doi: 10.2478/v10062-011-0003-8

ANNALES
UNIVERSITATIS MARIAE CURIE-SKŁODOWSKA
LUBLIN – POLONIA

VOL. LXV, NO. 1, 2011 SECTIO A 21–31

TYMOTEUSZ CHOJECKI

On the central limit theorem for some birth and death processes

ABSTRACT. Suppose that $\{X_n,\, n \ge 0\}$ is a stationary Markov chain and V is a certain function on a phase space of the chain, called an observable. We say that the observable satisfies the central limit theorem (CLT) if $Y_n := N^{-1/2}\sum_{n=0}^{N} V(X_n)$ converge in law to a normal random variable, as $N \to +\infty$. For a stationary Markov chain with the L^2 spectral gap the theorem holds for all V such that $V(X_0)$ is centered and square integrable, see Gordin [7]. The purpose of this article is to characterize a family of observables V for which the CLT holds for a class of birth and death chains whose dynamics has no spectral gap, so that Gordin's result cannot be used and the result follows from an application of Kipnis–Varadhan theory.

1. Introduction. Suppose that $\{X_n,\, n \ge 0\}$ is a stationary Markov chain defined over a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and V is a certain function, called an observable, given over the phase of the chain such that $\mathbb{E}V(X_0) = 0$ and $\mathbb{E}V^2(X_0) < +\infty$. Here $\mathbb{E}$ is the mathematical expectation corresponding to $\mathbb{P}$. We say that the chain satisfies the central limit theorem (CLT) if the random variables

$$Y_n := N^{-1/2}\sum_{n=0}^{N} V(X_n) \tag{1}$$

2000 *Mathematics Subject Classification.* 60J10, 60F05.

Key words and phrases. Central limit theorem, Markov chain, Lamperti's problem, birth and death processes, Kipnis–Varadhan theory, spectral gap.

converge in law to a normal random variable, as $N \to +\infty$. Characterization of Markov chains and the class of observables for which the CLT holds, is one of the fundamental problems in probability theory. One of the first results of this type has been the CLT proved by Doeblin [4] for chains whose transition probabilities satisfy what is now called *Doeblin condition*. For stationary Markov chains with the L^2 spectral gap the theorem has been proved by Gordin [7]. A remarkable result giving a complete characterization of reversible Markov chains, i.e. such that for any $N \geq 0$ the laws of $(X_0, X_1 \ldots, X_N)$ and of $(X_N, X_{N-1}, \ldots, X_0)$ are identical, satisfying the CLT has been proved by Kipnis and Varadhan in their seminal article [8], see also De Masi et al. [3]. It has been shown in [8] that the CLT holds for such chains if V satisfies

$$D^2(V) := \lim_{N \to +\infty} \frac{1}{N} \mathbb{E} \left[\sum_{n=0}^{N} V(X_n) \right]^2 < +\infty. \tag{2}$$

One can also prove, see [8], that the limit appearing in (2) always exists, being finite or infinite for any reversible chain. Sometimes, however, it is not easy to verify the condition directly.

It can be shown that any ergodic and Markov chain with a finite phase space has a spectral gap, so the CLT is valid by an application of the Gordin's result. According to our knowledge, the examples of chains not having the spectral gap property, yet satisfying the theorem, concern the situation when the phase space is uncountable, e.g. tagged particle in a simple exclusion process, random walks in random environments, diffusions in random media etc., see for instance [3, 9]. One of the latest review articles about CLT for tagged particles and diffusion in random environment is [11]. The objective of this paper is to show an application of the Kipnis–Varadhan theory in the perhaps simplest possible case (outside finite chains), namely to a reversible chain with a countable phase space but with no spectral gap (the Gordin's result cannot be used then). An example like this is furnished by a birth and death chain from Lamperti's problem (see the definition in Section 2), whose phase space is the set of non-negative integers. In the situation considered in the present article we also give a necessary and sufficient explicit condition for an observable V so that (2) holds. The problem of CLT for the trajectory of the chain has been solved in Menshikov and Wade article [10].

