doi: 10.2478/v10062-011-0011-8

ANNALES
UNIVERSITATIS MARIAE CURIE-SKŁODOWSKA
LUBLIN – POLONIA

VOL. LXV, NO. 2, 2011 SECTIO A 29–44

BODO DITTMAR and MAREN HANTKE

About a Pólya–Schiffer inequality

Dedicated to the memory of Professor Jan G. Krzyż

Abstract. For simply connected planar domains with the maximal conformal radius 1 it was proven in 1954 by G. Pólya and M. Schiffer that for the eigenvalues λ of the fixed membrane for any n the following inequality holds

$$\sum_{k=1}^{n} \frac{1}{\lambda_k} \geq \sum_{k=1}^{n} \frac{1}{\lambda_k^{(o)}},$$

where $\lambda^{(o)}$ are the eigenvalues of the unit disk. The aim of the paper is to give a sharper version of this inequality and for the sum of all reciprocals to derive formulas which allow in some cases to calculate exactly this sum.

1. Introduction. Let $D \subset \mathbb{R}^2$ be a bounded simply connected domain. We consider the following eigenvalue problems [1]: the eigenvalue problem of the fixed membrane

$$\begin{aligned} \Delta u + \lambda u &= 0 \text{ in } D \\ u &= 0 \text{ on } \partial D \end{aligned} \tag{1}$$

and the eigenvalue problem of the free membrane

$$\begin{aligned} \Delta v + \mu v &= 0 \text{ in } D \\ \frac{\partial v}{\partial n} &= 0 \text{ on } \partial D, \end{aligned} \tag{2}$$

where n stands for the normal to ∂D, λ and μ for the eigenvalue parameters. It is well known that there exist infinitely many eigenvalues with finite

2000 *Mathematics Subject Classification.* Primary 35P15; Secondary 35P05.
Key words and phrases. Membrane eigenvalues, sums of reciprocal eigenvalues.

multiplicity:

$$0 < \lambda_1 \le \lambda_2 \le \lambda_3 \dots ,$$
$$0 = \mu_1 < \mu_2 \le \mu_3 \dots \ .$$

The aim of this paper is to sharpen the following isoperimetric inequality proven by G. Pólya and M. Schiffer in 1954 [9] for the eigenvalues of the fixed membrane. For any n,

$$\sum_{k=1}^{n} \frac{1}{\lambda_k} \ge \sum_{k=1}^{n} \frac{1}{\lambda_k^{(o)}} , \tag{3}$$

where λ_k are the fixed membrane eigenvalues of a domain D with maximal conformal radius 1 and $\lambda_k^{(o)}$ are the fixed membrane eigenvalues of the unit disk. Many authors dealt with this problem, among others J. Hersch, C. Bandle, R. Laugesen and C. Morpurgo [1, 7, 4]. For the free membrane eigenvalues (3) was proven by the first author in [2, 3, 5]. On the one hand, we prove a sharper version of this inequality for the fixed and free membrane problem and on the other hand, we are able to give formulas for the sum of all reciprocals containing only the coefficients of the series expansion of the conformal mapping. In some cases we are able to calculate exactly this sum. Besides we prove some monotonicity results.

2. Fixed membrane problem. The eigenvalue problem of the fixed membrane (1) in a planar domain D is conformally equivalent to the following problem in the unit disk U

$$\Delta u + \lambda u |f'(z)|^2 = 0 \text{ in } U,$$
$$u\big|_{\partial U} = 0,$$
$$\int_U u_i u_j |f'(z)|^2 dA_z = \delta_{ij}, \ \ i,j = 1,2,\dots \ ,$$

where $f(z)$ denotes the conformal mapping from U onto D, u_i denote the eigenfunctions and δ_{ij} the Kronecker delta. We use the same notation for the transplanted eigenfunctions.

2.1. Isoperimetric inequalities. Our goal is the following.

Theorem 1. *Let $u_k^{(o)}$ be the eigenfunctions of the fixed membrane problem in the unit disk, $\lambda_k^{(o)}$ the corresponding eigenvalues and let $f(z) = z + a_2 z^2 + \dots$ be a conformal mapping of the unit disk onto D with the eigenvalues λ_k. Then, for any $n \ge 1$ we have*

$$\sum_{k=1}^{n} \frac{1}{\lambda_k} \ge \sum_{k=1}^{n} \frac{1}{\lambda_k^{(o)}} + \sum_{k=1}^{n} \frac{1}{\lambda_k^{(o)}} \sum_{j=2}^{\infty} j^2 |a_j|^2 \int_U {u_k^{(o)}}^2 r^{2j-2} dA.$$

In order to prove this theorem we need some lemmas and we follow the basic ideas in [5].

Lemma 1.

$$\max\int_U\int_U G(z,\zeta)|f'(z)|h(z)|f'(\zeta)|h(\zeta)dA_z dA_\zeta=\frac{1}{\lambda_n}\,,$$

where the maximum is taken over all function $h\in L^2(U)$ with

$$\int_U hu_j|f'(z)|dA_z=0,$$

$j=1,2,\ldots,n-1$, $\int_U h^2dA=1$. *$G(z,\zeta)$ denotes the Green's function of the unit disk. Equality holds for $h=u_n|f'(z)|$.*

Proof. We have [2, (2.9)],

$$G(z,\zeta)|f'(z)||f'(\zeta)|=\sum_{j=1}^{\infty}\frac{u_j(z)|f'(z)|u_j(\zeta)|f'(\zeta)|}{\lambda_j}$$

which yields

$$\begin{aligned}\int_U\int_U G(z,\zeta)|f'(z)|h(z)|f'(\zeta)|h(\zeta)dA_z dA_\zeta&=\sum_{n}^{\infty}\frac{(\int_U u_j(z)h(z)|f'(z)|dA)^2}{\lambda_j}\\&\le\frac{1}{\lambda_n}\int_U h^2dA=\frac{1}{\lambda_n}\end{aligned}$$

for a function h satisfying the conditions given in the lemma. We have equality if $h=u_n(z)|f'(z)|$. □

Lemma 2.

$$\max_{L_n}\min\int_U\int_U G(z,\zeta)|f'(z)|h(z)|f'(\zeta)|h(\zeta)dA_z dA_\zeta=\frac{1}{\lambda_n}\,,$$

where the maximum is taken over all n-dimensional linear spaces $L_n\subset L^2(U)$ and the minimum is taken over all $h\in L_n$, $||h||_{L^2(U)}=1$. Equality occurs for $L_n=\{u:u=c_1u_1|f'(z)|+\cdots+c_nu_n|f'(z)|,\ c_j\in\mathbb{R}\}$.

