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Integral formula for secantoptics and its application

Abstract. Some properties of secantoptics of ovals defined by Skrzypiec in 2008 were proved by Mozgawa and Skrzypiec in 2009. In this paper we generalize to this case results obtained by Cieślak, Miernowski and Mozgawa in 1996 and derive an integral formula for an annulus bounded by a given oval and its secantoptic. We describe the change of the area bounded by a secantoptic and find the differential equation for this function. We finish with some examples illustrating the above results.

1. Introduction. Throughout this paper an oval will be a plane, simple, closed curve given by the equation

$$z(t) = p(t)e^{it} + \dot{p}(t)ie^{it} \quad \text{for} \quad t \in [0, 2\pi], \tag{1.1}$$

where $p(t)$, called the support function of an oval, is of class C^3 and the function $R(t) = p(t) + \ddot{p}(t)$ is positive for all $t \in [0, 2\pi]$. Note that the function $R(t) = p(t) + \ddot{p}(t)$ is the curvature radius of the curve $z(t)$ at the point t and that the support function $p(t)$ can be extended to a periodic function on $\mathbb{R}$ with the period 2π. Let C be an oval and let $\beta \in [0, \pi)$, $\gamma \in [0, \pi-\beta)$ and $\alpha \in (\beta+\gamma, \pi)$ be fixed angles. In [14] we defined the notion of secantoptic $C_{\alpha,\beta,\gamma}$ of an oval C as the set of intersection points $z_{\alpha,\beta,\gamma}(t)$ of secants $s_1(t)$ and $s_2(t)$ to C for every $t \in [0, 2\pi]$. This construction is

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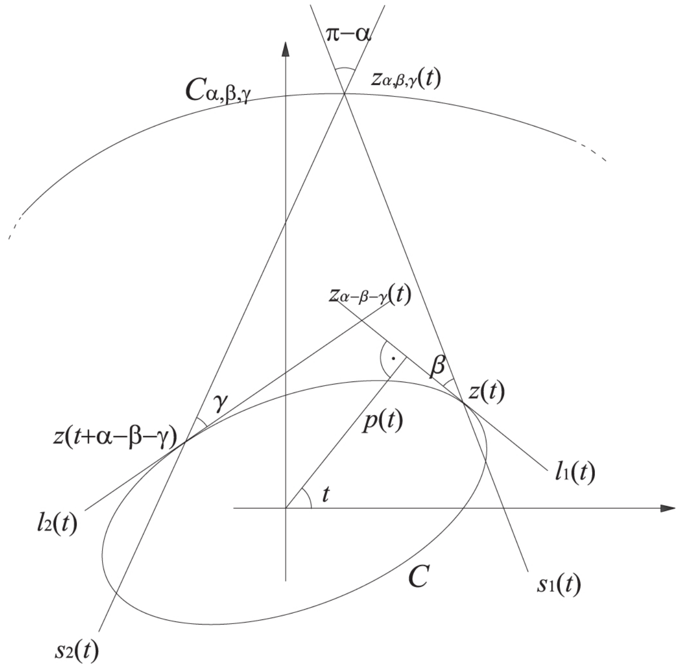

FIGURE 1. A construction of a secantoptic

shown in Figure 1. The parametrization of a secantoptic $C_{\alpha,\beta,\gamma}$ of an oval C is given by

$$z_{\alpha,\beta,\gamma}(t) = (p(t) + \lambda(t)\sin\beta + i(\dot{p}(t) + \lambda(t)\cos\beta))e^{it} \quad \text{for} \quad t \in [0, 2\pi],$$

where

$$\begin{aligned}\lambda(t) = \frac{1}{\sin\alpha}(&-p(t)\cos(\alpha-\beta) - \dot{p}(t)\sin(\alpha-\beta) \\ &+ p(t+\alpha-\beta-\gamma)\cos\gamma + \dot{p}(t+\alpha-\beta-\gamma)\sin\gamma).\end{aligned} \tag{1.2}$$

Let C be a fixed oval. We denote by $e(C)$ the exterior of C and by ζ a half line from $z(0)$ in direction $ie^{-i\beta}$. The mapping

$$F_{\beta,\gamma} : (\beta+\gamma, \pi) \times (0, 2\pi) \mapsto e(C) \setminus \zeta \tag{1.3}$$

is given by the formula

$$F_{\beta,\gamma}(\alpha, t) = z_{\alpha,\beta,\gamma}(t). \tag{1.4}$$

The Jacobian $J(F_{\beta,\gamma})$ of $F_{\beta,\gamma}$ at (α, t) is given by

$$J(F_{\beta,\gamma}) = \frac{1}{\sin\alpha}(R(t+\alpha-\beta-\gamma)\sin\gamma - \mu(t))(R(t)\sin\beta + \lambda(t)) > 0. \tag{1.5}$$

