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On a result by Clunie and Sheil-Small

Dedicated to Professor Bogdan Bojarski on the occasion of his 80th birthday

Abstract. In 1984 J. Clunie and T. Sheil-Small proved ([2, Corollary 5.8]) that for any complex-valued and sense-preserving injective harmonic mapping F in the unit disk $\mathbb{D}$, if $F(\mathbb{D})$ is a convex domain, then the inequality $|G(z_2) - G(z_1)| < |H(z_2) - H(z_1)|$ holds for all distinct points $z_1, z_2 \in \mathbb{D}$. Here H and G are holomorphic mappings in $\mathbb{D}$ determined by $F = H + \overline{G}$, up to a constant function. We extend this inequality by replacing the unit disk by an arbitrary nonempty domain Ω in $\mathbb{C}$ and improve it provided F is additionally a quasiconformal mapping in Ω.

Introduction. Let Ω be a nonempty domain in $\mathbb{C}$. Throughout the paper we always assume that $F : \Omega \to \mathbb{C}$ is a sense-preserving injective harmonic mapping in Ω of the following form

$$F(z) = H(z) + \overline{G(z)}\,, \quad z \in \Omega\,, \tag{0.1}$$

where H and G are holomorphic mappings in Ω. Note that if Ω is a simply connected domain, then each harmonic mapping F in Ω has a decomposition (0.1) up to a constant function; cf. e.g. [4]. From the classical Lewy's theorem it follows that the Jacobian $\mathrm{J}[F]$ does not vanish on Ω; cf. [3]. Since

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F is sense-preserving,

$$|H'(z)|^2 - |G'(z)|^2 = |\partial F(z)|^2 - |\bar{\partial} F(z)|^2 = \mathrm{J}[F](z) > 0\,, \quad z \in \Omega\,, \tag{0.2}$$

where $\partial \coloneqq \frac{1}{2}(\partial_x - \mathrm{i}\partial_y)$ and $\bar{\partial} \coloneqq \frac{1}{2}(\partial_x + \mathrm{i}\partial_y)$ are the so-called formal derivatives operators. Therefore the complex dilatation

$$\mu_F(z) \coloneqq \frac{\bar{\partial} F(z)}{\partial F(z)} = \frac{\overline{G'(z)}}{H'(z)}\,, \quad z \in \Omega\,, \tag{0.3}$$

is well defined and for every nonempty set $E \subset \Omega$, put

$$\|\mu_F\|_E \coloneqq \sup_{z \in E} |\mu_F(z)| \leq 1\,. \tag{0.4}$$

Since G'/H' is a holomorphic mapping, we conclude from the maximum principle that for every nonempty compact set $E \subset \Omega$,

$$\|\mu_F\|_E < 1\,. \tag{0.5}$$

Let $\mathbb{D}(a, r)$ stand for the Euclidean disk with the center at $a \in \mathbb{C}$ and the radius $r > 0$, i.e. $\mathbb{D}(a, r) \coloneqq \{z \in \mathbb{C} : |z - a| < r\}$. In particular $\mathbb{D} \coloneqq \mathbb{D}(0, 1)$ is the unit disk.

The classical result by J. Clunie and T. Sheil-Small [2, Corollary 5.8] reads as follows.

Theorem A. *If* $\Omega = \mathbb{D}$ *and* $F(\mathbb{D})$ *is a convex domain, then*

$$|G(z_2) - G(z_1)| < |H(z_2) - H(z_1)|\,, \quad z_1, z_2 \in \mathbb{D}\,,\, z_1 \neq z_2\,. \tag{0.6}$$

A little bit stronger version of Theorem A was proved in [1, Lemma 2.1]. Under the additional assumption that F is a quasiconformal mapping, the conclusion (0.6) can be improved; cf. Theorem 2.1 and Corollary 2.3 in Section 2 which are main results of this paper. To this end we show in Section 1 several auxiliary properties involving the functions H and G with the function F. In Section 3 we present several applications of the results from the previous sections. They deal with the quasiconformality of the function F and Lipschitz type relationships between the functions F and H.

All results in this paper are strictly related to the ones presented by the second named author during the XVI-th Conference on Analytic Functions and Related Topics, June 26–29, 2011 Chełm (Poland).

A more general case where the convexity of $F(\mathbb{D})$ is replaced by the so-called *α-convexity of* $F(\mathbb{D})$ is studied in [5].

1. Auxiliary properties of harmonic mappings. In this section we study the holomorphic mappings H and G associated with the mapping F by the equality (0.1).