As far as the structure of our paper is concerned, in the next section we will introduce some basic terms and present three theorems which are our main results. The remaining three sections deal with the proofs of these theorems.

2. Preliminaries and statements of the main results.

2.1. Generalities. Assume that $\{X_n,\ n \geq 0\}$ is a Markov chain whose state space is $\mathbb{Z}_+ = \{0, 1, 2, \ldots\}$. It means that there exists a function $p : \mathbb{Z}_+ \times \mathbb{Z}_+ \to [0, 1]$ such that $\int_{\mathbb{Z}_+} p(x, y)dy = 1$ for all $x \in \mathbb{Z}_+$ and

$$\mathbb{P}\big[X_{n+1} = x_{n+1} | X_0 = x_0, \ldots, X_n = x_n\big] = p(x_n, x_{n+1}).$$

Here $\int_{\mathbb{Z}_+} f(x)a(dx) := \sum_0^{+\infty} f(x)a(x)$ for an arbitrary $f : \mathbb{Z}_+ \to \mathbb{R}$ and $a : \mathbb{Z}_+ \to [0, +\infty)$. We will write $\int_{\mathbb{Z}_+} f(x)dx$ when $a(x) \equiv x$. Then

$$Pf(x) = \int_{\mathbb{Z}_+} p(x, y)f(y)dy, \quad \forall\, f \in B_b(\mathbb{Z}_+)$$

is called a transition operator. Here $B_b(\mathbb{Z}_+)$ denotes the space of all bounded functions on $\mathbb{Z}_+$. Suppose that $\pi : \mathbb{Z}_+ \to (0, 1]$ is a strictly positive probability measure, i.e. $\int_{\mathbb{Z}_+} \pi(x)dx = 1$. It is assumed to be reversible and ergodic with respect to the chain. Ergodicity means that for any bounded f equality $Pf = f$ implies that f is constant π a.s. We say that the chain is irreducible if for any $x, y \in \mathbb{Z}_+$ exists $n \geq 1$ such that $p^n(x, y) > 0$, where $p^n(x, y)$ denotes the probability of going from x to y in n steps.

Remark 2.1. It is well known ([5], p. 338) that a stationary, irreducible Markov chain with a countable state space is ergodic.

Reversibility, on the other hand, means that the detailed balance condition holds, i.e.:

$$p(x, y)\pi(x) = p(y, x)\pi(y), \quad \forall\, x, y \in \mathbb{Z}_+. \tag{3}$$

This condition is equivalent to the fact that P can be extended to a bounded and symmetric contraction on $L^2(\pi)$ – the space of all f such that $||f||_\pi^2 = \int_{\mathbb{Z}_+} f^2(x)\pi(dx) < +\infty$. In consequence, the spectrum of P lies in $[-1, 1]$. Note that $\lambda_0 = 1$ is the largest eigenvalue of P corresponding to an eigenfuction $f_0(x) \equiv 1$. Let

$$\lambda_1 := \sup\left[\langle Pf, f\rangle_\pi;\ \int_{\mathbb{Z}_+} f(x)\pi(dx) = 0,\ ||f||_\pi = 1\right]. \tag{4}$$

We say that the chain has the spectral gap property, when $\lambda_1 < 1$. Here $\langle\cdot,\cdot\rangle_\pi$ is the scalar product corresponding to $\|\cdot\|_\pi$.

Now we will formulate our first main result. It is a simple criterion for continuity of the spectrum at 1. For any $x \in \mathbb{Z}_+$ let us define $\hat{\lambda}_0^{(x)}$ by

$$\hat{\lambda}_0^{(x)} = \sup[\langle Pf, f\rangle_\pi;\ f(x) = 0,\ ||f||_\pi = 1]. \tag{5}$$

We note that $\hat{\lambda}_0^{(x)}$ is the largest eigenvalue for the "reduced" operator $\hat{P} = \Pi_x P \Pi_x$, where Π_x is the orthogonal projection onto the subspace $H_x^{(\pi)} := [f \in L^2(\pi) : f(x) = 0]$.