Expressions in brackets are important for further purposes, so we denote them by

$$L(t) = \lambda(t) + R(t)\sin\beta, \tag{1.6}$$

$$M(t) = \mu(t) - R(t+\alpha-\beta-\gamma)\sin\gamma. \tag{1.7}$$

We may, as it was shown in [12], express a secantoptic of an oval C as an isoptic of pair of its evolutoids. If

$$\psi_{-\beta}(t) = p(t+\beta)\cos\beta - \dot{p}(t+\beta)\sin\beta$$

and

$$\psi_{\gamma}(t) = p(t-\gamma)\cos\gamma + \dot{p}(t-\gamma)\sin\gamma$$

are the support functions of evolutoids $\Gamma_{-\beta}$ and Γ_γ, then the equation of secantoptic $C_{\alpha,\beta,\gamma}$ of an oval C is given by

$$z_\alpha^{\Gamma_{-\beta}\Gamma_\gamma}(t) = \psi_{-\beta}(t)e^{it} + \left(\psi_\gamma(t+\alpha)\frac{1}{\sin\alpha} - \psi_{-\beta}(t)\cot\alpha\right)ie^{it}.$$

2. Integral formula for annulus. Let $\beta \in [0,\pi)$, $\gamma \in [0, \pi-\beta)$ and $a \in (\beta+\gamma, \pi)$. Let (x,y) be a point in the annulus $CC_{a,\beta,\gamma}$ and let s be a secant line to the oval C passing through (x,y) and points (x_1,y_1) and (x_2,y_2) on C, where $(x_1,y_1) \in C$ is closer to (x,y) than (x_2,y_2) or $(x_1,y_1) = (x_2,y_2)$. Let the secant s be such that after rotation about an angle β around $(x_1,y_1) \in C$ we get the tangent line to C. Let $t_1(x,y) = \tau(x,y) + R\sin\beta$, where $\tau(x,y)$ denotes the distance between (x,y) and (x_1,y_1), and let R be a curvature radius of an oval C at (x_1,y_1).

Theorem 2.1. *Let C be a given oval and $C_{a,\beta,\gamma}$ its secantoptic. Then the following integral formula holds*

$$\iint\limits_{CC_{a,\beta,\gamma}} \frac{dxdy}{t_1} = L_C\left(\frac{\cos\gamma - \cos\beta\cos a}{\sin a} - \sin\beta\right), \tag{2.1}$$

where $CC_{a,\beta,\gamma}$ denotes the annulus contained between C and $C_{a,\beta,\gamma}$ and $L_C = \int_0^{2\pi} p(t)dt$ is the perimeter of C.

Proof. Let us consider the integral of $\frac{1}{t_1}$ in the annulus $CC_{a,\beta,\gamma}$, where $a \in (\beta+\gamma,\pi)$. After changing variables from (x,y) to (α,t) by means of diffeomorphism $F_{\beta,\gamma}$ we obtain

$$\iint\limits_{CC_{a,\beta,\gamma}} \frac{dxdy}{t_1} = \int\limits_0^{2\pi}\int\limits_{\beta+\gamma}^{a} \frac{-M(t)}{\sin\alpha}d\alpha dt.$$

The integrand can be expressed in terms of the support function of a given oval in the following way

$$\frac{-M(t)}{\sin\alpha} = \frac{-1}{\sin^2\alpha} p(t+\alpha-\beta-\gamma)\cos\alpha\cos\gamma + \frac{\sin\gamma}{\sin\alpha}\ddot{p}(t+\alpha-\beta-\gamma) + \frac{\sin(\alpha-\gamma)}{\sin^2\alpha}\dot{p}(t+\alpha-\beta-\gamma) + \frac{\cos\beta}{\sin^2\alpha}p(t) + \frac{\sin\beta}{\sin^2\alpha}\dot{p}(t).$$

It is well known ([13]) that the length of a curve parametrized by a support function is given by

$$L_C = \int_0^{2\pi} p(t)dt. \tag{2.2}$$

Hence

$$\int_0^{2\pi} p(t+\alpha-\beta-\gamma)dt = L_C \tag{2.3}$$

and

$$\int_0^{2\pi} \ddot{p}(t+\alpha-\beta-\gamma)dt = \int_0^{2\pi} \dot{p}(t+\alpha-\beta-\gamma)dt = \int_0^{2\pi} \dot{p}(t)dt = 0. \tag{2.4}$$

Therefore we obtain

$$\iint\limits_{CC_{a,\beta,\gamma}} \frac{dxdy}{t_1} = L_C\left(\int_{\beta+\gamma}^{a} -\frac{\cos\alpha\cos\gamma}{\sin^2\alpha}d\alpha + \int_{\beta+\gamma}^{a}\frac{\cos\beta}{\sin^2\alpha}d\alpha\right). \tag{2.5}$$