Lemma 1.1. *Suppose that $z_1, z_2 \in \Omega$ are points such that $z_1 \neq z_2$ and*

$$\{(1-t)F(z_1) + tF(z_2) : 0 \leq t \leq 1\} \subset F(E) \tag{1.1}$$

for a certain compact set $E \subset \Omega$, i.e. the line segment with endpoints $F(z_1)$ and $F(z_2)$ is included in $F(E)$. Then

$$-\frac{k}{1-k} \leq \operatorname{Re} \frac{\overline{G}(z_2) - \overline{G}(z_1)}{F(z_2) - F(z_1)} \leq \frac{k}{1+k} \tag{1.2}$$

as well as

$$\frac{1}{1+k} \leq \operatorname{Re} \frac{H(z_2) - H(z_1)}{F(z_2) - F(z_1)} \leq \frac{1}{1-k}\,, \tag{1.3}$$

where $k \coloneqq \|\mu_F\|_E$. Moreover, the following inequalities hold

$$\Big|\operatorname{Im} \frac{H(z_2) - H(z_1)}{F(z_2) - F(z_1)}\Big| = \Big|\operatorname{Im} \frac{\overline{G}(z_2) - \overline{G}(z_1)}{F(z_2) - F(z_1)}\Big| \leq \frac{k}{1-k^2} \tag{1.4}$$

$$\Big|\frac{\overline{G}(z_2) - \overline{G}(z_1)}{F(z_2) - F(z_1)}\Big| \leq \frac{k}{1-k} \tag{1.5}$$

$$\Big|\frac{H(z_2) - H(z_1)}{F(z_2) - F(z_1)}\Big| \leq \frac{1}{1-k}\,. \tag{1.6}$$

Proof. Take arbitrary distinct points $z_1, z_2 \in \Omega$ satisfying (1.1). Then the function

$$[0;1] \ni t \mapsto \gamma(t) \coloneqq (1-t)F(z_1) + tF(z_2) \tag{1.7}$$

has the following properties:

$$\gamma(0) = F(z_1)\,, \quad \gamma(1) = F(z_2) \quad \text{and} \quad \gamma([0;1]) \subset F(E)\,. \tag{1.8}$$

Hence $\sigma \coloneqq F^{-1} \circ \gamma$ is an arc in E joining z_1 with z_2. Then by (0.5) we see that

$$|\mu_F(\sigma(s))| \leq \|\mu_F\|_E = k < 1\,, \quad 0 \leq s \leq 1\,. \tag{1.9}$$

Write $w \coloneqq F(z_2) - F(z_1)$. Using the following formulas

$$\begin{aligned} \partial F^{-1}(F(z)) &= \frac{\overline{\bar\partial F}}{\mathrm{J}[F](z)} = \frac{\overline{H'(z)}}{|H'(z)|^2 - |G'(z)|^2} \\ \bar\partial F^{-1}(F(z)) &= -\frac{\bar\partial F}{\mathrm{J}[F](z)} = -\frac{\overline{G'(z)}}{|H'(z)|^2 - |G'(z)|^2} \end{aligned} \tag{1.10}$$

we obtain

$$\begin{aligned}G(z_2)-G(z_1) &= \int_\sigma G'(z)\mathrm{d}z = \int_0^1 G'(\sigma(s))\frac{\mathrm{d}}{\mathrm{d}s}\sigma(s)\mathrm{d}s\\ &= \int_0^1 G'(\sigma(s))[\partial F^{-1}(\gamma(s))\gamma'(s)+\bar\partial F^{-1}(\gamma(s))\overline{\gamma'(s)}]\mathrm{d}s\\ &= \int_0^1 G'(\sigma(s))[\partial F^{-1}(F(\sigma(s)))w+\bar\partial F^{-1}(F(\sigma(s)))\overline{w}]\mathrm{d}s\\ &= \int_0^1 G'(\sigma(s))\Big[\frac{\overline{H'(\sigma(s))}}{\mathrm{J}[F](\sigma(s))}w-\frac{\overline{G'(\sigma(s))}}{\mathrm{J}[F](\sigma(s))}\overline{w}\Big]\mathrm{d}s\\ &= \int_0^1 \frac{G'(\sigma(s))\overline{H'(\sigma(s))}w-|G'(\sigma(s))|^2\overline{w}}{|H'(\sigma(s))|^2-|G'(\sigma(s))|^2}\mathrm{d}s\\ &= \int_0^1 \frac{\mu_F(\sigma(s))\frac{G'(\sigma(s))}{\overline{G'(\sigma(s))}}w-|\mu_F(\sigma(s))|^2\overline{w}}{1-|\mu_F(\sigma(s))|^2}\mathrm{d}s\,.\end{aligned}$$