Theorem 2.2 (A criterion for continuity of the spectrum at 1). *Suppose that the chain $\{X_n,\, n \geq 0\}$ is reversible, irreducible and there exists $x \in \mathbb{Z}_+$ for which $\hat{\lambda}_0^{(x)} = 1$. Then 1 is not an isolated point of the spectrum of the transition operator P.*

The proof of this theorem is presented in Section 3.

2.2. Birth and death processes. We recall the definition of a birth and death process (see [5], p. 295, Example 3.4). In our setting it is a Markov chain on countable state space $\mathbb{Z}_+ := \{0, 1, 2, \dots, \}$ with the transition probabilities satisfying that $p(x, y) = 0$ if and only if $|x - y| \neq 1$. We also require that $p_0 := p(0, 1) = 1$ and

$$p(x, x+1) = p_x, \qquad p(x, x-1) = q_x,\ x \geq 1,$$

are all strictly positive. Of course we have $p_x + q_x = 1$ for all $x \geq 1$. In this case there is the measure $\tilde{\pi}(x) = \prod_{k=1}^{x}(p_{k-1}/q_k)$ which is reversible and unique up to a multiplicative constant ([5], p. 301, Example 4.4). This measure is not necessarily finite. In fact [5], p. 306, Theorem 4.5, it is infinite if and only if the chain is recurrent but not positive recurrent. In case it is positive recurrent we have $Z := \int_{\mathbb{Z}_+} \tilde{\pi}(x)dx < +\infty$ ([5], p. 307, Theorem 4.7) and then $\pi(x) := Z^{-1}\tilde{\pi}(x)$ is a unique invariant law of the chain. A reader can find more information about this class and more details about this chain in volume I of Feller's monography [6].

Our second goal is to find the relation between transition probabilities and spectral gap property in the birth and death process.

Theorem 2.3. *Let $\{X_n\ , n \geq 0\}$ be a birth and death process with transition probabilities as above. We have three possible situations then:*
(i) *if $\lim_{x\to+\infty} p_x = p$, $\lim_{x\to+\infty} q_x = q$ and $p < q$, then the chain is positive recurrent and has the spectral gap property,*
(ii) *if $p_x = 1/2 - c_x$ and $q_x = 1/2 + c_x$, where*

$$0 < c_* = \liminf_{x\to\infty} c_x x^\alpha \leq \limsup_{x\to\infty} c_x x^\alpha = c^* < +\infty, \tag{6}$$

and $\alpha \in (0, 1)$, then we have the positive recurrence but we do not have the spectral gap property,
(iii) *if* (6) *holds but for $\alpha > 1$, then the chain is recurrent, but not positive recurrent.*

The proof of this result is presented in Section 4.

Remark 2.4. From Section I.12, p. 71–76 of [2] we know that, when $\alpha = 1$, then both positive recurrence and null recurrence may occur. It depends on the constants c_*, c^*.

Remark 2.5. Theorem 2.3 can be interpreted as follows. If we have a strong drift to the left, i.e. the local drift $D_x := p_x - q_x$ satisfies $\limsup_{x\to\infty} D_x < 0$, then the chain is positive recurrent and has the spectral gap property. When we have a weaker drift to the left but $D_x \sim -c/x^\alpha$, for some $c > 0$ and $\alpha \in (0,1)$, then the chain is positive recurrent but does not have the spectral gap. Finally, when we further increase the probability of going to the right so that $D_x \sim -c/x^\alpha$, for $\alpha > 1$ and some $c > 0$, then the chain loses the property of the positive recurrence.

The most interesting case of the previous theorem is part (ii). In this situation we wish to characterize the class of observables for which the random variables (1) satisfy the CLT. The necessary and sufficient condition for this can be stated as follows.