If in the first integral we substitute $\sin\alpha = x$,

$$\iint\limits_{CC_{a,\beta,\gamma}} \frac{dxdy}{t_1} = L_C\left(-\cos\gamma\int_{\sin(\beta+\gamma)}^{\sin a}\frac{dx}{x^2} - \cos\beta\int_{\beta+\gamma}^{a}\frac{-1}{\sin^2\alpha}d\alpha\right),$$

then we obtain the following integral formula

$$\iint\limits_{CC_{a,\beta,\gamma}} \frac{dxdy}{t_1} = L_C\left(\frac{\cos\gamma - \cos\beta\cos a}{\sin a} - \sin\beta\right). \qquad \square$$

If $\beta=\gamma$, then the formula (2.1) simplifies to

$$\iint\limits_{CC_{a,\beta,\beta}} \frac{dxdy}{t_1} = L_C\left(\tan\frac{a}{2}\cos\beta - \sin\beta\right) = L_{\Gamma_{-\beta}}\left(\tan\frac{a}{2} - \tan\beta\right). \tag{2.6}$$

For $\beta = \gamma = 0$ we get the formula known from [3] for isoptics

$$\iint\limits_{CC_a} \frac{dxdy}{t_1} = L_C \tan\frac{a}{2}. \tag{2.7}$$

3. The area bounded by a secantoptic. In this section, using a suitable function, we describe the change of an area of a domain bounded by a secantoptic $C_{\alpha,\beta,\gamma}$ of an oval C, where α is a variable. We derive the differential equation for this function and, using the formula (2.1), we estimate the value of its first derivative at the left end of the domain. This reasoning is a generalization of results obtained for isoptics in [3].

Theorem 3.1. *The area of a domain bounded by a secantoptic $C_{\alpha,\beta,\gamma}$ of an oval C, where $\beta \in [0,\pi)$ and $\gamma \in [0, \pi - \beta)$ are fixed and α is from the interval $(\beta + \gamma, \pi)$, can be described by a function*

$$\begin{aligned} A_{\beta,\gamma}(\alpha) = \frac{1}{2\sin^2\alpha} \int\limits_0^{2\pi} \big(&\Psi^2_{-\beta}(t-\beta) + \Psi^2_{\gamma}(t+\alpha-\beta) \\ &- 2\Psi_{-\beta}(t-\beta)\Psi_\gamma(t+\alpha-\beta)\cos\alpha \\ &- \dot\Psi_{-\beta}(t-\beta)\Psi_\gamma(t+\alpha-\beta)\sin\alpha \\ &+ \dot\Psi_\gamma(t+\alpha-\beta)\Psi_{-\beta}(t-\beta)\sin\alpha\big)dt. \end{aligned} \tag{3.1}$$

Proof. Let us recall that if D is a domain bounded by a smooth curve K, then the following Green formula holds

$$\iint\limits_D dxdy = \frac{1}{2}\int\limits_K xdy - ydx.$$

We may use this formula to describe the area of a domain bounded by a secantoptic $C_{\alpha,\beta,\gamma}$ of an oval C

$$A_{\beta,\gamma}(\alpha) = \frac{1}{2}\int\limits_0^{2\pi} [z_{\alpha,\beta,\gamma}(t), \dot z_{\alpha,\beta,\gamma}(t)]dt. \tag{3.2}$$

Since

$$z_{\alpha,\beta,\gamma}(t) = (p(t) + \lambda(t)\sin\beta + i(\dot p(t) + \lambda(t)\cos\beta))e^{it},$$

then

$$\dot z_{\alpha,\beta,\gamma}(t) = ((\dot\lambda(t)\sin\beta - \lambda(t)\cos\beta + i(R(t) + \dot\lambda(t)\cos\beta + \lambda(t)\sin\beta))e^{it}.$$

Hence

$$\begin{aligned} [z_{\alpha,\beta,\gamma}(t), \dot z_{\alpha,\beta,\gamma}(t)] = {}& p(t)R(t) + \dot\lambda(t)(p(t)\cos\beta - \dot p(t)\sin\beta) \\ &+ \lambda(t)(p(t)\sin\beta + R(t)\sin\beta + \dot p(t)\cos\beta) + \lambda^2(t). \end{aligned}$$

In terms of the support function of the oval C we have

$$\begin{aligned}\sin^2\alpha\,[z_{\alpha,\beta,\gamma}(t),\dot z_{\alpha,\beta,\gamma}(t)] = {}&(p(t)\cos\beta-\dot p(t)\sin\beta)^2\\&+(p(t+\alpha-\beta-\gamma)\cos\gamma+\dot p(t+\alpha-\beta-\gamma)\sin\gamma)^2\\&+\ddot p(t)\sin\alpha\sin\beta(p(t+\alpha-\beta-\gamma)\cos\gamma\\&+\dot p(t+\alpha-\beta-\gamma)\sin\gamma)\\&+\ddot p(t+\alpha-\beta-\gamma)\sin\alpha\sin\gamma(p(t)\cos\beta-\dot p(t)\sin\beta)\\&-2\dot p(t+\alpha-\beta-\gamma)\cos\alpha\sin\gamma(p(t)\cos\beta-\dot p(t)\sin\beta)\\&-\dot p(t)\sin\alpha\cos\beta(p(t+\alpha-\beta-\gamma)\cos\gamma\\&+\dot p(t+\alpha-\beta-\gamma)\sin\gamma)\\&-2p(t+\alpha-\beta-\gamma)\cos\alpha\cos\gamma(p(t)\cos\beta-\dot p(t)\sin\beta)\\&+\dot p(t+\alpha-\beta-\gamma)\sin\alpha\cos\gamma(p(t)\cos\beta-\dot p(t)\sin\beta).\end{aligned}$$

