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Möbius invariant Besov spaces on the unit ball of $\mathbb{C}^n$

Dedicated to the memory of Professor Jan G. Krzyż

Abstract. We give new characterizations of the analytic Besov spaces B_p on the unit ball $\mathbb{B}$ of $\mathbb{C}^n$ in terms of oscillations and integral means over some Euclidian balls contained in $\mathbb{B}$.

1. Introduction. Let $\mathbb{B} = \{z \in \mathbb{C}^n : |z| < 1\}$ denote the open unit ball in $\mathbb{C}^n$ and $H(\mathbb{B})$ be the set of all holomorphic functions on $\mathbb{B}$. By $Aut(\mathbb{B})$ we mean the group of all automorphisms of $\mathbb{B}$. It is known that $Aut(\mathbb{B})$ is generated by the unitary operators and involutions of the form

$$\varphi_w(z) = \frac{w - P_w(z) - s_w Q_w(z)}{1 - \langle z, w\rangle},$$

where $w \in \mathbb{B}$, $s_w = (1 - |w|^2)^{1/2}$, P_w is the orthogonal projection of $\mathbb{C}^n$ to the subspace spanned by w, i.e.

$$P_w(z) = \frac{\langle z, w\rangle}{|w|^2} w \quad \text{for } w \neq 0, \quad \text{and } P_0(z) = 0,$$

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and $Q_w = I - P_w$ (see, e.g. [13, 16] for definition and properties of the automorphism group of $\mathbb{B}$). The mapping φ_a is called the Möbius transformation. It is known that $\rho(z,w) = |\varphi_z(w)|$ is a metric on $\mathbb{B}$, the so-called pseudo-hyperbolic metric (see, e.g. [9, 15, 16]).

Let dv be the Lebesgue measure on $\mathbb{B}$ normalized so that $v(\mathbb{B}) = 1$ and let $d\tau(z) = \frac{dv(z)}{(1-|z|^2)^{n+1}}$ be the invariant measure on $\mathbb{B}$.

For $f \in H(\mathbb{B})$, set

$$Q_f(z) = \sup_{0 \neq x \in \mathbb{C}^n} \frac{|\langle \nabla f(z), \overline{x} \rangle|}{H_z(x,x)^{1/2}}, \quad z \in \mathbb{B},$$

where $\nabla f(z) = (\partial f/\partial z_1, \partial f/\partial z_2, \dots, \partial f/\partial z_n)$ is the complex gradient of f and $H_z(x,x)$ is the Bergman metric on $\mathbb{B}$, that is

$$H_z(x,x) = \frac{n+1}{2} \frac{(1-|z|^2)|x|^2 + |\langle x, z \rangle|^2}{(1-|z|^2)^2}.$$

The Möbius invariant Besov space B_p, $1 < p \leq \infty$, consists of all holomorphic functions on $\mathbb{B}$ for which $Q_f \in L^p(\mathbb{B}, d\tau)$. In the case $p = \infty$ the space B_∞ is the Bloch space $\mathcal{B}$; so

$$\mathcal{B} = B_\infty = \{f \in H(\mathbb{B}) : \|f\|_{\mathcal{B}} < \infty\},$$

where

$$\|f\|_{\mathcal{B}} = \sup_{z \in \mathbb{B}} Q_f(z).$$

If $1 < p \leq \infty$ the space B_p is the Banach space with the norm

$$\|f\|_{B_p} = |f(0)| + (p-1)\|Q_f\|_{L^p(d\tau)}.$$

Hahn and Youssfi [3] proved that for $n > 1$ the Besov space B_p is nontrivial and contains all polynomials if and only if $p > 2n$. Moreover, it is known that for $f \in H(\mathbb{B})$, the following conditions are equivalent

(i) $f \in B_p$,
(ii) $|\nabla f(z)|(1-|z|^2) \in L^p(\mathbb{B}, d\tau)$,
(iii) $|\widetilde{\nabla} f(z)| \in L^p(\mathbb{B}, d\tau)$ where $|\widetilde{\nabla} f(z)| = |\nabla (f \circ \varphi_z)(0)|$.