Hence

$$\overline{(G(z_2)-G(z_1))} = \int_0^1 \frac{\overline{\mu_F(\sigma(s))}\frac{\overline{G'(\sigma(s))}}{G'(\sigma(s))}\overline{w}-|\mu_F(\sigma(s))|^2 w}{1-|\mu_F(\sigma(s))|^2}\mathrm{d}s\,. \tag{1.11}$$

Combining (1.11) with (1.9), we get

$$\begin{aligned}\mathrm{Re}\Big[\frac{1}{w}\overline{(G(z_2)-G(z_1))}\Big] &\le \int_0^1 \frac{\big|\overline{\mu_F(\sigma(s))}\frac{\overline{G'(\sigma(s))}}{G'(\sigma(s))}\frac{\overline{w}}{w}\big|-|\mu_F(\sigma(s))|^2}{1-|\mu_F(\sigma(s))|^2}\mathrm{d}s\\ &= \int_0^1 \frac{(|\mu_F(\sigma(s))|-|\mu_F(\sigma(s))|^2)}{1-|\mu_F(\sigma(s))|^2}\mathrm{d}s\\ &= \int_0^1 \frac{|\mu_F(\sigma(s))|}{1+|\mu_F(\sigma(s))|}\mathrm{d}s\\ &\le \frac{\|\mu_F\|_E}{1+\|\mu_F\|_E} = \frac{k}{1+k}\,,\end{aligned}$$

which yields the second inequality in (1.2). On the other hand, we conclude from (1.11) and (1.9) that

$$\begin{aligned}\mathrm{Re}\Big[\frac{1}{w}\overline{(G(z_2)-G(z_1))}\Big] &\ge \int_0^1 \frac{-|\overline{\mu_F(\sigma(s))}|-|\mu_F(\sigma(s))|^2}{1-|\mu_F(\sigma(s))|^2}\mathrm{d}s\\ &= -\int_0^1 \frac{|\overline{\mu_F(\sigma(s))}|}{1-|\mu_F(\sigma(s))|}\mathrm{d}s \ge -\frac{\|\mu_F\|_E}{1-\|\mu_F\|_E}\\ &= -\frac{k}{1-k}\,,\end{aligned}$$

which yields the first inequality in (1.2).

From (0.1) it follows that $F(z_2)-F(z_1) = H(z_2)-H(z_1)+\overline{G}(z_2)-\overline{G}(z_1)$, and hence

$$\operatorname{Re}\frac{H(z_2)-H(z_1)}{F(z_2)-F(z_1)} = 1-\operatorname{Re}\frac{\overline{G}(z_2)-\overline{G}(z_1)}{F(z_2)-F(z_1)}\ ;$$
$$\operatorname{Im}\frac{H(z_2)-H(z_1)}{F(z_2)-F(z_1)} = -\operatorname{Im}\frac{\overline{G}(z_2)-\overline{G}(z_1)}{F(z_2)-F(z_1)}\,.$$

This together with (1.2) yields the inequalities (1.3) and the equality in (1.4). From (1.11) we conclude that

$$\begin{aligned}\operatorname{Im}\Big[\frac{1}{w}\overline{(G(z_2)-G(z_1))}\Big] &= \int_0^1 \frac{\operatorname{Im}\big[\overline{\mu_F(\sigma(s))\frac{G'(\sigma(s))}{G'(\sigma(s))}}\frac{\overline{w}}{w}\big]-\operatorname{Im}|\mu_F(\sigma(s))|^2}{1-|\mu_F(\sigma(s))|^2}\mathrm{d}s\\ &\le \int_0^1\frac{|\mu_F(\sigma(s))|}{1-|\mu_F(\sigma(s))|^2}\mathrm{d}s \le \frac{k}{1-k^2}\,,\end{aligned}$$

which yields the inequality in (1.4). Applying (1.11) once again, we see that

$$\Big|\frac{1}{w}\overline{(G(z_2)-G(z_1))}\Big| \le \int_0^1 \frac{|\mu_F(\sigma(s))|+|\mu_F(\sigma(s))|^2}{1-|\mu_F(\sigma(s))|^2}\mathrm{d}s \le \frac{k}{1-k}\,,$$

