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# Lagrangians and Euler morphisms on fibered-fibered frame bundles from projectable-projectable classical linear connections

Abstract. We classify all $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$-natural operators $A$ transforming projectable-projectable torsion-free classical linear connections $\nabla$ on fibered-fibered manifolds $Y$ of dimension $(m_1, m_2, n_1, n_2)$ into $r$th order Lagrangians $A(\nabla)$ on the fibered-fibered linear frame bundle $L^{\text{fib-fib}}(Y)$ on $Y$. Moreover, we classify all $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$-natural operators $B$ transforming projectable-projectable torsion-free classical linear connections $\nabla$ on fibered-fibered manifolds $Y$ of dimension $(m_1, m_2, n_1, n_2)$ into Euler morphism $B(\nabla)$ on $L^{\text{fib-fib}}(Y)$. These classifications can be expanded on the $k$th order fibered-fibered frame bundle $L^{\text{fib-fib},k}(Y)$ instead of $L^{\text{fib-fib}}(Y)$.

**1. Introduction.** Lagrangians and Euler morphisms are important tools in the variational calculus. Several physical theories are using Euler–Lagrange equations, which are related with the Euler morphism of an $r$th order Lagrangian on a fibered manifold.

The idea of Lagrangians and Euler morphisms in the case of fibered manifolds was described in [2]. The aim of the present note is the generalization of results which were reached in [1] to the case of fibered-fibered manifolds.

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**2. Fibered-fibered manifolds.** A fibered-fibered manifold $Y$ is any commutative diagram

$$\begin{array}{ccc} Y & \xrightarrow{\pi} & X \\ {\scriptstyle q}\big\downarrow & & \big\downarrow{\scriptstyle p} \\ N & \xrightarrow{\pi_0} & M \end{array}$$

where maps $\pi, \pi_0, q, p$ are surjective submersions and induced map $Y \to X \times_M N$, $y \mapsto (\pi(y), q(y))$ is a surjective submersion. A fibered-fibered manifold has dimension $(m_1, m_2, n_1, n_2)$ if $\dim Y = m_1 + m_2 + n_1 + n_2$, $\dim X = m_1 + m_2$, $\dim N = m_1 + n_1$, $\dim M = m_1$. For two fibered-fibered manifolds $Y_1, Y_2$ of the same dimension $(m_1, m_2, n_1, n_2)$, a morphism $f\colon Y_1 \to Y_2$ is quadruple of local diffeomorphisms $f\colon Y_1 \to Y_2$, $f_1\colon X_1 \to X_2$, $f_2\colon N_1 \to N_2$, $f_0\colon M_1 \to M_2$ such that all squares of the cube in question are commutative [3].

All fibered-fibered manifolds of the given dimension $(m_1, m_2, n_1, n_2)$ and their all morphisms form the category which we denote by $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$.

Every object from the category $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$ is locally isomorphic to the standard fibered-fibered manifold

$$\begin{array}{ccc} \mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \times \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} & \longrightarrow & \mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \\ \big\downarrow & & \big\downarrow \\ \mathbb{R}^{m_1} \times \mathbb{R}^{n_1} & \longrightarrow & \mathbb{R}^{m_1} \end{array}$$

which we denote by $\mathbb{R}^{m_1,m_2,n_1,n_2}$, where arrows are obvious projections.

For fibered-fibered manifold $Y$ we have the fibered-fibered linear frame bundle

$$L^{\text{fib-fib}}(Y) = \left\{ j^1_{(0,0,0,0)}\psi \mid \psi\colon \mathbb{R}^{m_1,m_2,n_1,n_2} \to Y \text{ is an } \mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}\text{-map} \right\}$$