If we use support functions of evolutoids $\Gamma_{-\beta}$ and Γ_γ of the oval C, then we get

$$\begin{aligned}\sin^2\alpha\,[z_{\alpha,\beta,\gamma}(t),\dot z_{\alpha,\beta,\gamma}(t)] = {}&\Psi^2_{-\beta}(t-\beta)+\Psi^2_\gamma(t+\alpha-\beta)\\&-2\Psi_{-\beta}(t-\beta)\Psi_\gamma(t+\alpha-\beta)\cos\alpha\\&-\dot\Psi_{-\beta}(t-\beta)\Psi_\gamma(t+\alpha-\beta)\sin\alpha\\&+\dot\Psi_\gamma(t+\alpha-\beta)\Psi_{-\beta}(t-\beta)\sin\alpha.\end{aligned}\tag{3.3}$$

Hence

$$\begin{aligned}A_{\beta,\gamma}(\alpha) = {}&\frac{1}{2\sin^2\alpha}\int_0^{2\pi}\big(\Psi^2_{-\beta}(t-\beta)+\Psi^2_\gamma(t+\alpha-\beta)\\&-2\Psi_{-\beta}(t-\beta)\Psi_\gamma(t+\alpha-\beta)\cos\alpha\\&-\dot\Psi_{-\beta}(t-\beta)\Psi_\gamma(t+\alpha-\beta)\sin\alpha\\&+\dot\Psi_\gamma(t+\alpha-\beta)\Psi_{-\beta}(t-\beta)\sin\alpha\big)dt.\end{aligned}\tag{3.4}$$

The formula (3.3) is a generalization of the formula

$$\begin{aligned}[z_\alpha(t),\dot z_\alpha(t)] = \frac{1}{\sin^2\alpha}&\big(p^2(t)+p^2(t+\alpha)-2p(t)p(t+\alpha)\cos\alpha\\&-\dot p(t)p(t+\alpha)\sin\alpha+\dot p(t+\alpha)p(t)\sin\alpha\big)\end{aligned}$$

for isoptics, known from [3]. □

Note that

$$\int_0^{2\pi}\dot\Psi_\gamma(t+\alpha-\beta)\Psi_{-\beta}(t-\beta)dt = -\int_0^{2\pi}\dot\Psi_{-\beta}(t-\beta)\Psi_\gamma(t+\alpha-\beta)dt$$

and if $\beta=\gamma$, then

$$\int_0^{2\pi}\Psi_{-\beta}^2(t-\beta)dt=\int_0^{2\pi}\Psi_\gamma^2(t+\alpha-\beta)dt$$
$$=\cos^2\beta\int_0^{2\pi}p^2(t)dt+\sin^2\beta\int_0^{2\pi}\dot{p}^2(t)dt.$$

Hence for $\beta=\gamma$ the expression $A_{\beta,\gamma}(\alpha)$ can be written in the following form

$$\begin{aligned}A_{\beta,\beta}(\alpha)=\frac{1}{\sin^2\alpha}\int_0^{2\pi}\Big(\Psi_{-\beta}^2(t-\beta)-\Psi_\beta(t+\alpha-\beta)\big(\dot{\Psi}_{-\beta}(t-\beta)\sin\alpha\\ +\Psi_{-\beta}(t-\beta)\cos\alpha\big)\Big)dt\end{aligned}\tag{3.5}$$

similar to

$$A(\alpha)\sin^2\alpha=\int_0^{2\pi}\left(p^2(t)-p(t+\alpha)\left(\dot{p}(t)\sin\alpha+p(t)\cos\alpha\right)\right)dt$$

for isoptics, known from [3].

Since we have assumed that the support function of an oval C is of class C^3, then the support functions of its evolutoids $\Psi_{-\beta}(t)$ and $\Psi_\gamma(t)$ are of class C^2 and we may differentiate the function $A_{\beta,\gamma}(\alpha)$.