which leads to (1.5). Using the formulas (1.10), we have

$$\begin{aligned}H(z_2)-H(z_1) &= \int_\sigma H'(z)\mathrm{d}z = \int_0^1 H'(\sigma(s))\frac{\mathrm{d}}{\mathrm{d}s}\sigma(s)\mathrm{d}s\\ &= \int_0^1 H'(\sigma(s))[\partial F^{-1}(\gamma(s))\gamma'(s)+\bar\partial F^{-1}(\gamma(s))\overline{\gamma'(s)}]\mathrm{d}s\\ &= \int_0^1 H'(\sigma(s))[\partial F^{-1}(F(\sigma(s)))\gamma'(s)+\bar\partial F^{-1}(F(\sigma(s)))\overline{\gamma'(s)}]\mathrm{d}s\\ &= \int_0^1 H'(\sigma(s))\Big[\frac{\overline{H'(\sigma(s))}}{\mathrm{J}[F](\sigma(s))}w-\frac{\overline{G'(\sigma(s))}}{\mathrm{J}[F](\sigma(s))}\overline{w}\Big]\mathrm{d}s\\ &= \int_0^1\frac{|H'(\sigma(s))|^2w-H'(\sigma(s))\overline{G'(\sigma(s))}\overline{w}}{|H'((\sigma(s)))|^2-|G'((\sigma(s)))|^2}\mathrm{d}s\\ &= \int_0^1\frac{w-\overline{\mu_F(\sigma(s))\frac{G'(\sigma(s))}{G'(\sigma(s))}}\overline{w}}{1-|\mu_F(\sigma(s))|^2}\mathrm{d}s\,.\end{aligned}$$

Hence and by (1.9) we see that

$$\begin{aligned}|H(z_2)-H(z_1)| &\le \int_0^1\frac{|w|+|\mu_F(\sigma(s))||\overline{w}|}{1-|\mu_F(\sigma(s))|^2}\mathrm{d}s = \int_0^1\frac{|w|}{1-|\mu_F(\sigma(s))|}\mathrm{d}s\\ &\le \frac{|w|}{1-\|\mu_F\|_E} = \frac{|F(z_2)-F(z_1)|}{1-k}\,,\end{aligned}$$

which leads to (1.6), and the proof is complete. □

Corollary 1.2. *If $F(\Omega)$ is a convex domain and F is a quasiconformal mapping, then for all distinct points $z_1, z_2 \in \Omega$ the inequalities* (1.2)–(1.6) *hold with $k := \|\mu_F\|_\Omega$.*

Proof. Let $z_1, z_2 \in \Omega$ be arbitrarily chosen points such that $z_1 \neq z_2$. Since the set $F(\Omega)$ is convex and the inverse mapping F^{-1} is continuous, we deduce that

$$E := F^{-1}\big(\{(1-t)F(z_1) + tF(z_2) : 0 \le t \le 1\}\big)$$

is a compact subset of Ω. By (0.4), $\|\mu_F\|_E \le \|\mu_F\|_\Omega = k < 1$. Since

$$t \mapsto \frac{t}{1+t}\,, \quad t \mapsto \frac{1}{1-t}\,, \quad t \mapsto \frac{t}{1-t}\,, \quad t \mapsto \frac{t}{1-t^2} \quad \text{and} \quad t \mapsto \frac{-1}{1+t}$$

are increasing functions of $t \in [0;1)$, Lemma 1.1 shows that all the inequalities (1.2)–(1.6) hold with $k := \|\mu_F\|_\Omega$. □

2. Variants of J. Clunie and T. Sheil-Small inequality. As an application of Lemma 1.1 we can derive the following improvement of Theorem A by J. Clunie and T. Sheil-Small.

Theorem 2.1. *If $V \subset F(\Omega)$ is a nonempty convex set, then*

$$|G(z_2) - G(z_1)| \le S(\|\mu_F\|_U)|H(z_2) - H(z_1)|\,, \quad z_1, z_2 \in U\,, \tag{2.1}$$

where $U := F^{-1}(V)$ and $S : [0;1] \to \mathbb{R}$ is the function defined by the formula

$$S(k) := k \cdot \sqrt{\frac{(1-k)^2 + 1}{(1-k)^2 + k^2}}\,, \quad 0 \le k \le 1\,. \tag{2.2}$$