with the jet target projection $\pi_Y\colon L^{\text{fib-fib}}(Y) \to Y$,

$$\pi_Y(j^1_{(0,0,0,0)}\psi) = \psi(0,0,0,0),$$

where $(0,0,0,0) \in \mathbb{R}^{m_1+m_2+n_1+n_2}$. The bundle $L^{\text{fib-fib}}(Y)$ is a principal bundle over $Y$ with a structure group $G^1_{m_1,m_2,n_1,n_2} = L^{\text{fib-fib}}_{(0,0,0,0)}(\mathbb{R}^{m_1,m_2,n_1,n_2})$ acting on the right on $L^{\text{fib-fib}}(Y)$ by the composition of jets. Every $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$-map $f\colon Y \to Y_1$ induces a fibered map (a principal bundle morphism) $L^{\text{fib-fib}}(f)\colon L^{\text{fib-fib}}(Y) \to L^{\text{fib-fib}}(Y_1)$ over $f$ by the composition of jets $L^{\text{fib-fib}}(f)\big(j^1_{(0,0,0,0)}\psi\big) = j^1_{(0,0,0,0)}(f \circ \psi)$. The correspondence $L^{\text{fib-fib}}\colon \mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2} \rightsquigarrow \mathcal{FM}$ is a bundle functor [2].

**3. Lagrangians and natural operators transforming connections into Lagrangians.** An $r$th order Lagrangian on a fibered manifold $p$ :

$X \to M$ is a base preserving morphism $\lambda\colon J^r(X) \to \wedge^m T^*M$ from the $r$-jet prolongation bundle

$$J^r(X) = \{j^r_x\sigma \mid \sigma\colon M \to X \text{ is a local section of } p\colon X \to M,\ x \in M\}$$

into the bundle $\wedge^m T^*M$ of $m = \dim M$-forms on $M$ [2].

A classical linear connection $\widetilde{\nabla}$ on a fibered manifold $p\colon X \to M$ is projectable if there exists a (unique) classical linear connection $\underline{\widetilde{\nabla}}$ on $M$ such that a connection $\underline{\widetilde{\nabla}}$ is $p$-related with a connection $\widetilde{\nabla}$, that is $Tp \circ (\widetilde{\nabla}_W Z) = (\underline{\widetilde{\nabla}}_{\underline{W}}\underline{Z}) \circ p$, where $W$ and $Z$ are projectable vector fields on $X$, which are $p$-related with vector fields $\underline{W}$ and $\underline{Z}$ on $M$.

Let $Y$ be a fibered-fibered manifold

$$\begin{array}{ccc} Y & \xrightarrow{\pi} & X \\ {\scriptstyle q}\big\downarrow & & \big\downarrow{\scriptstyle p} \\ N & \xrightarrow{\pi_0} & M \end{array}$$

We say that a projectable classical linear connection $\nabla$ on $Y$ is projectable-projectable if there exists a unique projectable classical linear connection $\underline{\nabla}$ on $X$ such that a connection $\underline{\nabla}$ is $\pi$-related with a connection $\nabla$.

In this paper we study the problem how a projectable-projectable torsion-free classical linear connection $\nabla$ on a fibered-fibered manifold $Y$ of dimension $(m_1, m_2, n_1, n_2)$ can induce an $r$th order Lagrangian $A(\nabla)$ on $\pi_Y\colon L^{\text{fib-fib}}(Y) \to Y$ in the canonical way. To this aim we must determine $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$-natural operators

$$A\colon Q_\tau^{\text{proj-proj}} \to \big(J^r L^{\text{fib-fib}}, \wedge^m T^*\big),$$

where $m = m_1 + m_2 + n_1 + n_2$, in the sense of [2].

We describe completely all such natural operators $A$ in question.

An $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$-natural operator

$$A\colon Q_\tau^{\text{proj-proj}} \to \big(J^r L^{\text{fib-fib}}, \wedge^m T^*\big)$$

(where $m = m_1 + m_2 + n_1 + n_2$) sending projectable-projectable torsion-free classical linear connections $\nabla$ on fibered-fibered manifolds $Y$ of dimension $(m_1, m_2, n_1, n_2)$ into $r$th order Lagrangians $A(\nabla)$ on the fibered-fibered linear frame bundle $\pi_Y\colon L^{\text{fib-fib}}(Y) \to Y$ for $Y$ is the family of $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$-invariant regular operators

$$A_Y\colon Q_\tau^{\text{proj-proj}}(Y) \to Lagr^r\big(L^{\text{fib-fib}}(Y)\big)$$