Theorem 3.2. *The function $A_{\beta,\gamma}(\alpha)$ given by formula (3.4) for $\beta\in[0,\pi)$, $\gamma\in[0,\pi-\beta)$ and $\alpha\in(\beta+\gamma,\pi)$ satisfies the following differential equation*

$$A'_{\beta,\gamma}(\alpha)\sin\alpha+2A_{\beta,\gamma}(\alpha)\cos\alpha=G(\alpha),\tag{3.6}$$

where

$$G(\tau)=\int_0^{2\pi}(\Psi_{-\beta}(t-\beta)\Psi_\gamma(t+\tau-\beta)-\dot{\Psi}_{-\beta}(t-\beta)\dot{\Psi}_\gamma(t+\tau-\beta))dt\tag{3.7}$$

for $\tau\in[\beta+\gamma,\pi]$. Moreover, if $\beta\neq 0$ or $\gamma\neq 0$, then

$$0\le A'_{\beta,\gamma}((\beta+\gamma)^+)\le L_C\max_{t\in[0,2\pi]}R(t)\frac{\sin\beta\sin\gamma}{\sin(\beta+\gamma)}.\tag{3.8}$$

Proof. Let

$$A_{\beta,\gamma}(\alpha)\sin^2\alpha=I(\alpha),\tag{3.9}$$

where

$$I(\alpha) = \frac{1}{2}\int_0^{2\pi} (\Psi^2_{-\beta}(t-\beta) + \Psi^2_\gamma(t+\alpha-\beta)$$
$$- 2\Psi_{-\beta}(t-\beta)\Psi_\gamma(t+\alpha-\beta)\cos\alpha$$
$$- \dot{\Psi}_{-\beta}(t-\beta)\Psi_\gamma(t+\alpha-\beta)\sin\alpha$$
$$+ \dot{\Psi}_\gamma(t+\alpha-\beta)\Psi_{-\beta}(t-\beta)\sin\alpha)dt.$$

Differentiating the formula (3.9), we obtain

$$A'_{\beta,\gamma}(\alpha)\sin^2\alpha + 2A_{\beta,\gamma}(\alpha)\sin\alpha\cos\alpha = I'(\alpha). \tag{3.10}$$

Hence

$$A'_{\beta,\gamma}(\alpha)\sin\alpha + 2A_{\beta,\gamma}(\alpha)\cos\alpha = I'(\alpha)\frac{1}{\sin\alpha} \tag{3.11}$$

and we can assume that

$$G(\alpha) = I'(\alpha)\frac{1}{\sin\alpha}. \tag{3.12}$$

After straightforward calculations we get

$$I'(\alpha) = \sin\alpha\int_0^{2\pi} (\Psi_{-\beta}(t-\beta)\Psi_\gamma(t+\alpha-\beta) - \dot{\Psi}_{-\beta}(t-\beta)\dot{\Psi}_\gamma(t+\alpha-\beta)))dt$$

and

$$G(\alpha) = \int_0^{2\pi} (\Psi_{-\beta}(t-\beta)\Psi_\gamma(t+\alpha-\beta) - \dot{\Psi}_{-\beta}(t-\beta)\dot{\Psi}_\gamma(t+\alpha-\beta)))dt. \tag{3.13}$$

Now we estimate the right-hand side derivative of $A_{\beta,\gamma}$ at $\beta+\gamma$. Using the integral formula for annulus (2.1), we get

$$L_C\left(\frac{\cos\gamma - \cos\beta\cos a}{\sin a} - \sin\beta\right) = \iint_{CC_{a,\beta,\gamma}} \frac{dxdy}{t_1}$$
$$\geq \frac{1}{\max_{t\in[0,2\pi]} L(a,t)} \iint_{CC_{a,\beta,\gamma}} dxdy$$
$$= \frac{A_{\beta,\gamma}(a) - A_{\beta,\gamma}(\beta+\gamma)}{\max_{t\in[0,2\pi]} L(a,t)},$$

which can be rewritten as

$$\begin{aligned} 0 &\leq A_{\beta,\gamma}(a) - A_{\beta,\gamma}(\beta+\gamma) \\ &\leq L_C\left(\frac{\cos\gamma - \cos\beta\cos a}{\sin a} - \sin\beta\right)\max_{t\in[0,2\pi]} L(a,t) \end{aligned} \tag{3.14}$$

and hence we obtain the following estimation of difference quotient for the function $A_{\beta,\gamma}$

$$\begin{aligned} 0 \leq& \frac{A_{\beta,\gamma}(a) - A_{\beta,\gamma}(\beta+\gamma)}{a-(\beta+\gamma)} \\ \leq& \frac{\max_{t\in[0,2\pi]} L(a,t)}{a-(\beta+\gamma)} L_C \left(\frac{\cos\gamma - \cos\beta\cos a}{\sin a} - \sin\beta \right). \end{aligned} \tag{3.15}$$

To get the estimation of the right-hand side derivative of $A_{\beta,\gamma}$ at $\beta+\gamma$ we have to calculate the value of the limit

$$l = \lim_{a\to(\beta+\gamma)^+} \frac{1}{a-(\beta+\gamma)} L_C \left(\frac{\cos\gamma - \cos\beta\cos a}{\sin a} - \sin\beta \right) \max_{t\in[0,2\pi]} L(a,t).$$