Proof. Fix $z_1, z_2 \in U$. If $z_1 = z_2$, then the inequality in (2.1) is obvious. Therefore we may assume that $z_1 \neq z_2$. Let γ be the function defined by (1.7). By the convexity of V,

$$\gamma([0;1]) \subset V \subset F(\Omega)\,.$$

Since the inverse mapping F^{-1} is continuous and $\gamma([0;1])$ is a compact set, the set $E := F^{-1}(\gamma([0;1]))$ is compact subset of Ω. Furthermore,

$$\gamma([0;1]) = F(E) \quad \text{and} \quad E \subset U\,. \tag{2.3}$$

By (0.5), $k := \|\mu_F\|_E < 1$. Setting

$$a := \mathrm{Re}\,\frac{\overline{G}(z_2) - \overline{G}(z_1)}{F(z_2) - F(z_1)}\,, b := \mathrm{Re}\,\frac{H(z_2) - H(z_1)}{F(z_2) - F(z_1)} \text{ and } c := \mathrm{Im}\,\frac{H(z_2) - H(z_1)}{F(z_2) - F(z_1)}$$

we deduce from (0.1) that $a + b = 1$,

$$\mathrm{Im}\,\frac{\overline{G}(z_2) - \overline{G}(z_1)}{F(z_2) - F(z_1)} = -c \quad \text{and} \quad \left|\frac{\overline{G}(z_2) - \overline{G}(z_1)}{F(z_2) - F(z_1)}\right|^2 = (1-b)^2 + c^2\,. \tag{2.4}$$

From Lemma 1.1 it follows that

$$\frac{1}{2} < \frac{1}{1+k} \le b \le \frac{1}{1-k} \quad \text{and} \quad c \le \frac{k}{1-k^2}\,. \tag{2.5}$$

Hence

$$0 \le \Big(b - \frac{1}{1+k}\Big)\Big(\frac{1}{1-k} - b\Big) = -b^2 + \frac{2b}{1-k^2} - \frac{1}{1-k^2}\,,$$

and consequently

$$b^2 \le \frac{2b}{1-k^2} - \frac{1}{1-k^2} = \frac{2b-1}{1-k^2}\,.$$

Combining this with the last inequality in (2.5), we get

$$b^2 + c^2 \le \frac{2b-1}{1-k^2} + \frac{k^2}{(1-k^2)^2} = \frac{(2b-1)(1-k^2)+k^2}{(1-k^2)^2}\,.$$

Hence and by the first inequality in (2.5) we have

$$\frac{(1-b)^2+c^2}{b^2+c^2} = 1 - \frac{2b-1}{b^2+c^2} \le 1 - \frac{(2b-1)(1-k^2)^2}{(2b-1)(1-k^2)+k^2}\,. \tag{2.6}$$

From the first inequality in (2.5) it follows that

$$(2b-1)(1-k^2) \ge \Big(\frac{2}{1+k} - 1\Big)(1-k)(1+k) = (1-k)^2\,.$$

Combining this with (2.6) and (2.2), we see that

$$\frac{(1-b)^2+c^2}{b^2+c^2} \le 1 - \frac{(1-k)^2(1-k^2)}{(1-k)^2+k^2} = \frac{(1-k)^2k^2+k^2}{(1-k)^2+k^2} = S(k)^2\,.$$

Applying now (2.4), we obtain

$$\begin{aligned}\left|\frac{G(z_2)-G(z_1)}{F(z_2)-F(z_1)}\right|^2 &= \left|\frac{\overline{G}(z_2)-\overline{G}(z_1)}{F(z_2)-F(z_1)}\right|^2 = \frac{(1-b)^2+c^2}{b^2+c^2}(b^2+c^2) \\ &\le S(k)^2(b^2+c^2) = S(k)^2\left|\frac{H(z_2)-H(z_1)}{F(z_2)-F(z_1)}\right|^2\,.\end{aligned} \tag{2.7}$$

By (0.4) and the second inclusion in (2.3), $k = \|\mu_F\|_E \le \|\mu_F\|_U$. Since S is an increasing function, we see that $S(k) \le S(\|\mu_F\|_U)$. This together with (2.7) yields (2.1), which is the desired conclusion. □

Remark 2.2. From the formula (2.2) it follows easily that S is a strictly increasing continuous function in $[0;1]$ and

$$0 = S(0) < k < S(k) < S(1) = 1\,, \quad 0 < k < 1\,. \tag{2.8}$$

Corollary 2.3. *If $F(\Omega)$ is a convex domain and G is not a constant function, then $k := \|\mu_F\|_\Omega > 0$ and*

$$|G(z_2) - G(z_1)| < S(k)|H(z_2) - H(z_1)|\,, \quad z_1, z_2 \in \Omega\,,\ z_1 \ne z_2\,. \tag{2.9}$$

