qquad\qquad\qquad u(0, y) = u(1, y) = 0$$

($y \in [-\pi, \pi]$), where $b(x)$ is a real twice continuously differentiable scalar function, and K_0 is a matrix-valued function defined on $[-\pi, \pi]$ with $\|K_0(x)\|_{\mathbb{C}^n} \in L^2([-\pi, \pi], \mathbb{C})$. Here $\|A\|_{\mathbb{C}^n}$ means the operator (spectral)

norm of an $n \times n$-matrix A. In the considered case $a_j = j^2$, $B(x)u = b(x)u$, and

(9.3) $$C(x)u(x,y) = C_0 u(x,y) = \int_{-\pi}^{\pi} K(y-s)u(x,s)ds$$

does not depend on x. Let

$$K_0(y) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \tau_k e^{iky}$$

be the Fourier expansion with the matrix Fourier coefficients

$$\tau_k = \int_{-\pi}^{\pi} K_0(s)e^{-iks}ds.$$

We have $C_0 e^{ikx} = \tau_k e^{ikx}$. Let d_{jk} be an eigenvector of τ_k, corresponding to an eigenvalue $\lambda_j(\tau_k)$ $(j = 1, \ldots, n)$. Then

$$C_0 e^{ikx} d_{jk} = \int_{-\pi}^{\pi} K_0(x-s)d_{jk}e^{iks}ds = e^{ikx}\tau_k d_{jk} = e^{ikx}\lambda_j(\tau_k)d_{jk}.$$

So the spectrum of C_0 consists of the points

$$\lambda_j(\tau_k) \ (k = 0, \pm 1, \pm 2, \ldots ; j = 1, \ldots, n).$$

In addition, $Z(x) = -B'(x) + B^2(x) + C(x) = -b'(x) + b^2(x) + C_0$. Hence, $[Z(x), B(x)]_c = 0$, and $\|Z'(x)\|_H = \sqrt{2\pi}|b''(x) - 2b(x)b'(x)|$. Take $M = Z(0) = -b'(0) + b^2(0) + C_0$. Then the spectrum of M consists of the points

$$\lambda_{jk}(M) = b'(0) + b^2(0) + \lambda_j(\tau_k) \quad (k = 0, \pm 1, \pm 2, \ldots ; j = 1, \ldots, n).$$

Furthermore, since C_0 and $B(x)$ commute, we have $m_0 = 1$. By (5.8)

$$q \le \hat{q} := \int_0^1 \|[Z(s), B(s)]_c + Z'(s)\|_H ds = \int_0^1 |b''(x) - 2b(x)b'(x)|dx.$$

Moreover,

$$N_2^2(M_I) = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} N_2^2(K_{0I}(x-s))dx \ ds,$$

where $M_I, K_{0I}(x)$ are the imaginary Hermitian components of M and $K_0(x)$, respectively. Thanks to Corollary 8.8, the spectrum of problem (9.1), (9.2) lies in the union of the sets

$$\{z \in \mathbb{C} : |z - \lambda_{jk}(M) - m^2| \le r(\hat{q})\}$$

$(j = 1, 2, \ldots, n; \ k = 0, \pm 1, \pm 2, \ldots; \ m = 1, 2, \ldots)$. Here $r(\hat{q})$ is the unique positive root of equation (8.8) with $p = 2$, $b_2 = \sqrt{2}$ and $q = \hat{q}$. Besides, the bound (8.11) is valid.

In particular, if $1 + \inf_{j,k} \operatorname{Re} \lambda_{j,k}(M) > r(\hat{q})$, then E is stable. If $1 + \inf_{j,k} \operatorname{Re} \lambda_{j,k}(M) < -\hat{q}$, then it is unstable.

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