Proof. For $h\in L^2(U)$ which satisfies the conditions of Lemma 1 we get

$$\begin{aligned}\frac{1}{\lambda_n}&\ge\int_U\int_U G(z,\zeta)|f'(z)|h(z)|f'(\zeta)|h(\zeta)dA_z dA_\zeta\\&\ge\min\int_U\int_U G(z,\zeta)|f'(z)|h(z)|f'(\zeta)|h(\zeta)dA_z dA_\zeta\,.\end{aligned}$$

In every n-dimensional subspace there exists such a function h and consequently,

$$\max_{L_n}\min\int_U\int_U G(z,\zeta)|f'(z)|h(z)|f'(\zeta)|h(\zeta)dA_z dA_\zeta\le\frac{1}{\lambda_n}.$$

We take the space $L_n = \{u = c_1 u_1|f'(z)| + \cdots + c_n u_n|f'(z)|,\ c_j \in \mathbb{R}\}$ and obtain the opposite inequality, which completes the proof. □

Lemma 3.

$$\sum_1^n \frac{1}{\lambda_j} = \max_{L_n} \sum_1^n \int_U \int_U G(z,\zeta)|f'(z)|h_i(z)|f'(\zeta)|h_i(\zeta)dA_z dA_\zeta\,,$$

where $\{h_i\}_{i=1}^n$ *is a basis of* L_n *satisfying the orthonormality conditions* $\int_U h_i h_j dA = \delta_{ij}$. *Equality holds for* $L_n = \{u : u = c_1 u_1|f'(z)| + \cdots + c_n u_n|f'(z)|,\ c_j \in \mathbb{R}\}$.

Proof. There exist a function $h_n \in L_n$ with $||h_n|| = 1$,

$$\int_U h_n|f'(z)|u_j(z)dA = 0,$$

$j = 1, \ldots, n-1$, and a function $h_{n-1} \in L_n$ with $||h_{n-1}|| = 1$,

$$\int_U h_{n-1}|f'(z)|u_j(z)dA = 0,$$

$j = 1, \ldots, n-2$, $\int_U h_n h_{n-1} dA = 0$, etc. Finally there exists a function $h_1 \in L_n$ with $||h_1|| = 1$, $\int_U h_1 h_j dA = 0, j = 2, \ldots, n$. From Lemma 1 it follows that

$$\frac{1}{\lambda_j} \geq \int_U \int_U G(z,\zeta)|f'(z)|h_j(z)|f'(\zeta)|h_j(\zeta)dA_z dA_\zeta,\ j = 1, \ldots, n,$$

which establishes the inequality in the lemma. Taking $h_j = u_j|f'(z)|$, $j = 1, \ldots, n$, we obtain the equality in the last inequality, which completes the proof. □

Proof of Theorem 1. We use Lemma 3 with h_j replaced by $h_j|f'|$ and obtain

$$\sum_1^n \frac{1}{\lambda_j} = \max_{L_n} \sum_1^n \int_U \int_U G(z,\zeta)|f'(z)|^2 h_i(z)|f'(\zeta)|^2 h_i(\zeta)dA_z dA_\zeta$$

with $\int_U h_i h_j |f'(z)|^2 dA = \delta_{ij}$, $i, j = 1, \ldots, n$. We will show that there exists a set of functions $\{h_j\}_{j=1}^n$ satisfying the condition mentioned above and

$$h_j = \sum_{i=1}^j c_{ji} u_i^{(o)},\ c_{jj} \neq 0.$$

With $c_{11} \neq 0$ the function $h_1 = c_{11} u_1^{(o)}$ satisfies $\int_U h_1 |f'(z)|^2 dA = 1$. We choose c_{21}, c_{22} such that $\int_U h_2 h_1 |f'(z)|^2 dA = 0$ and $\int_U h_2^2 |f'(z)|^2 dA = 1$ and evidently $c_{22} \neq 0$. In general, it is easy to see that there exist constants $c_{j1}, \ldots, c_{jj}$ such that $\int_U h_j h_i |f'(z)|^2 dA = \delta_{ij}, i = 1, \ldots, j$. We now proceed by induction that $c_{jj} \neq 0$, $j = 1, \dots, n$, then a consequence of $c_{jj} = 0$ is $c_{j-1j-1} = 0$. The Hilbert–Schmidt theorem yields

$$\int_U G(z,\zeta)|f'(z)|^2 h_j(z)dA_z = \sum_1^\infty b_{jk}\frac{u_k^{(o)}(\zeta)}{\lambda_k^{(o)}}, \quad b_{jk} = \int_U h_j u_k^{(o)}|f'(z)|^2 dA$$

which implies that

$$\int_U\int_U G(z,\zeta)|f'(z)|^2 h_j(z)|f'(\zeta)|^2 h_j(\zeta)dA_z dA_\zeta = \sum_{k=1}^\infty \frac{b_{jk}^2}{\lambda_k^{(o)}}.$$