Proof. Take arbitrary points $z_1, z_2 \in \Omega$ such that $z_1 \ne z_2$. As in the proof of Theorem 1.1 we see that the function γ defined by (1.7) satisfies the equality in (2.3) with the compact set $E := F^{-1}(\gamma([0;1]))$. Moreover, $\|\mu_F\|_E \le \|\mu_F\|_\Omega = k$. Then by Lemma 1.1,

$$\operatorname{Re}\frac{H(z_2)-H(z_1)}{F(z_2)-F(z_1)} \ge \frac{1}{1+k} > 0\,.$$

Hence $H(z_2) \ne H(z_1)$, and therefore H is an injective mapping. Then the function

$$\Omega \ni z \mapsto \omega(z) := \begin{cases} \frac{G(z)-G(z_1)}{H(z)-H(z_1)} & \text{as } z \in \Omega \setminus \{z_1\}\,, \\ \frac{G'(z_1)}{H'(z_1)} & \text{as } z = z_1\,, \end{cases} \tag{2.10}$$

is well defined. Since H is an injective mapping, we see that ω is a holomorphic function in Ω. By Theorem 2.1,

$$|\omega(z)| \le S(k)\,, \quad z \in \Omega \setminus \{z_1\}\,. \tag{2.11}$$

Suppose that $|\omega(z_2)| = S(k)$. Then by the maximum principle for holomorphic functions we deduce that ω is a constant function, and so $\omega(z) = \lambda$ for a certain $\lambda \in \mathbb{C}$ satisfying $|\lambda| = S(k)$. Taking into account (2.10), we conclude that

$$\frac{G(z)-G(z_1)}{H(z)-H(z_1)} = \lambda\,, \quad z \in \Omega \setminus \{z_1\}\,.$$

Hence $F(z) = H(z) + \overline{\lambda H(z)} + \overline{G(z_1) - \lambda H(z_1)}$ for $z \in \Omega$, and consequently

$$k = \|\mu_F\|_\Omega = |\lambda| = S(k)\,.$$

Combining this with (2.8) we see that $k = 0$ or $k = 1$. If $k = 1$, then for every $z \in \Omega$,

$$\mathrm{J}[F](z) = |\partial F(z)|^2 - |\bar\partial F(z)|^2 = |H'(z)|^2 - |\overline{\lambda}\,\overline{H'(z)}|^2 = (1-|\lambda|^2)|H'(z)|^2 = 0\,,$$

which is impossible. Therefore $k = 0$, and then (2.11) yields $\omega(z) = 0$ as $z \in \Omega$. Hence $G(z) - G(z_1) = 0$ as $z \in \Omega$. This means that G is a constant function, which contradicts the assumption. Thus $|\omega(z_2)| \ne S(k)$, which together with (2.11) leads to $|\omega(z_2)| < S(k)$. Then (2.9) follows from (2.10), which completes the proof. □

Remark 2.4. Note that the inequality in (2.9) of Corollary 2.3 reduces to J. Clunie and T. Sheil-Small's inequality in (0.6) provided $\Omega := \mathbb{D}$ and $k := \|\mu_F\|_\mathbb{D} = 1$. If $k < 1$, then the inequality in (2.9) is stronger than the one in (0.6). Therefore Corollary 2.3 essentially improves Theorem A.

Corollary 2.5. *If $F(\Omega)$ is a convex domain, then F is a quasiconformal mapping if and only if there exists a constant L such that $0 \le L < 1$ and*

$$|G(z_2) - G(z_1)| \le L|H(z_2) - H(z_1)|\,, \quad z_1, z_2 \in \Omega\,. \tag{2.12}$$

Moreover, if the condition (2.12) holds for some $L \in [0;1)$, then F is a quasiconformal mapping with $\|\mu_F\|_\Omega \le L$.

Proof. If F is a quasiconformal mapping, then $k := \|\mu_F\|_\Omega < 1$, and Theorem 2.1 implies the condition (2.12) with $L := S(k) < 1$. Conversely, fix $z \in \Omega$. From the condition (2.12) it follows that

$$\Big|\frac{G(\zeta) - G(z)}{\zeta - z}\Big| \le L\Big|\frac{H(\zeta) - H(z)}{\zeta - z}\Big|, \quad \zeta \in \Omega \setminus \{z\}.$$

A passage to the limit with ζ tending to z implies that $|G'(z)| \le L|H'(z)|$. Since z is an arbitrary point in Ω, we see that

$$|\mu_F(z)| = |G'(z)/H'(z)| = |G'(z)|/|H'(z)| \le L, \quad z \in \Omega.$$

Hence $\|\mu_F\|_\Omega \le L < 1$, and consequently F is a quasiconformal mapping, which completes the proof. □

3. Examples of applications. In what follows we derive several applications of the results from the previous sections, dealing with the quasiconformality of the function F and Lipschitz type relationships between the functions F and H.