We apply l'Hôpital's rule and calculate the limit

$$\begin{aligned} l_H = \lim_{a\to(\beta+\gamma)^+} L_C \Bigg(& \frac{\cos\beta - \cos\gamma\cos a}{\sin^2 a} \max_{t\in[0,2\pi]} L(a,t) \\ & + \left(\frac{\cos\gamma - \cos\beta\cos a}{\sin a} - \sin\beta \right) \max_{t\in[0,2\pi]} \frac{\partial L(a,t)}{\partial a} \Bigg). \end{aligned}$$

Note that

$$\lim_{a\to(\beta+\gamma)^+} \max_{t\in[0,2\pi]} L(a,t) = \lim_{a\to(\beta+\gamma)^+} \max_{t\in[0,2\pi]} (\lambda(a,t) + R(t)\sin\beta). \tag{3.16}$$

Since for fixed $\beta \in [0,\pi)$, $\gamma \in [0,\pi-\beta)$, $a \in (\beta+\gamma,\pi)$ and $t \in [0,2\pi]$ expressions $\lambda(a,t)$ and $R(t)\sin\beta$ are nonnegative and $R(t)\sin\beta$ does not depend on a, then

$$\lim_{a\to(\beta+\gamma)^+} \max_{t\in[0,2\pi]} L(a,t) = \lim_{a\to(\beta+\gamma)^+} \max_{t\in[0,2\pi]} \lambda(a,t) + \max_{t\in[0,2\pi]} R(t)\sin\beta. \tag{3.17}$$

Suppose that the value $\max_{t\in[0,2\pi]} \lambda(a,t)$ is taken for some $t_1 \in [0,2\pi]$ and that $\beta \neq 0$ or $\gamma \neq 0$. Then from (1.2) we have

$$\begin{aligned} \lim_{a\to(\beta+\gamma)^+} \max_{t\in[0,2\pi]} \lambda(a,t_1) = & \lim_{a\to(\beta+\gamma)^+} \Big(\frac{1}{\sin a} (-p(t_1)\cos(a-\beta) \\ & - \dot{p}(t_1)\sin(a-\beta) + p(t_1+a-\beta-\gamma)\cos\gamma \\ & + \dot{p}(t_1+a-\beta-\gamma)\sin\gamma) \Big) \\ = & \frac{1}{\sin(\beta+\gamma)} (-p(t_1)\cos\gamma - \dot{p}(t_1)\sin\gamma \\ & + p(t_1)\cos\gamma + \dot{p}(t_1)\sin\gamma) = 0. \end{aligned}$$

Therefore

$$\lim_{a\to(\beta+\gamma)^+} \max_{t\in[0,2\pi]} L(a,t) = \max_{t\in[0,2\pi]} R(t)\sin\beta \tag{3.18}$$

and

$$\lim_{a\to(\beta+\gamma)^+}\left(\left(\frac{\cos\gamma-\cos\beta\cos a}{\sin a}-\sin\beta\right)\max_{t\in[0,2\pi]}\frac{\partial L(a,t)}{\partial a}\right)=0.$$

Finally we get

$$l_H = L_C \max_{t\in[0,2\pi]} R(t)\frac{\sin\beta\sin\gamma}{\sin(\beta+\gamma)}. \tag{3.19}$$

If $\beta\in(0,\pi)$, $\gamma\in(0,\pi-\beta)$ and $t\in[0,2\pi]$ then the limit l_H has real, nonnegative value. If $\beta=\gamma=0$, i.e. for isoptics of ovals, we have $l_H=0$. Hence from l'Hôpital's rule

$$l=\begin{cases} L_C \max_{t\in[0,2\pi]} R(t)\frac{\sin\beta\sin\gamma}{\sin(\beta+\gamma)}, & \text{if } \beta\neq 0 \text{ or } \gamma\neq 0,\\ 0, & \text{if } \beta=\gamma=0\ . \end{cases}$$

and we get (3.8). □

Let us try to illustrate on some examples the meaning of this estimation.

Example 3.1. Consider a circle C given by a support function $p(t)=re^{it}$. The equation of its secantoptic $C_{\alpha,\beta,\gamma}$ is of the form

$$\begin{aligned} z_{\alpha,\beta,\gamma}(t)=\frac{re^{it}}{\sin\alpha}&\left((\cos\beta\sin(\alpha-\beta)+\sin\beta\cos\gamma)\right.\\ &\left.+i\cos\beta(\cos\gamma-\cos(\alpha-\beta))\right). \end{aligned} \tag{3.20}$$