The proofs can be found in [3, 8, 16].

The following results for the space B_p are reminiscences of Holland and Walsh characterization of the Bloch space [6].

In the case $n = 1$ Stroethoff [14] proved that for $2 < p < \infty$,

$$f \in B_p \Leftrightarrow \int_{\mathbb{B}} \int_{\mathbb{B}} \left| \frac{f(z) - f(w)}{z - w} \right|^p (1-|z|^2)^{\frac{p}{2}} (1-|w|^2)^{\frac{p}{2}} d\tau(w) d\tau(z) < \infty.$$

This equivalence has been generalized to the unit ball case in [8], where the following result has been obtained. If $2n < p < \infty$, then

$$f \in B_p \Leftrightarrow$$

$$\int_{\mathbb{B}}\int_{\mathbb{B}}\left(\frac{|f(z)-f(w)|}{|w-P_w(z)-s_wQ_w(z)|}\right)^p(1-|z|^2)^{\frac{p}{2}}(1-|w|^2)^{\frac{p}{2}}d\tau(w)d\tau(z)<\infty. \tag{1}$$

Let $B(a,r)$ denote a Euclidian ball of radius r and centered at $a \in \mathbb{C}^n$. For $a \in \mathbb{B}$ and $0 < r < 1$ let

$$E(a,r) = \{z \in \mathbb{B} : |\varphi_a(z)| < r\} = \varphi_a(B(0,r))$$

be the pseudo-hyperbolic (or Bergman) metric ball centered at z. Then $E(a,r)$ is an elipsoid in $\mathbb{C}^n$. We will often use the following property of $E(a,r)$.

There exists a positive constant C (dependent on r, but not on z and a) such that

$$C^{-1}(1-|z|^2) \le |1-\langle z,a\rangle| \le C(1-|a|^2) \tag{2}$$

for all $z \in E(a,r)$.

Using equivalence (1), we easily obtain the following

Theorem 1. *Assume that $f \in H(\mathbb{B})$ and $2n < p < \infty$. Then $f \in B_p$ if and only if*

$$\int_{\mathbb{B}}\int_{\mathbb{B}}\left(\frac{|f(z)-f(w)|}{|1-\langle z,w\rangle|}\right)^p(1-|z|^2)^{\frac{p}{2}}(1-|w|^2)^{\frac{p}{2}}d\tau(w)d\tau(z)<\infty. \tag{3}$$

Proof. Assume that for $f \in H(\mathbb{B})$ condition (3) is satisfied. Since

$$|\widetilde{\nabla} f(z)|^p \le C\int_{E(z,r)}\frac{|f(w)-f(z)|^p}{|1-\langle w,z\rangle|^{n+1}}dv(w), \quad \text{(see, e.g. [8])}$$

and for $w \in E(z,r)$,

$$1-r^2 < 1-|\varphi_z(w)|^2 = \frac{(1-|z|^2)(1-|w|^2)}{|1-\langle w,z\rangle|^2}, \tag{4}$$

we get, using (2),

$$\begin{aligned}
&\int_{\mathbb{B}}|\widetilde{\nabla} f(z)|^p d\tau(z)\\
&\quad\le C\int_{\mathbb{B}}\int_{E(z,r)}\frac{|f(w)-f(z)|^p}{|1-\langle w,z\rangle|^{n+1}}\frac{(1-|z|^2)^{p/2}(1-|w|^2)^{p/2}}{|1-\langle w,z\rangle|^p}dv(w)d\tau(z)\\
&\quad\le C\int_{\mathbb{B}}\int_{E(z,r)}\frac{|f(w)-f(z)|^p}{(1-|w|^2)^{n+1}}\frac{(1-|z|^2)^{p/2}(1-|w|^2)^{p/2}}{|1-\langle w,z\rangle|^p}dv(w)d\tau(z)\\
&\quad\le C\int_{\mathbb{B}}\int_{\mathbb{B}}\frac{|f(w)-f(z)|^p(1-|z|^2)^{p/2}(1-|w|^2)^{p/2}}{|1-\langle w,z\rangle|^p}d\tau(w)d\tau(z).
\end{aligned}$$