Consequently,

$$\begin{aligned}\sum_1^n \frac{1}{\lambda_j} &\geq \sum_1^n \int_U\int_U G(z,\zeta)|f'(z)|^2 h_i(z)|f'(\zeta)|^2 h_i(\zeta)dA_z dA_\zeta \\ &= \sum_1^n\sum_{k=1}^\infty \frac{b_{jk}^2}{\lambda_k^{(o)}} \geq \sum_1^n\sum_1^n \frac{b_{jk}^2}{\lambda_k^{(o)}} = \sum_{k=1}^n\left(\sum_{j=1}^n b_{jk}^2\right)\frac{1}{\lambda_k^{(o)}}.\end{aligned}$$

It is proven that the lower-triangular matrix of the coefficients c_{ij} is nonsingular and we obtain $u_k^{(o)} = \sum_1^k g_{ki}h_i$. We have $b_{jk} = \int_U |f'(z)|^2 h_j u_k^{(o)} dA = \int_U |f'(z)|^2 h_j \sum_1^k g_{ki}h_i dA = g_{kj}$ because of the orthogonality of the functions h_j. It follows that

$$\sum_{j=1}^n b_{jk}^2 = \sum_{j=1}^n g_{kj}^2 = \int_U {u_k^{(o)}}^2 |f'(z)|^2 dA.$$

For a radial eigenfunction we have

$$\int_U {u_k^{(o)}}^2 |f'(z)|^2 dA = 1 + \sum_2^\infty j^2|a_j|^2\int_U {u_k^{(o)}}^2 r^{2(j-1)}dA.$$

For a non-radial eigenfunction $u_k^{(o)}$ there are two eigenfunctions $u_k^{(o)}, u_{k+1}^{(o)}$ with the same eigenvalue such that ${u_k^{(o)}}^2 + {u_{k+1}^{(o)}}^2$ is radial. It follows that

$$\int_U\left({u_k^{(o)}}^2 + {u_{k+1}^{(o)}}^2\right)|f'(z)|^2 dA = 2 + \sum_2^\infty j^2|a_j|^2\int_U\left({u_k^{(o)}}^2 + {u_{k+1}^{(o)}}^2\right)r^{2j-2}dA.$$

For the non-radial eigenfunctions we take the eigenfunction with

$$\int_U {u_k^{(o)}}^2 |f'(z)|^2 dA \geq 1 + \sum_2^\infty j^2|a_j|^2\int_U {u_k^{(o)}}^2 r^{2(j-1)}dA$$

which establishes the theorem. □

We can cancel the eigenfunctions of the unit disk if we take the sum over all eigenvalues.

Theorem 2. *Let $u_k^{(o)}$ be the eigenfunctions of the fixed membrane problem in the unit disk, $\lambda_k^{(o)}$ the corresponding eigenvalues and let $f(z) = z + a_2 z^2 + \dots$ be a conformal mapping of the unit disk onto D with the eigenvalues λ_k. Then*

$$\sum_1^\infty \frac{1}{\lambda_j^2} \geq -\sum_1^\infty \frac{1}{\lambda_j^{o2}} + 2\int_U |f'(z)|^2 \int_U G^2(z,\zeta)dA_\zeta dA_z$$
$$= \sum_1^\infty \frac{1}{\lambda_j^{o2}} + 4\pi \sum_2^\infty n^2 |a_n|^2 \int_0^1 r^{2n-1} h(r)dr,$$
$$h(r) = \int_U G^2(z,\zeta)dA_\zeta.$$

Proof. We have [2, (2.7)],

$$\sum_1^\infty \frac{1}{\lambda_j^2} = \sum_1^\infty \frac{\int_U (\nabla G_j)^2 dA}{\lambda_j^{(o)}}$$

with

$$G_j(\zeta) = \int_U G(z,\zeta) u_j^{(o)}(z) |f'(z)|^2 dA_z.$$

Furthermore [2, (2.23)]

$$\int_U (\nabla G_j)^2 dA \geq -\frac{1}{\lambda_j^{(o)}} + \frac{2}{\lambda_j^{(o)}} \int_U {u_j^{(o)}}^2 |f'(z)|^2 dA$$

which gives after changing summation and integration

$$\sum_1^\infty \frac{\int_U (\nabla G_j)^2 dA}{\lambda_j^{(o)}} \geq -\sum_1^\infty \frac{1}{{\lambda_j^{(o)}}^2} + 2\sum_1^\infty \frac{\int_U {u_j^{(o)}}^2 |f'(z)|^2 dA}{{\lambda_j^{(o)}}^2}$$
$$= -\sum_1^\infty \frac{1}{{\lambda_j^{(o)}}^2} + 2\int_U |f'(z)|^2 \left(\int_U G^2(z,\zeta)dA_\zeta \right) dA_z\,,$$

where we have used

$$\int_U G^2(z,\zeta)dA_\zeta = \sum_1^\infty \frac{{u_j^{(o)}}^2(z)}{{\lambda_j^{(o)}}^2}.$$

The last identity in the theorem holds because $\int_U G^2(z,\zeta)dA_\zeta$ is radial and $|f'(z)|^2 = 1 + 4|a_2|^2 + \dots + n^2|a_n|^2 + \dots +$ terms in z^m, $m \geq 1$. □

Remark 1. By equality [2, (2.27)] we have

$$\int_U G^2(z,\zeta)dA_\zeta = \frac{\pi}{2} - \frac{3}{4}\pi r^2 + \pi \sum_1^\infty \frac{r^{2n+2}}{n(n+1)} - \pi \sum_2^\infty \frac{r^{2n}}{n^2-1},\ |z| = r.$$

2.2. Formula for the sum of all reciprocals. The aim of this section is to give a formula for the sum of all reciprocal eigenvalues of the fixed membrane problem for any bounded simply connected domain D. This formula makes explicitly use of coefficients of the series expansion of the conformal mapping from the unit disk U to the domain D.