Theorem 2.6. *Let $V : \mathbb{Z}_+ \to \mathbb{R}$ be a zero-mean function in $L^2(\pi)$ and $\{X_n,\ n \geq 0\}$ is the chain from Theorem 2.3 (ii). Then, Y_n given by* (1) *satisfies* (2) *if and only if*

$$\int_{\mathbb{Z}_+} \frac{dx}{\pi(x)} \left[\int_{0 \leq y \leq x} V(y)\pi(dy) \right]^2 < \infty.$$

3. Proof of Theorem 2.2. In order to make our calculations easier, we change our space $L^2(\pi)$ into space ℓ^2 corresponding to the counting measure on $\mathbb{Z}_+$, while $\langle \cdot, \cdot \rangle$, $||\cdot||$ denote the respective scalar product and the ℓ^2 norm.

We will introduce some terms which are useful throughout the proof. Let us denote by $A = [a(x,y)]_{x,y \in \mathbb{Z}_+}$ a matrix with

$$a(x,y) := \pi^{1/2}(x) p(x,y) \pi(y)^{-1/2}.$$

Note that the definition means that $A = DPD^{-1}$, where

$$D = \mathrm{diag}[\pi^{1/2}(0), \pi^{1/2}(1), \ldots],$$

i.e. the operators are unitary equivalent. In particular the above means that

$$\lambda_1 = \sup[\langle Af, f \rangle;\ \|f\| = 1,\ f \in \ell^2_0]. \tag{7}$$

Observe that A is a symmetric matrix. Moreover, for any f with $||f|| \leq 1$ we have

$$\begin{aligned}
|\langle Af, f \rangle| &:= \left| \int_{\mathbb{Z}_+} \int_{\mathbb{Z}_+} a(x,y) f(x) f(y) dx dy \right| \\
&= \left| \int_{\mathbb{Z}_+} \int_{\mathbb{Z}_+} \pi^{1/2}(x) p(x,y) \pi(y)^{-1/2} f(x) f(y) dx dy \right| \\
&\leq \left\{ \int_{\mathbb{Z}_+} \int_{\mathbb{Z}_+} \pi(x) p(x,y) \pi^{-1}(y) f(y)^2 dx dy \right\}^{1/2} \left\{ \int_{\mathbb{Z}_+} \int_{\mathbb{Z}_+} p(x,y) f(x)^2 dx dy \right\}^{1/2} \\
&\leq ||f||^2 \leq 1.
\end{aligned}$$

The spectrum of A is also contained in $[-1,1]$, in fact because of the unitary equivalence, it coincides with the spectrum of P. Note that $f_* :=$

$(\pi^{1/2}(0), \pi^{1/2}(1), \ldots)$ is an eigenvector that corresponds to an eigenvalue $\lambda_0 = 1$. Denote by ℓ_0^2 the space consisting of $f \in \ell^2$ such that $\langle f, f_* \rangle = 0$.

Denote by A' matrix $A' = \Pi_x A \Pi_x$. We can easily check that

$$\hat{\lambda}_0^{(x)} = \sup[\langle Af, f\rangle;\ \|f\| = 1,\ f \in H_x], \tag{8}$$

where $H_x := [f \in \ell^2 : f(x) = 0]$ and $\hat{\lambda}_0^{(x)}$ in (8) is defined in (5).

If the supremum is attained at certain $f^{(0)}$, such that $\|f^{(0)}\| = 1$, $f^{(0)} \in H_x$, then we would have to have $Af^{(0)} = f^{(0)}$ and that would mean $Pg^{(0)} = g^{(0)}$, where $g^{(0)} = D^{-1} f^{(0)}$. This, however, would imply $g^{(0)} = c1$ for some constant c, or equivalently $f^{(0)} = c[\pi^{1/2}(0), \pi^{1/2}(1), \ldots]$. Since $f^{(0)}(x) = 0$ we would have $c = 0$, which leads to a contradiction. The above means that 1 is not in the point spectrum of A' and since it does belong to the spectrum it must be in its continuous part. We show that the above implies that

$$\lambda_1 = 1. \tag{9}$$

Suppose otherwise, i.e. $\lambda_1 < 1$. Indeed, suppose that $f^{(n)} \in H_x$ are such that $\|f^{(n)}\| = 1$ and

$$\langle Af^{(n)}, f^{(n)} \rangle \to 1.$$

Denote by Q the orthogonal projection onto ℓ_0^2. We have

$$f^{(n)} = \alpha_n f_* + Qf^{(n)}$$

and $\|Qf^{(n)}\|^2 = 1 - \alpha_n^2$. Since $\langle Af_*, Qf^{(n)} \rangle = 0$ we have from (9)

$$1 \leftarrow \langle Af^{(n)}, f^{(n)} \rangle \le \alpha_n^2 + \lambda_1 (1 - \alpha_n^2) \to 1.$$