Theorem 3.1. *If $F(\Omega)$ is a convex domain, then*

$$|F(z_2) - F(z_1)| \le 2|H(z_2) - H(z_1)|, \quad z_1, z_2 \in \Omega. \tag{3.1}$$

If additionally $k := \|\mu_F\|_\Omega < 1$, then

$$\frac{|F(z_2) - F(z_1)|}{1+k} \le |H(z_2) - H(z_1)| \le \frac{|F(z_2) - F(z_1)|}{1-k}, \quad z_1, z_2 \in \Omega. \tag{3.2}$$

Proof. Suppose that $F(\Omega)$ is a convex domain and fix $z_1, z_2 \in \Omega$. If $z_1 = z_2$, then the inequalities in (3.1) and (3.2) are obvious. Therefore we may assume that $z_1 \ne z_2$. As in the proof of Theorem 2.1 we see that the function γ defined by (1.7) satisfies the equality in (2.3) with the compact set $E := F^{-1}(\gamma([0;1]))$. Moreover, $\|\mu_F\|_E \le \|\mu_F\|_\Omega = k$. Then from the first inequality in (1.3) it follows that

$$\frac{1}{1+k} \le \frac{1}{1+\|\mu_F\|_E} \le \operatorname{Re}\frac{H(z_2) - H(z_1)}{F(z_2) - F(z_1)} \le \frac{|H(z_2) - H(z_1)|}{|F(z_2) - F(z_1)|}.$$

This yields the inequality in (3.1) for any $k \le 1$ and the first inequality in (3.2) as $k < 1$. Assume now that $k < 1$. Then the inequality (1.6) implies that

$$\frac{|H(z_2) - H(z_1)|}{|F(z_2) - F(z_1)|} \le \frac{1}{1-\|\mu_F\|_E} \le \frac{1}{1-k},$$

which yields the second inequality in (3.2). □

Let us recall that for all $L_1, L_2 > 0$, a mapping $f : \Omega \to \mathbb{C}$ is:

(i) L_2-*Lipschitz* if

$$|f(z_2) - f(z_1)| \le L_2|z_2 - z_1|, \quad z_1, z_2 \in \Omega; \tag{3.3}$$

(ii) L_1-*coLipschitz* if

$$\frac{1}{L_1}|z_2 - z_1| \le |f(z_2) - f(z_1)|\,, \quad z_1, z_2 \in \Omega\,; \tag{3.4}$$

(iii) L_2, L_1-*biLipschitz* if f is simultaneously a L_2-Lipschitz and L_1-coLipschitz mapping.

A mapping $f : \Omega \to \mathbb{C}$ is said to be: *Lipschitz*, *coLipschitz* and *biLipschitz* provided f is respectively: L_2-Lipschitz for a certain $L_2 > 0$, L_1-coLipschitz for a certain $L_1 > 0$ and L_2, L_1-biLipschitz for some $L_1, L_2 > 0$.

From Corollary 2.5 we can see that F is a biLipschitz mapping if and only if H is a biLipschitz mapping provided F is a quasiconformal mapping and $F(\Omega)$ is a convex domain. However, from Theorem 3.1 we can derive the following more precise result.

Corollary 3.2. *Suppose that $k := \|\mu_F\|_\Omega < 1$ and $F(\Omega)$ is a convex domain. Then for every $L > 0$:*

(i) *If F is L-Lipschitz, then H is $L/(1-k)$-Lipschitz;*
(ii) *If F is L-coLipschitz, then H is $L(1+k)$-coLipschitz;*
(iii) *If H is L-Lipschitz, then F is $L(1+k)$-Lipschitz;*
(iv) *If H is L-coLipschitz, then F is $L/(1-k)$-coLipschitz.*

In particular, F is a biLipschitz mapping if and only if H is a biLipschitz mapping.