We have [2],

$$\sum_{j=1}^{\infty} \frac{1}{\lambda_j^2} = \int_U \int_U G^2(z,\zeta)|f'(z)|^2|f'(\zeta)|^2 \, dA_z \, dA_\zeta \, . \tag{4}$$

Regarding the singularity of the Green's function we write for $|\zeta| < |z|$,

$$G(z,\zeta) = \frac{1}{2\pi}\left(-\ln|z| - \ln\left|1-\frac{\zeta}{z}\right| + \ln|1-z\bar{\zeta}|\right)$$

and obtain

$$\begin{aligned} G^2(z,\zeta) = \frac{1}{4\pi^2}\Bigg(&\ln^2|z| + \ln^2|1-\frac{\zeta}{z}| + \ln^2|1-z\bar{\zeta}| + 2\ln|z|\ln\left|1-\frac{\zeta}{z}\right| \\ &-2\ln|z|\ln|1-z\bar{\zeta}| - 2\ln\left|1-\frac{\zeta}{z}\right|\ln|1-z\bar{\zeta}|\Bigg) \, . \end{aligned} \tag{5}$$

We use symmetry of the Green's function

$$\sum_{j=1}^{\infty} \frac{1}{\lambda_j^2} = 2\int_{0<|z|<1}\int_{0<|\zeta|<|z|} G^2(z,\zeta)|f'(z)|^2|f'(\zeta)|^2 \, dA_z \, dA_\zeta . \tag{6}$$

Due to (5) the sum (4) consists of 6 summands

$$\sum_{j=1}^{\infty} \frac{1}{\lambda_j^2} = I_1 + I_2 + I_3 + I_4 - I_5 - I_6 \, . \tag{7}$$

For the explicit calculation of the integrals it is more convenient to use polar coordinates, but we forbear from using new notations. For the expansion series of f we use the following notations

$$|f'(s,\theta)|^2 = \sum_{n=0}^{\infty} a_{0,n}s^n + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(a_{m,n}\cos m\theta + b_{m,n}\sin m\theta)s^n \, . \tag{8}$$

By some elementary calculations we obtain [6],

$$
\begin{aligned}
I_1 &= 2\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{a_{0,n}a_{0,m}}{(n+2)(n+m+4)^3},\\
I_2 &= \frac{1}{4}\sum_{k=2}^{\infty}\left(\frac{1}{k}\sum_{n=1}^{k-1}\frac{1}{n}\right)\sum_{m=1}^{\infty}\sum_{l=1}^{\infty}\frac{a_{k,m}a_{k,l}+b_{k,m}b_{k,l}}{(k+m+2)(m+l+4)}\\
&+\frac{1}{2}\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\sum_{l=0}^{\infty}\frac{a_{0,m}a_{0,l}}{n^2(2n+m+2)(m+l+4)}\\
&+\frac{1}{4}\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\sum_{m=1}^{\infty}\sum_{l=1}^{\infty}\frac{a_{k,m}a_{k,l}+b_{k,m}b_{k,l}}{n(k+n)(2n+k+m+2)(m+l+4)},\\
I_3 &= \frac{1}{4}\sum_{k=2}^{\infty}\left(\frac{1}{k}\sum_{n=1}^{k-1}\frac{1}{n}\right)\sum_{m=1}^{\infty}\sum_{l=1}^{\infty}\frac{a_{k,m}a_{k,l}+b_{k,m}b_{k,l}}{(k+m+2)(2k+m+l+4)}\\
&+\frac{1}{2}\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\sum_{l=0}^{\infty}\frac{a_{0,m}a_{0,l}}{n^2(2n+m+2)(4n+m+l+4)}\\
&+\frac{1}{4}\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\sum_{m=1}^{\infty}\sum_{l=1}^{\infty}\frac{a_{k,m}a_{k,l}+b_{k,m}b_{k,l}}{n(k+n)(2n+k+m+2)(4n+2k+m+l+4)},\qquad(9)\\
I_4 &= \frac{1}{2}\sum_{k=1}^{\infty}\sum_{m=1}^{\infty}\sum_{l=1}^{\infty}\frac{a_{k,m}a_{k,l}+b_{k,m}b_{k,l}}{k(k+m+2)(m+l+4)^2},\\
I_5 &= \frac{1}{2}\sum_{k=1}^{\infty}\sum_{m=1}^{\infty}\sum_{l=1}^{\infty}\frac{a_{k,m}a_{k,l}+b_{k,m}b_{k,l}}{k(k+m+2)(2k+m+l+4)^2},\\
I_6 &= \sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\sum_{l=0}^{\infty}\frac{a_{0,m}a_{0,l}}{n^2(2n+m+2)(2n+m+l+4)}\\
&+\frac{1}{4}\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\sum_{m=1}^{\infty}\sum_{l=1}^{\infty}\frac{a_{k,m}a_{k,l}+b_{k,m}b_{k,l}}{n(k+n)(2n+k+m+2)(2n+m+l+4)}\\
&+\frac{1}{4}\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\sum_{m=1}^{\infty}\sum_{l=1}^{\infty}\frac{a_{k,m}a_{k,l}+b_{k,m}b_{k,l}}{n(k+n)(2n+k+m+2)(2k+2n+m+l+4)}\\
&+\frac{1}{4}\sum_{n=1}^{k-1}\sum_{k=2}^{\infty}\sum_{m=1}^{\infty}\sum_{l=1}^{\infty}\frac{a_{k,m}a_{k,l}+b_{k,m}b_{k,l}}{n(k-n)(k+m+2)(2k-2n+m+l+4)}\ .
\end{aligned}
$$

Some simplifying manipulations lead to the following result.