for $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$-objects $Y$, where $Q_\tau^{\text{proj-proj}}(Y)$ is the space of all projectable-projectable torsion-free classical linear connections on $Y$ and $Lagr^r\big(L^{\text{fib-fib}}(Y)\big)$ is the space of all $r$th order Lagrangians on

$$\pi_Y\colon L^{\text{fib-fib}}(Y) \to Y.$$

The invariance means that if $\nabla \in Q_\tau^{\text{proj-proj}}(Y)$ and $\nabla_1 \in Q_\tau^{\text{proj-proj}}(Y_1)$ are $f$-related with respect to an $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$-map $f\colon Y \to Y_1$, then $A_Y(\nabla)$ and $A_{Y_1}(\nabla_1)$ are also $f$-related. The regularity of a natural operator $A_Y$ means that $A_Y$ transforms smoothly parametrized families of connections in $Q_\tau^{\text{proj-proj}}(Y)$ into smoothly parametrized families of Lagrangians in $Lagr^r\big(L^{\text{fib-fib}}(Y)\big)$.

To present an example of natural operator $A$ in question we need some preparations. Let $M$ be an $m$-manifold with a torsion-free classical linear connection $\underline{\widetilde{\nabla}}$. Given a linear frame $\underline{l} \in L(M)$, the linear isomorphism $\underline{l}\colon \mathbb{R}^m \to T_{\underline{x}}(M)$ defines a coordinate system in $T_{\underline{x}}(M)$. Therefore, $\underline{\widetilde{\nabla}}$-exponential map $Exp_{\underline{x}}^{\underline{\widetilde{\nabla}}}\colon T_{\underline{x}}(M) \to M$ being the diffeomorphism, defines a normal coordinate system $\underline{\varphi}$ with center $\underline{x}$ in $M$ by the composition $\underline{\varphi} = \underline{l}^{-1} \circ \big(Exp_{\underline{x}}^{\underline{\widetilde{\nabla}}}\big)^{-1}\colon M \to \mathbb{R}^m$. If $\underline{\varphi}_1\colon M \to \mathbb{R}^m$ is another $\underline{\widetilde{\nabla}}$-normal coordinate system on $M$ with center $\underline{x}$, then $\underline{\varphi}_1 = I \circ \underline{\varphi}$ for some linear isomorphism $I\colon \mathbb{R}^m \to \mathbb{R}^m$ [4].

Let $p\colon X \to M$ be a fibered manifold of dimension $(m,n)$ and let $x \in X_{\underline{x}}$, $\underline{x} \in M$. Let $\widetilde{\nabla}$ be a projectable torsion-free classical linear connection on $X$ with the underlying torsion-free classical linear connection $\underline{\widetilde{\nabla}}$ on $M$. Since a connection $\widetilde{\nabla}$ is $p$-related with $\underline{\widetilde{\nabla}}$, then $p$ sends $\widetilde{\nabla}$-geodesics into $\underline{\widetilde{\nabla}}$-geodesics. Consequently, the $\widetilde{\nabla}$-exponential map $Exp_x^{\widetilde{\nabla}}\colon T_x(X) \to X$ at $x$ is a local fibered diffeomorphism covering the $\underline{\widetilde{\nabla}}$-exponential map $Exp_{\underline{x}}^{\underline{\widetilde{\nabla}}}\colon T_{\underline{x}}(M) \to M$ at $\underline{x}$, where $T_x(X)$ is treated as a fibered manifold $Tp\colon T_x(X) \to T_{\underline{x}}(M)$.