Proof. The implications (i)–(iv) follow directly from the conditions (3.3), (3.4) and (3.2). The last statement is a direct conclusion from these implications. □

Theorem 3.3. *Suppose that $F(\Omega)$ is a convex domain. Then the following four conditions are equivalent to each other:*

(i) *F is a quasiconformal mapping;*
(ii) *there exists a constant L_1 such that $1 \le L_1 < 2$ and*

$$|F(z_2) - F(z_1)| \le L_1 |H(z_2) - H(z_1)|\,, \quad z_1, z_2 \in \Omega\,; \tag{3.5}$$

(iii) *there exists a constant $L_2 \ge 1$ such that*

$$|H(z_2) - H(z_1)| \le L_2 |F(z_2) - F(z_1)|\,, \quad z_1, z_2 \in \Omega\,; \tag{3.6}$$

(iv) *$H \circ F^{-1}$ is a biLipschitz mapping;*
(v) *$F \circ H^{-1}$ is a biLipschitz mapping.*

Moreover, the following implications hold: (3.5) $\implies$ $\|\mu_F\|_\Omega \le L_1 - 1$ *and* (3.6) $\implies$ $\|\mu_F\|_\Omega \le 1 - (1/L_2)$.

Proof. Suppose that the condition (i) holds, i.e. $k := \|\mu_F\|_\Omega < 1$. By Theorem 3.1 the first inequality in (3.2) holds, and consequently the condition (3.5) holds with $L_1 := 1 + k < 2$. Applying Theorem 3.1 once more, we see that the second inequality in (3.2) holds, and consequently the condition

(3.6) holds with $L_2 := 1/(1-k) \geq 1$. Both the inequalities in (3.2) imply the conditions (iv) and (v).

Conversely, the condition (v) clearly implies the one (iv). Next the condition (iv) yields the one (iii). It remains to prove the implications (ii) $\Longrightarrow$ (i) and (iii) $\Longrightarrow$ (i).

Fix $z \in \mathbb{D}$, $r > 0$ and $\theta \in \mathbb{R}$ and set $w := z + r\mathrm{e}^{\mathrm{i}\theta}$. Assume first that the condition (3.6) holds for a certain $L_2 \geq 1$. Then

$$\frac{1}{L_2}\Big|\frac{H(w)-H(z)}{w-z}\Big| \leq \Big|\frac{H(w)-H(z)}{w-z} + \frac{\overline{w-z}}{w-z}\,\frac{\overline{G(w)-G(z)}}{\overline{w-z}}\Big|,$$

and letting r tend to 0, we obtain

$$\frac{1}{L_2}|H'(z)| \leq |H'(z) + \mathrm{e}^{-2\mathrm{i}\theta}\overline{G'(z)}|.$$

Hence choosing suitably θ, we have

$$\frac{1}{L_2}|H'(z)| \leq |H'(z)| - |G'(z)|,$$

and thus

$$|G'(z)| \leq \Big(1 - \frac{1}{L_2}\Big)|H'(z)|.$$

Combining this with (0.2), we deduce that

$$|\mu_F(z)| = |G'(z)|/|H'(z)| \leq 1 - \frac{1}{L_2} < 1, \quad z \in \Omega. \tag{3.7}$$

Assume now that the condition (3.5) holds for a certain L_1 such that $1 \leq L_1 < 2$. Then

$$\Big|\frac{H(w)-H(z)}{w-z} + \frac{\overline{w-z}}{w-z}\,\frac{\overline{G(w)-G(z)}}{\overline{w-z}}\Big| \leq L_1\Big|\frac{H(w)-H(z)}{w-z}\Big|,$$

and letting r tend to 0, we obtain

$$|H'(z) + \mathrm{e}^{-2\mathrm{i}\theta}\overline{G'(z)}| \leq L_1|H'(z)|.$$

Hence choosing suitably θ, we have

$$|H'(z)| + |G'(z)| \leq L_1|H'(z)|,$$

and so

$$|\mu_F(z)| = |G'(z)|/|H'(z)| \leq L_1 - 1 < 1, \quad z \in \Omega. \tag{3.8}$$

Each of the conditions (3.7) and (3.8) means that F is a quasiconformal mapping. Moreover, the condition (3.7) implies that $\|\mu_F\|_\Omega \leq 1 - (1/L_2)$ and the one (3.8) yields $\|\mu_F\|_\Omega \leq L_1 - 1$, which completes the proof. □

Remark 3.4. Note that the proofs of implications (ii) $\Longrightarrow$ (i) and (iii) $\Longrightarrow$ (i) have a local character and do not require the assumption that $F(\Omega)$ is a convex domain. Therefore each of conditions (ii) and (iii) implies that F is a quasiconformal mapping without the convexity of the image $F(\Omega)$.

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