Hence (3) implies $f \in B_p$. The other implication follows from (1) and from the inequality

$$|w - P_w(z) - s_w Q_w(z)| \leq |1 - \langle z, w\rangle|, \quad z, w \in \mathbb{B}. \qquad \square$$

For $\alpha > -1$ we define the weighted volume measure $dv_\alpha(z) = c_\alpha(1 - |z|^2)^\alpha dv(z)$, where c_α is a positive constant such that $v_\alpha(\mathbb{B}) = 1$.

We remark that condition (3) can be written in the form

$$\int_{\mathbb{B}}\int_{\mathbb{B}} \left(\frac{|f(z) - f(w)|}{|1 - \langle z, w\rangle|}\right)^p dv_\alpha(z)dv_\alpha(w) < \infty,$$

where $\alpha = -n - 1 + p/2$.

Moreover, the inequality

$$|w - P_w(z) - s_w Q_w(z)| \leq |z - w|, \quad z, w \in \mathbb{B},$$

and equivalence (3) imply that if $f \in B_p$, then

$$\int_{\mathbb{B}}\int_{\mathbb{B}} \left(\frac{|f(z) - f(w)|}{|z - w|}\right)^p dv_\alpha(z)dv_\alpha(w) < \infty, \quad \alpha = -n - 1 + p/2. \tag{5}$$

We do not know if condition (5) is sufficient for f to belong to B_p. The sufficiency of (5) has been claimed in [4]. Unfortunately, the proof given there is not correct.

For $p = \infty$, condition (5) is understood as

$$\|f\|_{\tilde{\mathcal{B}}} = \sup_{z,w \in \mathbb{B}, z \neq w} \frac{|f(z) - f(w)|}{|z - w|}(1 - |z|^2)^{\frac{1}{2}}(1 - |w|^2)^{\frac{1}{2}} < \infty$$

and is necessary and sufficient for containment in the Bloch space $\mathcal{B}$ as shown in [12]. For the proof of the last result the authors [12] used the so-called conformal Möbius transformation. We also will discuss this transformation in the next section.

Recently, M. Pavlović [10, 11] considered a more general space of C^1 functions in the unit ball for which two Bloch norms can be defined as follows

$$\|f\|_{\mathcal{B}_1} = \sup_{x \in \mathbb{B}}(1 - |x|^2)\|df(x)\|, \tag{6}$$

$$\|f\|_{\mathcal{B}_2} = \sup_{x \in \mathbb{B}} \|\tilde{d}f(x)\|, \tag{7}$$

where $\|df(x)\|$ is the norm of the differential of f at x and $\|\tilde{d}f(x)\| = \|d(f \circ \varphi_x)(0)\|$. It is proved in [10, 11] that

$$\|f\|_{\mathcal{B}_1} = \|f\|_{\tilde{\mathcal{B}}}$$

and

$$\|f\|_{\mathcal{B}_2} = \sup_{z,w \in \mathbb{B}, z \neq w} \frac{|f(z) - f(w)|}{|w - P_w(z) - s_w Q_w(z)|}(1 - |z|^2)^{\frac{1}{2}}(1 - |w|^2)^{\frac{1}{2}}.$$

Here we get one more criterion for containment in the Bloch space. Namely, if $f \in H(\mathbb{B})$, then

$$f \in \mathcal{B} \Leftrightarrow \sup_{z,w\in\mathbb{B},z\neq w} \frac{|f(z)-f(w)|}{|1-\langle z,w\rangle|}(1-|z|^2)^{\frac{1}{2}}(1-|w|^2)^{\frac{1}{2}} < \infty.$$

Finally, it is worth noting that characterizations of weighted Bergman spaces on the unit ball in terms of double integrals of the functions $|f(z)-f(w)|/|1-\langle z,w\rangle|$ and $|f(z)-f(w)|/|z-w|$ have been recently obtained in [5] and [7].