This implies $\alpha_n^2 \to 1$ and in consequence $\|Qf^{(n)}\| \to 0$. Suppose that $\alpha_n \to 1$. This yields

$$\pi^{1/2}(x) \le \|f^{(n)} - f_*\| \to 0,$$

which is impossible. On the other hand, if $\alpha_n \to -1$ we have

$$\pi^{1/2}(x) \le \|f^{(n)} + f_*\| \to 0,$$

which is again impossible. Hence the conclusion of the theorem follows.

4. Proof of Theorem 2.3. We split our proof into two parts. The first one, called "strong drift to the left", considers the case (i) from Theorem 2.3, the second, "weaker drift to the left", deals with the cases (ii) and (iii) from the theorem. The main point is that in case (i), when the drift to the left is sufficiently strong, the chain has a spectral gap.

4.1. Strong drift to the left. In this case we have $\lim_{x\to+\infty} p_x = p$, $\lim_{x\to+\infty} q_x = q$ and $p < q$. First, we check whether the chain is positive recurrent, i.e. we verify that

$$Z := \int_{\mathbb{Z}_+} \tilde{\pi}(x)dx < \infty,$$

where

$$\tilde{\pi}(x) = \prod_{k=1}^{x} \frac{p_{k-1}}{q_k},$$

see Example 4.4, p. 301 of [5]. From assumption (i) we know that for all $\varepsilon > 0$, there exists $i_0 > 0$, such that for all $k \geq k_0$ we have

$$|p_k - p| < \varepsilon \wedge |q_k - q| < \varepsilon.$$

Now we can see that

$$\begin{aligned} \int_{\mathbb{Z}_+} \tilde{\pi}(x)dx &= \int_{0\leq x\leq k_0} \tilde{\pi}(x)dx + \tilde{\pi}(k_0) \sum_{k=k_0+1}^{\infty} \prod_{j=k_0+1}^{k} \frac{p_{j-1}}{q_j} \\ &\leq c \sum_{k=k_0+1}^{\infty} \left(\frac{p+\varepsilon}{q-\varepsilon}\right)^{k-k_0} < \infty \end{aligned}$$

for some $c > 0$, provided that $p + \varepsilon < q - \varepsilon$. Hence in this case we know from [5], p. 307, Theorem 4.7 that we have a positive recurrence.

Now we show the spectral gap property. From [1], see Theorem 1.5, p. 10, case 3, we have that $\lambda_1 < 1$ if and only if $\delta < \infty$, where

$$\delta := \sup_{x\geq 1} \int_{0\leq y\leq x-1} dy \left\{ [\tilde{\pi}(y)p_y]^{-1} \int_{x\leq y} \tilde{\pi}(y)dy \right\}.$$

Observe that

$$\begin{aligned} \int_{x\leq y} \tilde{\pi}(y)dy &= \tilde{\pi}(x) \int_{x+1\leq k} dk \prod_{l=x+1}^{k} \frac{p_{l-1}}{q_l} \\ &\leq \tilde{\pi}(x) \int_{x+1\leq k} \left(\frac{p+\varepsilon}{q-\varepsilon}\right)^{k-k_0} dk \leq C\tilde{\pi}(x) \end{aligned}$$

for some constant $C > 0$, provided that $p + \varepsilon < q - \varepsilon$. We can write then

$$\begin{aligned} \delta &\leq \frac{C}{p-\varepsilon} \sup_{x\geq 1} \int_{0\leq y\leq x-1} \frac{\tilde{\pi}(x)dy}{\tilde{\pi}(y)} = \frac{C}{p-\varepsilon} \sup_{x\geq 1} \int_{0\leq y\leq x-1} dy \prod_{k=y+1}^{x} \frac{p_{k-1}}{q_k} \\ &\leq \frac{C}{p-\varepsilon} \times \frac{(p+\varepsilon)(q-\varepsilon)^{-1}}{1-(p+\varepsilon)(q-\varepsilon)^{-1}} < +\infty. \end{aligned}$$