Using the first derivative of this parametrization

$$\dot z_{\alpha,\beta,\gamma}(t)=\frac{re^{it}}{\sin\alpha}\left(\cos\beta(\cos(\alpha-\beta)-\cos\gamma)+i(\cos\beta\sin(\alpha-\beta)+\sin\beta\cos\gamma)\right),$$

we may consider the expression

$$[z_{\alpha,\beta,\gamma}(t),\dot z_{\alpha,\beta,\gamma}(t)]=\frac{r^2}{\sin^2\alpha}(\cos^2\beta-2\cos\beta\cos\gamma\cos\alpha+\cos^2\gamma).$$

Using the Green formula, we get the function which describes the area of the annulus

$$\begin{aligned} A_{\beta,\gamma}(\alpha)&=\frac12\int_0^{2\pi}\frac{r^2}{\sin^2\alpha}(\cos^2\beta-2\cos\beta\cos\gamma\cos\alpha+\cos^2\gamma)dt\\ &=\pi\frac{r^2}{\sin^2\alpha}(\cos^2\beta-2\cos\beta\cos\gamma\cos\alpha+\cos^2\gamma) \end{aligned}$$

and we calculate its derivative $\dot A_{\beta,\gamma}(\alpha)$

$$\dot A_{\beta,\gamma}(\alpha)=\frac{2\pi r^2}{\sin^3\alpha}\big(\cos\beta\cos\gamma+\cos\beta\cos\gamma\cos^2\alpha-(\cos^2\beta+\cos^2\gamma)\cos\alpha\big).$$

The limit of this derivative at $\alpha = \beta + \gamma$ is equal to the value $\dot{A}_{\beta,\gamma}(\beta + \gamma)$ and equals

$$\dot{A}_{\beta,\gamma}(\beta+\gamma) = \frac{2\pi r^2 \sin\beta \sin\gamma}{\sin(\beta+\gamma)}.$$

Recall that for a circle C its perimeter is $L_C = 2\pi r$ and the curvature radius is $R(t) = r$. Therefore, a circle satisfies with equality the formula from Theorem 3.2.

Example 3.2. Now we are looking for an example of a curve for which

$$0 < \dot{A}_{\beta,\gamma}(\beta+\gamma) < L_C \max_{t\in[0,2\pi]} R(t) \frac{\sin\beta\sin\gamma}{\sin(\beta+\gamma)}. \tag{3.21}$$

We write the formula (3.1) in terms of the support function of a given oval C

$$\begin{aligned} A_{\beta,\gamma}(\alpha) = \frac{1}{2\sin^2\alpha}\int_0^{2\pi} &((\cos^2\beta + \cos^2\gamma)p^2(t) + (\sin^2\beta + \sin^2\gamma)\dot{p}^2(t) \\ &- 2p(t)p(t+\alpha-\beta-\gamma)\cos\alpha\cos\beta\cos\gamma \\ &+ 2p(t)\dot{p}(t+\alpha-\beta-\gamma)(\sin\alpha\cos\beta\cos\gamma - \cos\alpha\sin(\beta+\gamma)) \\ &+ 2\dot{p}(t)\dot{p}(t+\alpha-\beta-\gamma)(\cos\alpha\sin\beta\sin\gamma - \sin\alpha\sin(\beta+\gamma)) \\ &- 2\dot{p}(t)\ddot{p}(t+\alpha-\beta-\gamma)\sin\alpha\sin\beta\sin\gamma)dt. \end{aligned}$$

Let us consider an oval C, whose support function is given by $p(t) = a + b\cos 3t$, where $a > 8b$ and $b > 0$. We have then $\dot{p}(t) = -3b\sin 3t$, $\ddot{p}(t) = -9b\cos 3t$ and $R(t) > 0$ for each $t \in [0, 2\pi]$. For this oval C the function $A_{\beta,\gamma}(\alpha)$ can be written in the form

$$\begin{aligned} A_{\beta,\gamma}(\alpha) = \frac{1}{2\sin^2\alpha}\int_0^{2\pi} &\Big((\cos^2\beta + \cos^2\gamma)(a^2 + 2ab\cos 3t + b^2\cos^2 3t) \\ &+ (\sin^2\beta+\sin^2\gamma)9b^2\sin^2 3t - 2\big(a^2 + ab\cos 3(t+\alpha-\beta-\gamma) \\ &+ ab\cos 3t + b^2\cos 3t\cos 3(t+\alpha-\beta-\gamma)\big)\cos\alpha\cos\beta\cos\gamma \\ &- 6\big(ab\sin 3(t+\alpha-\beta-\gamma) - b^2\cos 3t\sin 3(t+\alpha-\beta-\gamma)\big) \\ &\quad\times(\sin\alpha\cos\beta\cos\gamma - \cos\alpha\sin(\beta+\gamma)) \\ &+ 18b^2\sin 3t\sin 3(t+\alpha-\beta-\gamma)(\cos\alpha\sin\beta\sin\gamma - \sin\alpha\sin(\beta+\gamma)) \\ &- 2\cdot 27b^2\sin 3t\cos 3(t+\alpha-\beta-\gamma)\sin\alpha\sin\beta\sin\gamma\Big)dt. \end{aligned}$$