If we compose $\big(Exp_x^{\widetilde{\nabla}}\big)^{-1}$ with a fibered linear isomorphism (fibered linear frame) $l\colon \mathbb{R}^{m,n} \to T_x(X)$ covering a linear frame $\underline{l}\colon \mathbb{R}^m \to T_{\underline{x}}(M)$, then we obtain a fibered $\widetilde{\nabla}$-normal coordinate system $\varphi = l^{-1} \circ \big(Exp_x^{\widetilde{\nabla}}\big)^{-1}\colon X \to \mathbb{R}^{m,n}$ with center $x$ covering a $\underline{\widetilde{\nabla}}$-normal coordinate system

$$\underline{\varphi} = \underline{l}^{-1} \circ \big(Exp_{\underline{x}}^{\underline{\widetilde{\nabla}}}\big)^{-1}\colon M \to \mathbb{R}^m$$

with center $\underline{x}$. If $\varphi_1\colon X \to \mathbb{R}^{m,n}$ is another fibered $\widetilde{\nabla}$-normal coordinate system on $X$ with center $x$, then $\varphi_1 = I \circ \varphi$ for some fibered linear isomorphism $I\colon \mathbb{R}^{m,n} \to \mathbb{R}^{m,n}$.

Quite similarly as above, if $\nabla$ is a projectable-projectable torsion-free classical linear connection on a fibered-fibered manifold $Y$ of dimension $(m_1, m_2, n_1, n_2)$ and $y \in Y$, then there exists a fibered-fibered $\nabla$-normal coordinate system $\varphi\colon Y \to \mathbb{R}^{m_1,m_2,n_1,n_2}$ with center $y$. If $\varphi_1\colon Y \to \mathbb{R}^{m_1,m_2,n_1,n_2}$ is another fibered-fibered $\nabla$-normal coordinate system with center $y$, then $\varphi_1 = I \circ \varphi$ for some fibered-fibered linear isomorphism $I\colon \mathbb{R}^{m_1,m_2,n_1,n_2} \to \mathbb{R}^{m_1,m_2,n_1,n_2}$.

**4. The first main result.** Let $Q^s_{\text{proj-proj}}$ be the vector space of all $s$-jets $j^s_{(0,0,0,0)}(\nabla)$ at $(0,0,0,0) \in \mathbb{R}^{m_1+m_2+n_1+n_2}$ of projectable-projectable torsion-free classical linear connections $\nabla$ on $\mathbb{R}^{m_1,m_2,n_1,n_2}$ satisfying the condition

$$\sum_{j,k=1}^{m_1+m_2+n_1+n_2} \nabla^i_{jk}(x)x^j x^k = 0 \quad \text{for} \quad i = 1,\dots,m_1+m_2+n_1+n_2,$$

where $\nabla^i_{jk}\colon \mathbb{R}^{m_1,m_2,n_1,n_2} \to \mathbb{R}$, for $i,j,k = 1,\dots,m_1+m_2+n_1+n_2$, are the Christoffel symbols of a connection $\nabla$ in the usual fibered-fibered coordinate system $x^1,\dots,x^{m_1+m_2+n_1+n_2}$ on $\mathbb{R}^{m_1,m_2,n_1,n_2}$.

Equivalently, $x^1,\dots,x^{m_1+m_2+n_1+n_2}$ are fibered-fibered $\nabla$-normal coordinates with center $(0,0,0,0)$. The equivalence is a simple consequence of the well-known equations of $\nabla$-geodesics and the fact that in the $\nabla$-normal coordinate system $\nabla$-geodesics passing through the center are straight lines.

Let $\pi_s\colon Q^\infty_{\text{proj-proj}} \to Q^s_{\text{proj-proj}}$ for $s = 1,2,\dots$ be the jet projections. Let $\pi^r_0\colon J^r\big(L^{\text{fib-fib}}(\mathbb{R}^{m_1,m_2,n_1,n_2})\big) \to L^{\text{fib-fib}}(\mathbb{R}^{m_1,m_2,n_1,n_2})$ be the jet projection. Denote $l_0 := j^1_{(0,0,0,0)}(id_{\mathbb{R}^{m_1,m_2,n_1,n_2}}) \in L^{\text{fib-fib}}(\mathbb{R}^{m_1,m_2,n_1,n_2})$.