4.2. Weaker drift to the left. Now we will be interested in the case when $p_x = 1/2 - c_x$, $q_x = 1/2 + c_x$, where

$$\liminf_{x\to\infty} c_x x^\alpha = c_* > 0,$$
$$\limsup_{x\to\infty} c_x x^\alpha = c^* < \infty,$$

so we can find positive constants K, D_1, D_2, such that for all $x \geq K$ we have

$$\frac{D_1}{x^\alpha} \geq c_x \geq \frac{D_1}{x^\alpha}.$$

We show that when $\alpha \in (0,1)$, then we do not have the spectral gap property but we have the positive recurrence. Also we show that when $\alpha > 1$, we do not have the positive recurrence. Thus, $\alpha = 1$ is a critical exponent, where the chain loses the positive recurrence.

It is easy to see that we can find some positive constant c, for which $p_{k-1}/q_k < 1 - c/k^\alpha$ for $k > K$. And for such c we have

$$-\log\frac{p_{k-1}}{q_k} > -\log\left(1 - \frac{c}{k^\alpha}\right) > \frac{c}{k^\alpha}, \quad k > K.$$

Hence, using the integral test for convergence we have

$$\tilde{\pi}(x) < \exp\left\{-c\sum_{k=1}^{x}\frac{1}{k^\alpha}\right\} < \tilde{c}\exp\left(-cx^{1-\alpha}\right), \quad x > K,$$

where $\tilde{c}$ denotes a positive constant. From the comparison test we see that $\int_{\mathbb{Z}_+}\tilde{\pi}(x)dx < +\infty$ when $\alpha < 1$ and the positive recurrence follows.

On the other hand, we also have some positive constant c', for which $p_{k-1}/q_k > 1 - c'/k^\alpha$ for $k > K$. But when $\alpha > 1$, we can easily check, using again the integral test for convergence, that

$$\tilde{\pi}(x) > \hat{c}\exp\left(-c''x^{1-\alpha}\right), \quad x > K,$$

where $\hat{c}$, c'' are other positive constants. Then $\tilde{\pi}(x)$ fails the necessary condition for convergence of the respective series.

Now we will show that we do not have the spectral gap property when $\alpha < 1$. To do so, we use Theorem 2.2 and show that 1 cannot be an isolated point of the spectrum. We choose $x = 0$ in condition (5). Denote by A' a symmetric matrix obtained from A by crossing out the 0th column and 0th row. We prove that $\sup_{||f||=1}\langle A'f, f\rangle = 1$. Let f_n have $n^{-1/2}$ on the first n coordinates and the rest of them vanishes, i.e. $f_n := [n^{-1/2}, n^{-1/2}, \ldots, n^{-1/2}, 0\ldots]$. A simple computation shows that

$$a(x,x+1) = \sqrt{p_x q_{x+1}} = \begin{cases} \sqrt{\frac{1}{2} + c_1}\,, & x = 0 \\ \sqrt{\frac{1}{4} + \frac{1}{2}c_{x+1} - \frac{1}{2}c_x - c_x c_{x+1}}\,, & x > 0 \end{cases}$$

$$a(x,x-1) = \sqrt{p_{x-1}q_x} = \begin{cases} \sqrt{\frac{1}{2}+c_1}\,, & x=1 \\ \sqrt{\frac{1}{4}+\frac{1}{2}c_x-\frac{1}{2}c_{x-1}-c_xc_{x-1}}\,, & x>1 \end{cases}$$

and

$$\begin{aligned}\langle A' f_n, f_n\rangle &= 2\int_{x\ge 1} a(x,x+1) f_n(x) f_n(x+1)dx \\ &= \frac{1}{n}\int_{1\le x\le n} \sqrt{1+2c_{x+1}-2c_x-c_xc_{x+1}}dx.\end{aligned}$$