Theorem 3. *For the eigenvalues of* (1) *the following equality holds*

$$\sum_{j=1}^{\infty}\frac{1}{\lambda_j^2}=\sum_{m=0}^{\infty}\sum_{l=0}^{\infty}(A_{m,l}+B_{m,l})\cdot a_{0,m}a_{0,l}$$
$$+\sum_{k=1}^{\infty}\sum_{m=1}^{\infty}\sum_{l=1}^{\infty}(C_{k,m,l}+D_{k,m,l})(a_{k,m}a_{k,l}+b_{k,m}b_{k,l})$$
$$+\sum_{k=2}^{\infty}\sum_{m=1}^{\infty}\sum_{l=1}^{\infty}E_{k,m,l}(a_{k,m}a_{k,l}+b_{k,m}b_{k,l})$$

with $a_{k,l}$ *and* $b_{k,l}$ *defined by* (8) *and the coefficients*

$$A_{m,l}=\frac{4}{(m+2)(m+l+4)^3},$$
$$B_{m,l}=\sum_{n=1}^{\infty}\frac{8}{(m+l+4)(2n+m+l+4)(4n+m+l+4)(2n+m+2)},$$
$$C_{k,m,l}=\frac{2}{(k+m+2)(m+l+4)^2(2k+m+l+4)}$$
$$+\frac{2}{(k+m+2)(m+l+4)(2k+m+l+4)^2},$$
$$D_{k,m,l}=\sum_{n=1}^{\infty}\frac{2}{(2n+k+m+2)(m+l+4)(2k+4n+m+l+4)(2n+m+l+4)}$$
$$+\sum_{n=1}^{\infty}\frac{2}{(2n+k+m+2)(m+l+4)(2k+4n+m+l+4)(2k+2n+m+l+4)},$$
$$E_{k,m,l}=\sum_{n=1}^{k-1}\frac{2}{(k+m+2)(m+l+4)(2n+m+l+4)(2k+m+l+4)}\,.$$

2.3. Torsional rigidity. For the torsional rigidity, by Pòlya and Schiffer [9, p. 330], we have

$$P=4\int_U\int_U G(z,\zeta)|f'(z)|^2|f'(\zeta)|^2\,dA_z\,dA_\zeta.$$

By similar calculations as in the previous subsection, we obtain

Theorem 4. *The torsional rigidity is given by*

$$\begin{aligned}P&=16\pi\sum_{m=0}^{\infty}\sum_{l=0}^{\infty}\frac{a_{0,m}a_{0,l}}{(m+2)(m+l+4)^2}\\&\quad+8\pi\sum_{k=1}^{\infty}\sum_{m=1}^{\infty}\sum_{l=1}^{\infty}\frac{a_{k,m}a_{k,l}+b_{k,m}b_{k,l}}{(k+m+2)(m+l+4)(2k+m+l+4)}.\end{aligned}\tag{10}$$

Another formula for the torsional rigidity in terms of the conformal mapping is given by Pòlya and Szegö [10, p. 120].

3. Free membrane problem. For the free membrane problem (2) the situation is more involved. The following theorem and lemma have been proved in [3].

Theorem 5. *Let $\mu_k^{(o)}$ be the eigenvalues of the free membrane problem in the unit disk and let μ_k the free membrane eigenvalues of the domain D. Then, for any $n \geq 2$ we have*

$$\sum_{k=2}^{n} \frac{1}{\mu_k} \geq \frac{\int_U \tilde{u}_k^2 |f'(z)|^2 dA}{\mu_k^{(o)}}.$$

Lemma 4. *For a radial eigenfunction $v_k^{(o)}$ we have*

$$\begin{aligned}\int_U \tilde{u}_k^2 |f'(z)|^2 dA &= \int_U {v_k^{(o)}}^2 |f'(z)|^2 dA - \frac{1}{A}\left(\int_U v_k^{(o)} |f'(z)|^2 dA\right)^2 \\ &\geq 1 + \frac{\pi}{A} \sum_2^{\infty} j^2 |a_j|^2 \int_U {v_k^{(o)}}^2 r^{2j-2} dA.\end{aligned}$$

For a non-radial eigenfunction $v_k^{(o)}$ we take the eigenfunctions $v_k^{(o)}$ and $v_{k+1}^{(o)}$ belonging to the same eigenvalue and have

$$\begin{aligned}\int_U (\tilde{u}_k^2 + \tilde{u}_{k+1}^2) |f'(z)|^2 dA &= \int_U \left({v_k^{(o)}}^2 + {v_{k+1}^{(o)}}^2\right) |f'(z)|^2 dA \\ &\quad - \frac{1}{A}\left(\int_U v_k^{(o)} |f'(z)|^2 dA\right)^2 - \frac{1}{A}\left(\int_U v_{k+1}^{(o)} |f'(z)|^2 dA\right)^2 \\ &\geq 2 + \sum_2^{k} j^2 |a_j|^2 \int_U \left({v_k^{(o)}}^2 + {v_{k+1}^{(o)}}^2\right) r^{2j-2} |f'(z)|^2 dA.\end{aligned}$$

These results yield an inequality similar to the isoperimetric inequality given in Theorem 1.