Let $J^r_{l_0}\big(L^{\text{fib-fib}}(\mathbb{R}^{m_1,m_2,n_1,n_2})\big)$ be the fibre of $\pi^r_0$ at $l_0$. Let $\mu\colon Q^\infty_{\text{proj-proj}} \times J^r_{l_0}\big(L^{\text{fib-fib}}(\mathbb{R}^{m_1,m_2,n_1,n_2})\big) \to \mathbb{R}$. We say that $\mu$ satisfies the finite determination property, if for any $\rho \in Q^\infty_{\text{proj-proj}}$ and $\sigma \in J^r_{l_0}\big(L^{\text{fib-fib}}(\mathbb{R}^{m_1,m_2,n_1,n_2})\big)$ we can find an open neighborhood $U \subset Q^\infty_{\text{proj-proj}}$ of $\rho$, open neighborhood $V \subset J^r_{l_0}\big(L^{\text{fib-fib}}(\mathbb{R}^{m_1,m_2,n_1,n_2})\big)$ of $\sigma$, a finite number $s$ and a smooth map $f\colon \pi_s(U) \times V \to \mathbb{R}$ such that $\mu = f \circ (\pi_s \times id_V)$ on $U \times V$.

We are in a position to present the following example of the operator $A$ in question.

**Example 1.** Let $\mu\colon Q^\infty_{\text{proj-proj}} \times J^r_{l_0}\big(L^{\text{fib-fib}}(\mathbb{R}^{m_1,m_2,n_1,n_2})\big) \to \mathbb{R}$ be a function satisfying the finite determination property. Given a projectable-projectable torsion-free classical linear connection $\nabla$ on a fibered-fibered manifold $Y$ of dimension $(m_1,m_2,n_1,n_2)$, we define the $r$th order Lagrangian

$$A^{\langle\mu\rangle}_Y(\nabla)\colon J^r\big(L^{\text{fib-fib}}(Y)\big) \to \wedge^m T^*(Y)$$

on $\pi_Y\colon L^{\text{fib-fib}}(Y) \to Y$ by

$$A^{\langle\mu\rangle}_Y(\nabla)(\sigma) := \mu\big(j^\infty_{(0,0,0,0)}(\varphi_*\nabla), J^r\big(L^{\text{fib-fib}}(\varphi)\big)(\sigma)\big) \cdot l^*_1 \wedge \ldots \wedge l^*_{m_1+m_2+n_1+n_2},$$

where $m = m_1+m_2+n_1+n_2$, $\sigma \in J^r_l\big(L^{\text{fib-fib}}(Y)\big)$, $l = j^r_{(0,0,0,0)}(\varphi^{-1}) \in \big(L^{\text{fib-fib}}(Y)\big)_y$, $y \in Y$, $l_i = T(\varphi^{-1})\big(\frac{\partial}{\partial x^i}|_{(0,0,0,0)}\big)$ for $i = 1,\dots,m_1+m_2+n_1+n_2$ is the basis in $T_y(Y)$ and $l^*_i$ for $i = 1,\dots,m_1+m_2+n_1+n_2$ is the dual basis in $T^*_y(Y)$ and $\varphi\colon Y \to \mathbb{R}^{m_1,m_2,n_1,n_2}$ is a fibered-fibered $\nabla$-normal coordinate system with center $y$ such that $J^r\big(L^{\text{fib-fib}}(\varphi)\big)(\sigma) \in J^r_{l_0}\big(L^{\text{fib-fib}}(\mathbb{R}^{m_1,m_2,n_1,n_2})\big)$.

The definition of $A^{\langle\mu\rangle}_Y(\nabla)(\sigma)$ is correct because $germ_y(\varphi)$ is uniquely determined.

Consequently, for given a projectable-projectable torsion-free classical linear connection $\nabla$ on $Y$, we have an $r$th order Lagrangian

$$A^{\langle\mu\rangle}_Y : J^r\big(L^{\text{fib-fib}}(Y)\big) \to \wedge^m T^*(Y), \quad \text{where } m = m_1 + m_2 + n_1 + n_2.$$

The family $A^{\langle\mu\rangle} : Q^{\text{proj-proj}}_\tau \rightsquigarrow \big(J^r L^{\text{fib-fib}}, \wedge^m T^*\big)$ of operators

$$A^{\langle\mu\rangle}_Y : Q^{\text{proj-proj}}_\tau(Y) \longrightarrow Lagr^r\big(L^{\text{fib-fib}}(Y)\big),$$
$$\nabla \longrightarrow A^{\langle\mu\rangle}_Y(\nabla)$$

for any $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$-object $Y$ is an $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$-natural operator.