Let $\varepsilon > 0$. Then, there exists i_0 such that for $i > i_0$

$$|2c_{i+1} - 2c_i - c_ic_{i+1}| < \varepsilon.$$

Hence,

$$\langle A' f_n, f_n\rangle \ge \sqrt{1-\varepsilon}, \text{when } n\to\infty.$$

Since $\varepsilon > 0$ was arbitrary we have $\hat{\lambda}_0^{(0)} = 1$ and, by Theorem 2.2, we do not have the spectral gap property.

5. Proof of Theorem 2.6. In this section we assume that the hypothesis of Theorem 2.3 part (ii) holds. We formulate a sufficient and necessary condition for an observable V, so that (2) holds for random variables given by (1).

We are going to use the result from [8], see (1.8) p. 3. According to that result, the necessary and sufficient condition for the validity of (2) is that $||V||_{-1}^2 < \infty$, where

$$||V||_{-1}^2 := \sup_{\varphi\in L^2(\pi)} \left\{ 2\langle V,\varphi\rangle_\pi - \langle (I-P)\varphi,\varphi\rangle_\pi \right\}. \tag{10}$$

In what follows, we will find the maximizer of this supremum by solving the Euler–Lagrange equation (14). The maximizer belongs to a certain Hilbert space, that we denote by $\mathcal{H}_1$, which is a bigger space than $L^2(\pi)$.

Observe that we have the following equality:

$$\langle (I-P)\varphi,\varphi\rangle_\pi = \frac{1}{2}\int_{\mathbb{Z}_+}\int_{\mathbb{Z}_+} p(x,y)(\varphi(y)-\varphi(x))^2 dy\pi(dx). \tag{11}$$

Denote the discrete gradient $\partial f(x) = f(x+1) - f(x)$ and its dual, with respect to the scalar product from ℓ^2, $\partial^* f(x) = f(x-1) - f(x)$. It is easy to see that when the supp$f$ is compact, then the following integration by parts formula holds

$$\int_{\mathbb{Z}_+} \partial f(x) g(x)dx = \int_{x\ge 1} f(x)\partial^* g(x)dx - f(0)g(0).$$

In our case we have

$$\begin{aligned}\mathcal{E}(\varphi) &:= \langle (I-P)\varphi, \varphi\rangle_\pi \\ &= \frac{1}{2}\int_{x\geq 1}\left[\left(\frac{1}{2}+c_x\right)\Big(\partial^*\varphi(x)\Big)^2+\left(\frac{1}{2}-c_x\right)\Big(\partial\varphi(x)\Big)^2\right]\pi(dx).\end{aligned}$$

We can find some positive constants c_1, c_2, for which

$$c_1\mathcal{E}_0(\varphi) \leq \mathcal{E}(\varphi) \leq c_2\mathcal{E}_0(\varphi),$$

where

$$\mathcal{E}_0(\varphi) := \int_{\mathbb{Z}_+}\big(\partial\varphi(x)\big)^2\pi(dx).$$

Then,

$$\sup_{\varphi\in L^2(\pi)}\big\{2\langle V,\varphi\rangle_\pi - c_2\mathcal{E}_0(\varphi)\big\} \leq ||V||_{-1}^2 \leq \sup_{\varphi\in L^2(\pi)}\big\{2\langle V,\varphi\rangle_\pi - c_1\mathcal{E}_0(\varphi)\big\}. \tag{12}$$