2. Characterizations in terms of oscillation and integral means. For $f \in H(\mathbb{B})$, $z \in \mathbb{B}$ and $0 < r < 1$ we put

$$\omega_r(f)(z) = \sup\{|f(z)-f(w)| : \ w \in E(z,r)\}$$

and

$$MO_r(f)(z) = \frac{1}{v(E(z,r))}\int_{E(z,r)} |f(w)-f_{z,r}|dv(w),$$

where

$$f_{z,r} = \frac{1}{v(E(z,r))}\int_{E(z,r)} f(u)dv(u).$$

$\omega_r(f)$ and $MO_r(f)$ are, respectively, the oscillation and the mean oscillation of f in the Bergman metric at the point z.

The following characterizations of the space B_p in terms of $\omega_r(f)$ and $MO_r(f)$ can be found in [16].

Theorem A. *Let $f \in H(\mathbb{B})$ and $2n < p$, and $0 < r < 1$. Then the following conditions are equivalent*

(i) $f \in B_p$,
(ii) $\omega_r(f) \in L^p(\mathbb{B}, d\tau)$,
(iii) $MO_r(f) \in L^p(\mathbb{B}, d\tau)$.

We will prove similar characterizations of B_p in terms of oscillations, but in a different metric. The metric will be connected with the conformal Möbius transformation on $\mathbb{B}$ given by

$$\varphi_a^c(z) = \frac{|z-a|^2 a - (1-|a|^2)(z-a)}{||a|z-a'|^2},$$

where $a \in \mathbb{B}$, $a' = \frac{a}{|a|}$ for $a \neq 0$ and $a' = (1,0,\ldots,0)$, when $a = 0$. The mapping φ_a^c is an involution automorphism of $\mathbb{B}$ such that $\varphi_a^c(0) = a$ and $\varphi_a^c(a) = 0$. Moreover,

$$|\varphi_a(z)| \leq |\varphi_a^c(z)|, \quad a, z \in \mathbb{B}.$$

Also, it is easy to check that

$$\frac{1-|\varphi_a^c(z)|^2}{|\varphi_a^c(z)|^2} = \frac{(1-|z|^2)(1-|a|^2)}{|z-a|^2}. \tag{8}$$

We refer the reader to [1] and [12] for further properties of φ_a^c.

Analogously to the Möbius transformations case, the formula $\rho^c(a,z) = |\varphi_a^c(z)|$ defines a metric on $\mathbb{B}$. We give the proof of this fact, probably known, because we do not know a reference. By the definition of φ_a^c, we get

$$|\varphi_a^c(z)| = \frac{|z-a|}{||a|z-a'|} = \frac{|a-z|}{||z|a-z'|} = |\varphi_z^c(a)|.$$

It is also obvious that

$$|\varphi_a^c(z)| = 0 \ \Leftrightarrow \ z = a.$$

The invariance of $\rho^c(a,z)$ under the conformal Möbius transformations follows immediately from formula (38) in [1]. So, we have

$$\rho^c(a,z) = |\varphi_a^c(z)| = |\varphi^c_{\varphi^c_w(a)}(\varphi^c_w(z))| = \rho^c(\varphi^c_w(a), \varphi^c_w(z)).$$

In view of this, it is enough to show that

$$\rho^c(a,z) \le |a| + |z|. \tag{9}$$

Using the inequality

$$1-(x+y)^2 \le \frac{\left(1-x^2\right)\left(1-y^2\right)}{(1+xy)^2},$$

for $x, y \in [0,1]$, (see, e.g. [15]), we obtain

$$\begin{aligned} 1-(|a|+|z|)^2 &\le \frac{\left(1-|a|^2\right)\left(1-|z|^2\right)}{(1+|a||z|)^2} \\ &\le \frac{\left(1-|a|^2\right)\left(1-|z|^2\right)}{||a|z-a'|^2} = 1-|\varphi_a^c(z)|^2, \end{aligned}$$

which proves (9).