3.1. Formulas for the sum of all reciprocals. Analogously to the procedure in Subsection 2.2 we now want to present a formula for the sum of all reciprocal eigenvalues of the free membrane problem (2). We know [3],

$$\begin{aligned}\sum_{j=2}^{\infty} \frac{1}{\mu_j^2} &= \int_U \int_U N^2(z,\zeta) |f'(z)|^2 |f'(\zeta)|^2 \, dA_z \, dA_\zeta + A^2 C^2 \\ &\quad - \frac{2}{A} \int_U \left(\int_U N(z,\zeta) |f'(\zeta)|^2 \, dA_\zeta\right)^2 |f'(z)|^2 \, dA_z\end{aligned} \tag{11}$$

with $A = \int_U |f'(z)|^2 \, dA_z$ the area of domain D. Furthermore, the constant C is given by $C = \frac{1}{A^2} \int_U \int_U N(z,\zeta) |f'(z)|^2 |f'(\zeta)|^2 \, dA_z \, dA_\zeta$. The Neumann's function

$$N(z,\zeta) = \frac{1}{2\pi}(-\ln|z-\zeta| - \ln|1 - z\bar{\zeta}|)$$

of the unit disk differs from the unit disk Green's function only in a sign of the second part. Therefore, we simply obtain

$$\int_U \int_U N^2(z,\zeta)|f'(z)|^2|f'(\zeta)|^2\, dA_z\, dA_\zeta = I_1 + I_2 + I_3 + I_4 + I_5 + I_6\,, \tag{12}$$

where we can use previous results. For the area of D we obtain

$$A = 2\pi \sum_{n=0}^{\infty} \frac{a_{0,n}}{2+n}\,. \tag{13}$$

Besides putting $z = re^{i\phi}$, we get

$$\begin{aligned} -\int_U N(z,\zeta)|f'(\zeta)|^2\, dA_\zeta &= -\sum_{n=0}^{\infty} \frac{a_{0,n}}{(n+2)^2} + \sum_{n=0}^{\infty} \frac{a_{0,n}}{(n+2)^2} r^{n+2} \\ &+ \sum_{\substack{m=1 \\ m\neq n-2}}^{\infty} \sum_{n=1}^{\infty} r^n \frac{m+2}{n(m-n+2)(m+n+2)} (a_{n,m}\cos n\phi + b_{n,m}\sin n\phi) \\ &- \sum_{\substack{m=1 \\ m\neq n-2}}^{\infty} \sum_{n=1}^{\infty} \frac{r^{m+2}}{(m+n+2)(m-n+2)} (a_{n,m}\cos n\phi + b_{n,m}\sin n\phi) \\ &+ \frac{1}{2}\sum_{n=3}^{\infty} \frac{r^n}{n^2}(a_{n,n-2}\cos n\phi + b_{n,n-2}\sin n\phi) \\ &- \frac{1}{2}\sum_{n=3}^{\infty} \frac{r^n}{n} \ln r (a_{n,n-2}\cos n\phi + b_{n,n-2}\sin n\phi) =: H(r,\phi)\,. \end{aligned} \tag{14}$$

Next, we find

$$\begin{aligned} A^2 C = -A \sum_{n=0}^{\infty} \frac{a_{0,n}}{(n+2)^2} &+ 2\pi \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{a_{0,n}a_{0,m}}{(n+2)^2(n+m+4)} \\ &+ \pi \sum_{\substack{m=1 \\ m\neq n-2}}^{\infty} \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \frac{(m+2)(a_{n,m}+b_{n,m})(a_{n,k}+b_{n,k})}{n(m-n+2)(m+n+2)(n+k+2)} \\ &+ \pi \sum_{\substack{m=1 \\ m\neq n-2}}^{\infty} \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \frac{(a_{n,m}+b_{n,m})(a_{n,k}+b_{n,k})}{(m+n+2)(m-n+2)(m+k+4)} \\ &+ \frac{\pi}{2}\sum_{n=3}^{\infty}\sum_{m=1}^{\infty} \frac{(a_{n,n-2}+b_{n,n-2})(a_{n,m}+b_{n,m})}{n^2(n+m+2)} \\ &+ \frac{\pi}{2}\sum_{n=3}^{\infty}\sum_{m=1}^{\infty} \frac{(a_{n,n-2}+b_{n,n-2})(a_{n,m}+b_{n,m})}{n(n+m+2)^2}\,. \end{aligned} \tag{15}$$

Summarizing, we obtain the following result.

Theorem 6. *For the eigenvalues of* (2) *it holds*

$$\sum_{j=2}^{\infty}\frac{1}{\mu_j^2}=\sum_{j=1}^{6}I_j+\frac{1}{A^2}(A^2C)^2+\int_0^1\int_0^{2\pi}H(r,\phi)^2\left(\sum_{n=0}^{\infty}a_{0,n}r^{(n+1)}\right.$$
$$\left.+\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(a_{m,n}\cos m\phi+a_{m,n}\sin m\phi)r^{(n+1)}\right)d\phi\,dr$$

with $a_{k,l}$ *and* $b_{k,l}$ *according to* (8) *and results in* (9), (13), (14), (15).

4. Examples.

4.1. Unit disk. For the unit disk we have $|f'|^2\equiv 1$. We obtain (compare [2, 2.27])

$$\sum_{j=1}^{\infty}\frac{1}{\lambda_j^{(0)^2}}=\frac{\pi^2}{48}-\frac{5}{32}$$

for the Dirichlet eigenvalues. For the Neumann problem we have [2, (3.47)],

$$\int_U\int_U N^2(z,\zeta)\,dA_z\,dA_\zeta=\frac{5}{192}(2\pi^2-15)$$

and finally we get

$$\sum_{j=2}^{\infty}\frac{1}{\mu_j^{(0)^2}}=\frac{5}{96}\pi^2-\frac{5}{12}\ .$$

Remark 2. By an analogous calculation we obtain [6]

$$\int_U\int_U\left(\int_U G(z,\eta)G(\zeta,\eta)\,dA_\eta\right)^2dA_\zeta\,dA_z=\sum_{j=1}^{\infty}\frac{1}{\lambda_j^{(0)^4}}$$
$$=\frac{1}{64}\left(\frac{\pi^4}{180}+\frac{5\pi^2}{18}-\frac{3491}{1728}-\zeta(3)\right),$$

where ζ denotes the ζ-function.