The main result of the present note is the following:

**Theorem 1.** *Any $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$-natural operator*

$$A\colon Q^{\text{proj-proj}}_\tau \rightsquigarrow \big(J^r L^{\text{fib-fib}}, \wedge^m T^*\big),$$

*where $m = m_1 + m_2 + n_1 + n_2$, is of the form $A = A^{\langle\mu\rangle}$ for a uniquely determined function $\mu\colon Q^\infty_{\text{proj-proj}} \times J^r_{l_0}\big(L^{\text{fib-fib}}(\mathbb{R}^{m_1,m_2,n_1,n_2})\big) \to \mathbb{R}$ satisfying the finite determination property.*

**Proof.** Let $A$ be a $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$-natural operator in question. We must define a map $\mu\colon Q^\infty_{\text{proj-proj}} \times J^r_{l_0}\big(L^{\text{fib-fib}}(\mathbb{R}^{m_1,m_2,n_1,n_2})\big) \to \mathbb{R}$ by

$$\mu\big(j^\infty_{(0,0,0,0)}(\nabla),\sigma\big) := \langle A_{\mathbb{R}^{m_1,m_2,n_1,n_2}}(\nabla)(\sigma), (l_0)_1 \wedge \ldots \wedge (l_0)_{m_1+m_2+n_1+n_2}\rangle,$$

where $l_0 = ((l_0)_1, \ldots, (l_0)_{m_1+m_2+n_1+n_2})$ is the basis in $T_{(0,0,0,0)}(\mathbb{R}^{m_1+m_2+n_1+n_2})$. Then by the non-linear Peetre theorem [2], $\mu$ satisfies the finite determination property. By the invariance of $A$ and $A^{\langle\mu\rangle}$ with respect to fibered-fibered normal coordinates we obtain $A = A^{\langle\mu\rangle}$. □

**Remark 1.** Quite similarly one can describe all $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$-natural operators $A\colon Q^{\text{proj-proj}}_\tau \to \big(J^r L^{\text{fib-fib},k}, \wedge^m T^*\big)$ transforming projectable-projectable torsion-free classical linear connections $\nabla$ on $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$-objects $Y$ into $r$th order Lagrangians $A_Y(\nabla)$ on $\pi^k_Y\colon L^{\text{fib-fib},k}(Y) \to Y$, where

$$L^{\text{fib-fib},k}(Y) := \Big\{ j^k_{(0,0,0,0)}(\psi) \mid \psi\colon \mathbb{R}^{m_1,m_2,n_1,n_2} \to Y \text{ is a local } \mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}\text{-map} \Big\}$$

is the fibered-fibered $k$th order frame bundle for $Y$. All such natural operators in question are of the form $A^{\langle\mu\rangle}$ for functions

$$\mu\colon Q^\infty_{\text{proj-proj}} \times J^r_{l_0}\big(L^{\text{fib-fib},k}(\mathbb{R}^{m_1,m_2,n_1,n_2})\big) \to \mathbb{R}$$

satisfying the obviously modified finite determination property, where $J^r_{l_0}\big(L^{\text{fib-fib},k}(\mathbb{R}^{m_1,m_2,n_1,n_2})\big)$ is the fiber of

$$J^r_{l_0}\big(L^{\text{fib-fib},k}(\mathbb{R}^{m_1,m_2,n_1,n_2})\big) \to L^{\text{fib-fib},k}(\mathbb{R}^{m_1,m_2,n_1,n_2})$$

at the $k$th order frame $l_0 = j^k_{(0,0,0,0)}(id_{\mathbb{R}^{m_1,m_2,n_1,n_2}}) \in L^{\text{fib-fib},k}(\mathbb{R}^{m_1,m_2,n_1,n_2})$. The generalized natural operators $A^{\langle\mu\rangle}$ can be defined quite similarly as in Example 1.