After some simplifications we get

$$\begin{aligned}A_{\beta,\gamma}(\alpha) = \frac{\pi}{2\sin^2\alpha}\Big(&2a^2(\cos^2\beta+\cos^2\gamma-2\cos\alpha\cos\beta\cos\gamma)\\
&+b^2\big(\cos^2\beta+\cos^2\gamma+9(\sin^2\beta+\sin^2\gamma)\\
&-2\cos\alpha\cos\beta\cos\gamma\cos 3(\alpha-\beta-\gamma)\\
&-6\sin 3(\alpha-\beta-\gamma)(\sin\alpha\cos\beta\cos\gamma-\cos\alpha\sin(\beta+\gamma))\\
&+18\cos 3(\alpha-\beta-\gamma)(\cos\alpha\sin\beta\sin\gamma-\sin\alpha\sin(\beta+\gamma))\\
&+54\sin 3(\alpha-\beta-\gamma)\sin\alpha\sin\beta\sin\gamma\big)\Big).\end{aligned}$$

The area bounded by C is equal to

$$\begin{aligned}A_{\beta,\gamma}(\beta+\gamma) = \frac{\pi}{2\sin^2(\beta+\gamma)}\Big(&2a^2(\cos^2\beta+\cos^2\gamma-2\cos^2\beta\cos^2\gamma\\
&+2\sin\beta\sin\gamma\cos\beta\cos\gamma)+b^2\big(\cos^2\beta+\cos^2\gamma\\
&+9(\sin^2\beta+\sin^2\gamma)-2\cos(\beta+\gamma)\cos\beta\cos\gamma\\
&+18(\cos(\beta+\gamma)\sin\beta\sin\gamma-\sin^2(\beta+\gamma))\big)\Big) = \pi(a^2-4b^2)\end{aligned}$$

and the derivative of the function $A_{\beta,\gamma}$ after some calculations can be written as

$$\begin{aligned}\dot{A}_{\beta,\gamma}(\alpha) = \frac{2\pi}{\sin^3\alpha}\Big(&a^2\big(\cos\beta\cos\gamma-\cos\alpha(\cos^2\beta+\cos^2\gamma)+\cos^2\alpha\cos\beta\cos\gamma\big)\\
&+b^2\big(-4\sin^2\alpha\cos 3(\alpha-\beta-\gamma)(\cos\beta\cos\gamma-9\sin\beta\sin\gamma)\\
&+12\sin^2\alpha\sin 3(\alpha-\beta-\gamma)\sin(\beta+\gamma)-\cos\alpha-4\cos\alpha(\sin^2\beta+\sin^2\gamma)\\
&+\cos^2\alpha\cos 3(\alpha-\beta-\gamma)(\cos\beta\cos\gamma-9\sin\beta\sin\gamma)\\
&+3\sin\alpha\cos\alpha\sin 3(\alpha-\beta-\gamma)(\cos\beta\cos\gamma-9\sin\beta\sin\gamma)\\
&-3\cos^2\alpha\sin 3(\alpha-\beta-\gamma)\sin(\beta+\gamma)\\
&+9\sin\alpha\cos\alpha\cos 3(\alpha-\beta-\gamma)\sin(\beta+\gamma)\big)\Big).\end{aligned}$$

At $\alpha=\beta+\gamma$ we get

$$\dot{A}_{\beta,\gamma}(\beta+\gamma) = \frac{2\pi(a^2+32b^2)\sin\beta\sin\gamma}{\sin(\beta+\gamma)}. \tag{3.22}$$

From the inequality (3.8) for a curve given by a support function $p(t) = a + b\cos 3t$ we have

$$L_C = \int_0^{2\pi} p(t)dt = \int_0^{2\pi}(a+b\cos 3t)dt = at\Big|_0^{2\pi} + \frac{b}{3}\sin 3t\Big|_0^{2\pi} = 2\pi a, \tag{3.23}$$

$$R(t) = a + b\cos 3t - 9b\cos 3t = a - 8b\cos 3t,$$

$$\max_{t\in[0,2\pi]} R(t) = a + 8b.$$

Hence the value of the derivative (3.22) and its estimation (3.8) lead us to the inequality

$$0 < \frac{2\pi(a^2 + 32b^2)\sin\beta\sin\gamma}{\sin(\beta+\gamma)} \leq 2\pi(a^2 + 8ab)\frac{\sin\beta\sin\gamma}{\sin(\beta+\gamma)} \tag{3.24}$$

which can be reduced to the form $4b \leq a$. Hence the inequality (3.24) is always satisfied if $a > 8b$ for the curve, but as we can see the value of $\dot{A}_{\beta,\gamma}(\beta+\gamma)$ is less and is not equal to $L_C \max_{t\in[0,2\pi]} R(t)\frac{\sin\beta\sin\gamma}{\sin(\beta+\gamma)}$ in the inequality (3.8).

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