For the torsional rigidity we obtain [10],

$$P=\frac{\pi}{2}\,.$$

4.2. Cardioid and similar domains. For the conformal mapping

$$f_n(z)=z+\frac{1}{n}z^n\qquad n\in N\,,\ n\geq 2$$

we get the image domains in the following Figure 1.

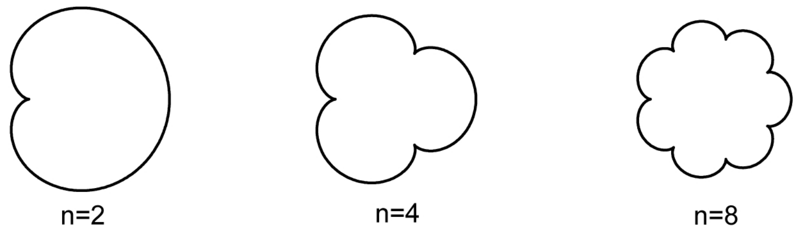

FIGURE 1. Images for $f_n(z)$

The function series

$$|f_n'(s,\theta)|^2 = 1 + s^{2(n-1)} + 2\cos(n-1)\theta s^{n-1}$$

are finite with $a_{0,0}^{(n)} = 1$, $a_{0,2(n-1)}^{(n)} = 1$ and $a_{n-1,n-1}^{(n)} = 2$. Therefore, we can explicitly calculate the sums (4) and (11). The results are given in Table 1.

n	$\sum_{j=1}^{\infty} \frac{1}{\lambda_j^2(n)}$	$\sum_{j=2}^{\infty} \frac{1}{\mu_j^2(n)}$
2	$\frac{3}{64}\pi^2 - \frac{551}{1536}$	$-\frac{349033}{414720} + \frac{15}{128}\pi^2$
3	$\frac{1}{27}\pi^2 - \frac{3817}{12960}$	$-\frac{5681479}{8709120} + \frac{5}{54}\pi^2$
4	$\frac{25}{768}\pi^2 - \frac{673343}{2580480}$	$-\frac{42418627}{71680000} + \frac{125}{1536}\pi^2$
5	$\frac{3}{100}\pi^2 - \frac{26917}{112000}$	$-\frac{277921733}{498960000} + \frac{3}{40}\pi^2$
6	$\frac{49}{1728}\pi^2 - \frac{54228619}{239500800}$	$-\frac{42228865589}{79106227200} + \frac{245}{3456}\pi^2$
7	$\frac{4}{147}\pi^2 - \frac{107001247}{494413920}$	$-\frac{8182651637}{15821245440} + \frac{10}{147}\pi^2$
8	$\frac{27}{1024}\pi^2 - \frac{43165500691}{206644838400}$	$-\frac{52752511370387}{104528550297600} + \frac{135}{2048}\pi^2$
9	$\frac{25}{972}\pi^2 - \frac{2030899264883}{10003708915200}$	$-\frac{14109961720724663}{28510570408320000} + \frac{125}{1944}\pi^2$
10	$\frac{121}{4800}\pi^2 - \frac{6647785909789}{33522128640000}$	$-\frac{8692886851705487}{17847181287936000} + \frac{121}{1920}\pi^2$
11	$\frac{3}{121}\pi^2 - \frac{607345599478309}{3123256725388800}$	$-\frac{34528128346868561}{71834904683942400} + \frac{15}{242}\pi^2$
12	$\frac{169}{6912}\pi^2 - \frac{3269956373169173}{17097894668574720}$	$-\frac{178545970440593961307}{375640745868586598400} + \frac{845}{13824}\pi^2$

TABLE 1. Sums for cardioid and related domains, $f_n(z) = z + \frac{1}{n}z^n$

Furthermore, we obtain the following monotonicity result.

Theorem 7. *Consider the conformal mapping* $f_n(z) = z + \frac{1}{n} z^n$, $n \geq 2$ *of the unit disk onto the domain* D_n *with the Dirichlet eigenvalues* $\lambda_j(n)$ *of* D_n. *Then the sum* $\sum_{j=1}^{\infty} \frac{1}{\lambda_j^2(n)}$ *is strictly decreasing in* n.

Proof. By some elementary estimates we obtain for $n \geq 3$,

$$\sum_{j=1}^{\infty} \frac{1}{\lambda_j^2(n)} - \sum_{j=1}^{\infty} \frac{1}{\lambda_j^2(n-1)} \geq \frac{40n^5 + 36n^4 - 78n^3 - 88n^2 - 23n + 2}{32n^4(n+1)^4(n+2)} > 0\,.$$

This proves our claim. For details on the estimates see [6]. □

The area of the image domains D_n is equal to $\pi(1+\frac{1}{n})$. Now we normalize the conformal mapping f_n, such that the area of the image is π. The normalized conformal mapping is given by

$$\tilde{f}_n(z) = \sqrt{\frac{n}{n+1}} \left(z + \frac{1}{n} z^n \right) .$$

The corresponding results for the eigenvalues for the Dirichlet and Neumann problem are given in Table 2.