**5. Euler morphisms and natural operators transforming connections into Euler morphisms.** We recall that the $r$th order Euler morphism on a fibered manifold $p\colon X \to M$ is a base preserving morphism $E\colon J^r(X) \to V^*(X) \otimes \wedge^m T^*(M)$, where $m = \dim M$. Here $V^*(X)$ denotes the vector bundle dual to the vertical vector bundle $V(X)$ for $X$. Special Euler morphisms can be obtained from Lagrangians by means of the well-known Euler operator [2], [5].

Quite similarly as for Lagrangians, we can describe completely all $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$-natural operators

$$B\colon Q^{\text{proj-proj}}_{\tau} \to \big(J^r L^{\text{fib-fib}}, V^* L^{\text{fib-fib}} \otimes \wedge^m T^*\big),$$

where $m = m_1+m_2+n_1+n_2$, transforming projectable-projectable torsion-free classical linear connections $\nabla$ on fibered-fibered manifold $Y$ of dimension $(m_1, m_2, n_1, n_2)$ into $r$th order Euler morphisms $B_Y(\nabla)$ on

$$\pi_Y\colon L^{\text{fib-fib}}(Y) \to Y.$$

**6. The second main result.**

**Example 2.** We consider a function

$$\mu\colon Q^{\infty}_{\text{proj-proj}} \times J^r_{l_0}\big(L^{\text{fib-fib}}(\mathbb{R}^{m_1,m_2,n_1,n_2})\big) \to \big(\mathcal{L}\big(G^1_{m_1,m_2,n_1,n_2}\big)\big)^*$$

satisfying the obviously modified finite determination property, where $\mathcal{L}(G^1_{m_1,m_2,n_1,n_2})$ denotes the Lie algebra of Lie group $G^1_{m_1,m_2,n_1,n_2}$. Given a projectable-projectable torsion-free classical linear connection on a fibered-fibered manifold $Y$ of dimension $(m_1, m_2, n_1, n_2)$, we define an $r$th order Euler morphism $B^{\langle\mu\rangle}_Y(\nabla)\colon J^r\big(L^{\text{fib-fib}}(Y)\big) \to V^*\big(L^{\text{fib-fib}}(Y)\big) \otimes \wedge^m T^*(Y)$, where $m = m_1 + m_2 + n_1 + n_2$, on $\pi_Y\colon L^{\text{fib-fib}}(Y) \to Y$ by

$$\big\langle B^{\langle\mu\rangle}_Y(\nabla)(\sigma), \eta^*_{|l}\big\rangle = \Big\langle \mu(j^{\infty}_{(0,0,0,0)}(\varphi_*\nabla), J^r\big(L^{\text{fib-fib}}(\varphi)\big)(\sigma)), \eta\Big\rangle\, l^*_1 \wedge \ldots \wedge l^*_{m_1+m_2+n_1+n_2}$$

for all $\sigma \in (J^r_l\big(L^{\text{fib-fib}}(Y)\big)$, $l = (l_1, \ldots, l_m) \in \big(L^{\text{fib-fib}}(Y)\big)_y$, $y \in Y$, where $m = m_1 + m_2 + n_1 + n_2$, $\eta \in \mathcal{L}(G^1_{m_1,m_2,n_1,n_2})$, where $\eta^*$ is the (vertical) fundamental vector field on the principal $G^1_{m_1,m_2,n_1,n_2}$-bundle $L^{\text{fib-fib}}(Y)$ corresponding to $\eta$ and $l^*_1, \ldots, l^*_m \in T^*_y Y$ is the dual basis to $l_1, \ldots, l_m \in T_y Y$

and $\varphi\colon Y \to \mathbb{R}^{m_1 m_2, n_1, n_2}$ is a fibered-fibered $\nabla$-normal coordinate system on $Y$ with center $y$ such that $\varphi(y) = (0,0,0,0) \in \mathbb{R}^{m_1+m_2+n_1+n_2}$ and $J^r\big(L^{\text{fib-fib}}(\varphi)\big)(\sigma) \in \big(J^r_{l_0}\big(L^{\text{fib-fib}}(\mathbb{R}^{m_1,m_2,n_1,n_2})\big)\big)$. The correspondence $B^{\langle\mu\rangle}\colon Q^{\text{proj-proj}}_\tau \rightsquigarrow \big(J^r L^{\text{fib-fib}}, V^* L^{\text{fib-fib}} \otimes \wedge^m T^*\big)$, where $m = m_1 + m_2 + n_1 + n_2$, is $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$-natural operator.