We can see therefore that $\|V\|_{-1} < +\infty$ if and only if the supremum of the functional $\Phi(\varphi)$ appearing on the right hand side of (12) is finite. Since the functional $\Phi(\cdot)$ is weakly upper semicontinuous on a Hilbert space

$$\mathcal{H}_1 := [\varphi : \mathcal{E}_0(\varphi) < +\infty],$$

and $\lim_{\|\varphi\|_{\mathcal{H}_1}\to+\infty}\Phi(\varphi) = -\infty$, it attains its maximum

$$\Phi_* = \sup[\Phi(\varphi) : \varphi \in L^2(\pi)] < +\infty \tag{13}$$

and its maximizer φ_* has to satisfy the Euler–Lagrange equation, which in this case reads

$$V(x)\pi(x) = \partial^*[\pi(x)\partial\varphi_*(x)], \quad \forall\, x \geq 1 \tag{14}$$

and $V(0) = -\partial\varphi(0)$, or equivalently

$$\pi(x)\partial\varphi(x) = -\pi(x)V(x) + \pi(x-1)\partial\varphi(x-1), \quad \forall\, x\geq 1$$

and $\partial\varphi(0) = -V(0)$. Note that from this equation we get

$$\pi(x)\partial\varphi(x) = -\int_{1\leq y\leq x}V(y)\pi(dy) + \pi(0)\partial\varphi(0) = -\int_{0\leq y\leq x}V(y)\pi(dy),$$

hence

$$\partial\varphi(x) = -\frac{1}{\pi(x)}\int_{0\leq y\leq x}V(y)\pi(dy).$$

The supremum in (13) equals

$$\Phi_* = \int_{\mathbb{Z}_+}[\partial\varphi(x)]^2\pi(dx) = \int_{\mathbb{Z}_+}\frac{dx}{\pi(x)}\left[\int_{0\leq y\leq x}V(y)\pi(dy)\right]^2$$

and the requirement that $\Phi_* < +\infty$ is the same requirement as $\|V\|_{-1} < +\infty$, thus the conclusion of Theorem 2.6 follows. Note that the fact that $\Phi_* < +\infty$ in particular implies that

$$\int_{\mathbb{Z}_+} V(y)\pi(dy) = 0.$$

Acknowledgement. I would like to express my deep gratitude to Professor Tomasz Komorowski, whose help and comments have been invaluable and without whom writing this paper would have been impossible, and to Professor Andrew Wade, who pointed out my error in part (iii) of Theorem 2.3.

References

[1] Chen, M., *Eigenvalues, Inequalities, and Ergodic Theory*, Springer-Verlag, London, 2005.

[2] Chung, K. L., *Markov Chains with Stationary Transition Probabilities*, 2nd edition, Springer-Verlag, Berlin, 1967.

[3] De Masi, A., Ferrari, P. A., Goldstein, S. and Wick, W. D., *An invariance principle for reversible Markov processes. Applications to random motions in random environments*, J. Statist. Phys. **55** (1989).

[4] Doeblin, W., *Sur deux proble'mes de M. Kolmogoroff concernant les chaines dénombrables*, Bull. Soc. Math. France **66** (1938), 210–220.

[5] Durrett, R., *Probability Theory and Examples*, Wadsworth Publishing Company, Belmont, 1996.

[6] Feller, W., *An Introduction to Probability Theory and its Applications*, Vol. II. Second edition, John Wiley & Sons, Inc., New York–London–Sydney, 1971.

[7] Gordin, M. I., *The central limit theorem for stationary processes*, Dokl. Akad. Nauk SSSR **188** (1969), 739–741 (Russian).

[8] Kipnis, C., Varadhan, S. R. S., *Central limit theorem for additive functionals of reversible Markov processes and applications to simple exclusions*, Comm. Math. Phys. **104**, no. 1 (1986), 1–19.

[9] Liggett, T., *Stochastic Interacting Systems: Contact, Voter and Exclusion Processes*, Grund. der Math. Wissen., 324, Springer-Verlag, Berlin, 1999.

[10] Menshikov, M., Wade, A.,*Rate of escape and central limit theorem for the supercritical Lamperti problem*, Stochastic Process. Appl. **120** (2010), 2078-–2099.

[11] Olla, S., *Notes on Central Limits Theorems for Tagged Particles and Diffusions in Random Environment*, Etats de la recherche: Milieux Aleatoires CIRM, Luminy, 2000.

Tymoteusz Chojecki
ul. Czeremchowa 12/26
20-807 Lublin
Poland
e-mail: chojecki.tymoteusz@gmail.com

Received October 21, 2010