n	$\sum_{j=1}^{\infty} \frac{1}{\tilde{\lambda}_j^2(n)}$	$\sum_{j=2}^{\infty} \frac{1}{\tilde{\mu}_j^2(n)}$
2	$\frac{1}{48}\pi^2 - \frac{551}{3456}$	$-\frac{349033}{933120} + \frac{5}{96}\pi^2$
3	$\frac{1}{48}\pi^2 - \frac{3817}{23040}$	$-\frac{5681479}{15482880} + \frac{5}{96}\pi^2$
4	$\frac{1}{48}\pi^2 - \frac{673343}{403200}$	$-\frac{42418627}{112000000} + \frac{5}{96}\pi^2$
5	$\frac{1}{48}\pi^2 - \frac{26917}{161280}$	$-\frac{277921733}{718502400} + \frac{5}{96}\pi^2$
6	$\frac{1}{48}\pi^2 - \frac{54228619}{325987200}$	$-\frac{42228865589}{107672364800} + \frac{5}{96}\pi^2$
7	$\frac{1}{48}\pi^2 - \frac{107001247}{645765120}$	$-\frac{8182651637}{20664483840} + \frac{5}{96}\pi^2$
8	$\frac{1}{48}\pi^2 - \frac{43165500691}{261534873600}$	$-\frac{52752511370387}{132293946470400} + \frac{5}{96}\pi^2$
9	$\frac{1}{48}\pi^2 - \frac{2030899264883}{12350257920000}$	$-\frac{14109961720724663}{35198235072000000} + \frac{5}{96}\pi^2$
10	$\frac{1}{48}\pi^2 - \frac{6647785909789}{40561775654400}$	$-\frac{8692886851705487}{21595089358402560} + \frac{5}{96}\pi^2$
11	$\frac{1}{48}\pi^2 - \frac{607345599478309}{3716933623603200}$	$-\frac{34528128346868561}{85489473342873600} + \frac{5}{96}\pi^2$
12	$\frac{1}{48}\pi^2 - \frac{3269956373169173}{20066279159646720}$	$-\frac{178545970440593961307}{440856153137438438400} + \frac{5}{96}\pi^2$

TABLE 2. Sums for normalized cardioid and related domains, $\tilde{f}_n(z) = \sqrt{\frac{n}{n+1}} \left(z + \frac{1}{n} z^n\right)$

We find the following monotonicity property for the eigenvalues.

Theorem 8. *Consider the conformal mapping* $\tilde{f}_n(z) = \sqrt{\frac{n}{n+1}}\left(z + \frac{1}{n}z^n\right)$*,* $n \geq 2$ *of the unit disk onto the domain* $\tilde{D}_n$ *with the Dirichlet eigenvalues* $\tilde{\lambda}_j(n)$ *of* $\tilde{D}_n$*. Then the sum* $\sum_{j=1}^{\infty} \frac{1}{\tilde{\lambda}_j^2(n)}$ *is strictly increasing in* n*.*

The proof is analogous to the previous one. We know from Luttinger [8] that the sum in the theorem is the greatest if the domain is a disk of area π.

Finally we find the torsional rigidity P for the cardioid and related domains D_n which are the images by the conformal mappings f_n,

$$P(D_n) = \frac{\pi}{2}\left(1 + \frac{4}{n^2} + \frac{1}{n^3}\right).$$

For the normalized conformal mapping $\tilde{f}_n$ we obtain the domains $\tilde{D}_n$ and

$$P(\tilde{D}_n) = \frac{\pi}{2}\left(\frac{n^2}{(n+1)^2} + \frac{4}{(n+1)^2} + \frac{1}{n(n+1)^2}\right).$$

It is easy to check that the torsional rigidity $P(D_n)$ is strictly decreasing in n, $n \geq 2$, and the torsional rigidity $P(\tilde{D}_n)$ strictly increasing in n for $n \geq 5$.

4.3. Regular polygons. Let g_n denote the conformal mapping of the unit disk onto the regular polygon E_n with maximal conformal radius 1, $n \geq 3$. Then the non-trivial coefficients of $|g_n'(z)|^2$ are given by

$$a_{0,2nj} = \left(\prod_{k=0}^{j-1} \frac{nk+2}{nk+n}\right)^2$$

and

$$a_{nm,2nj+nm} = 2\left(\prod_{k=0}^{j-1} \frac{nk+2}{nk+n}\right)\left(\prod_{k=0}^{j+m-1} \frac{nk+2}{nk+n}\right),$$

for details see [6, p. 61–65]. We obtain that the torsional rigidity $P(E_n)$ of E_n is strictly decreasing in n for $n \geq 3$.

References

[1] Bandle, C., *Isoperimetric Inequalities and Applications*, Pitman Publ., London, 1980.

[2] Dittmar, B., *Sums of reciprocal eigenvalues of the Laplacian*, Math. Nachr. **237** (2002), 45–61.

[3] Dittmar, B., *Sums of free membrane eigenvalues*, J. Anal. Math. **95** (2005), 323–332.

[4] Dittmar, B., *Eigenvalue problems and conformal mapping*, R. Kühnau (ed.), Handbook of Complex Analysis: Geometric Function Theory. Vol. 2, Elsevier, Amsterdam, 2005, pp. 669–686.

[5] Dittmar, B., *Free membrane eigenvalues*, Z. Angew. Math. Phys. **60** (2009), 565–568.

[6] Hantke, M., *Summen reziproker Eigenwerte*, Dissertation Martin-Luther-Universität, Halle–Wittenberg, 2006.

[7] Henrot, A., *Extremum problems for eigenvalues of elliptic operators*, Birkäuser, Basel–Boston–Berlin, 2006.
[8] Luttinger, J. M., *Generalized isoperimetric inequalities*, J. Mathematical Phys. **14** (1973), 586–593, ibid. **14** (1973), 1444–1447, ibid. **14** (1973), 1448–1450.
[9] Pólya, G., Schiffer, M., *Convexity of functionals by transplantation*, J. Analyse Math. **3** (1954), 245–345.
[10] Pólya, G., Szegö, G., *Isoperimetric Inequalities in Mathematical Physics*, Princeton University Press, Princeton, N. J., 1951.

Bodo Dittmar
Institut für Analysis
Martin-Luther-Universität
Halle-Wittenberg
D-06099 Halle (Saale)
Germany
e-mail: `bodo.dittmar@mathematik.uni-halle.de`

Maren Hantke
Institut für Analysis und Numerik
Otto-von-Guericke-Universität Magdeburg
D-39016 Magdeburg
Germany
e-mail: `maren.hantke@ovgu.de`

Received May 17, 2011