Similarly as Theorem 1 one can prove the following:

**Theorem 2.** *Any $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$-natural operator*

$$B\colon Q^{\text{proj-proj}}_\tau \rightsquigarrow \big(J^r L^{\text{fib-fib}}, V^* L^{\text{fib-fib}} \otimes \wedge^m T^*\big),$$

*where $m = m_1 + m_2 + n_1 + n_2$, is of the form $B = B^{\langle\mu\rangle}$ for some uniquely determined function*

$$\mu\colon Q^\infty_{\text{proj-proj}} \times J^r_{l_0}\big(L^{\text{fib-fib}}(\mathbb{R}^{m_1,m_2,n_1,n_2})\big) \to (\mathcal{L}(G^1_{m_1,m_2,n_1,n_2}))^*$$

*satisfying the modified finite determination property.*

**Proof.** Similarly as in the proof of Theorem 1 we define

$$\mu\colon Q^\infty_{\text{proj-proj}} \times J^r_{l_0}\big(L^{\text{fib-fib}}(\mathbb{R}^{m_1,m_2,n_1,n_2})\big) \to (\mathcal{L}(G^1_{m_1,m_2,n_1,n_2}))^*$$

by

$$\begin{aligned}&\langle \mu(j^\infty_{(0,0,0,0)}(\nabla), \sigma), \eta\rangle \\ &\quad = \langle B_{\mathbb{R}^{m_1,m_2,n_1,n_2}}(\nabla)(\sigma), \eta^*_{l_0} \otimes (l_0)_1 \wedge \ldots \wedge (l_0)_{m_1+m_2+n_1+n_2}\rangle,\end{aligned}$$

where

$$\eta \in \mathcal{L}(G^1_{m_1,m_2,n_1,n_2}),\ j^\infty_{(0,0,0,0)}(\nabla) \in Q^\infty_{\text{proj-proj}},\ \sigma \in J^r_{l_0}\big(L^{\text{fib-fib}}(\mathbb{R}^{m_1,m_2,n_1,n_2})\big),$$

$\eta^*$ is the fundamental vector field on $L^{\text{fib-fib}}(\mathbb{R}^{m_1,m_2,n_1,n_2})$ corresponding to $\eta \in \mathcal{L}(G^1_{m_1,m_2,n_1,n_2})$ and $l_0 = ((l_0)_1, \ldots, (l_0)_{m_1+m_2+n_1+n_2})$ is the basis in $T_{(0,0,0,0)}(\mathbb{R}^{m_1+m_2+n_1+n_2})$. Then $B = B^{\langle\mu\rangle}$. □

**Remark 2.** Quite similarly one can describe all $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$-natural operators $B\colon Q^{\text{proj-proj}}_\tau \to (J^r L^{\text{fib-fib},k}, V^* L^{\text{fib-fib},k} \otimes \wedge^m T^*)$, where $m = m_1+m_2+n_1+n_2$, transforming projectable-projectable torsion-free classical linear connections $\nabla$ on $(m_1, m_2, n_1, n_2)$-dimensional fibered-fibered manifolds $Y$ into Euler morphisms $B_Y(\nabla)$ on $\pi^k_Y\colon L^{\text{fib-fib},k}(Y) \to Y$ of fibered-fibered frames of order $k$ of $Y$. All such natural operators are of the form $B^{\langle\mu\rangle}$ for all

$$\mu\colon Q^\infty_{\text{proj-proj}} \times J^r_{l_0}\big(L^{\text{fib-fib},k}(\mathbb{R}^{m_1,m_2,n_1,n_2})\big) \to \mathcal{L}(G^k_{m_1,m_2,n_1,n_2})$$

satisfying the obviously modified finite determination property. The natural operators $B^{\langle\mu\rangle}$ can be constructed similarly as in Example 2.

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