For $a \in \mathbb{B}$ and $0 < r < 1$ let

$$E^c(a,r) = \{z \in \mathbb{B} : |\varphi_a^c(z)| < r\} = \varphi_a^c(B(0,r)).$$

The set $E^c(a,r)$ is a Euclidian ball in $\mathbb{R}^{2n}$ centered at $\frac{(1-r^2)a}{1-r^2|a|^2}$ and of the radius $\frac{(1-|a|^2)r}{1-r^2|a|^2}$. Note that if $z \in B(a, \frac{r}{2}(1-|a|^2))$, then

$$|\varphi_a^c(z)| = \frac{|z-a|}{||a|z-a'|} \le \frac{|z-a|}{|a'|-|a||z|} \le \frac{|z-a|}{1-|a|} \le \frac{2|z-a|}{1-|a|^2} < r.$$

It follows immediately that

$$B\left(a, \frac{r}{2}(1-|a|^2)\right) \subset E^c(a,r) \subset E(a,r). \tag{10}$$

Now, for $f \in H(\mathbb{B})$ and $z \in \mathbb{B}$, we define

$$\omega_r^c(f)(z) = \sup\{|f(z)-f(w)| : \ w \in E^c(z,r)\}$$

and

$$MO_r^c(f)(z) = \frac{1}{v(E^c(z,r))} \int_{E^c(z,r)} |f(w) - f_{z,r}^c| dv(w),$$

where

$$f_{z,r}^c = \frac{1}{v(E^c(z,r))} \int_{E^c(z,r)} f(u) dv(u).$$

We get the following analogue of Theorem A.

Theorem 2. *Let $2n < p < \infty$ and $0 < r < 1$. Then the following statements are equivalent*

(i) $f \in B_p$,
(ii) $\omega_r^c(f) \in L^p(\mathbb{B}, d\tau)$,
(iii) $MO_r^c(f) \in L^p(\mathbb{B}, d\tau)$.

Proof. (i)⇒(ii) If $f \in B_p$, then inclusion (10) and Theorem A imply that $\omega_r^c(f) \in L^p(\mathbb{B}, d\tau)$.
(ii)⇒(iii) Since

$$f(w) - f_{z,r}^c = f(w) - f(z) - (f_{z,r}^c - f(z))$$

and

$$f_{z,r}^c - f(z) = \frac{1}{v(E^c(z,r))} \int_{E^c(z,r)} (f(w) - f(z)) dv(w),$$

we get

$$MO_r^c(f)(z) \le \frac{2}{v(E^c(z,r))} \int_{E^c(z,r)} |f(w) - f(z)| dv(w) \le 2\omega_r^c(f)(z).$$

(iii)⇒(i) It follows from the subharmonicity of $|F|^p$, $F \in H(\mathbb{B})$, that for any $0 < s < 1$, $0 < p < \infty$ and $B(z,s) \subset \mathbb{B}$,

$$|\nabla F(z)|^p s^p \le C s^{-2n} \int_{B(z,s)} |F(w)|^p dv(w), \quad z \in \mathbb{B}. \tag{11}$$

Applying inequality (11) with $s = \frac{r}{2}(1 - |z|^2)$ to the function $F(w) = f(z + w) - f_{z,r}^c$ and using inclusion (10), we see that

$$|\nabla f(z)|(1 - |z|^2) \le C \int_{E^c(z,r)} |f(w) - f_{z,r}^c| \frac{dv(w)}{(1 - |w|^2)^{2n}} \le C MO_r^c(f)(z)$$

and the proof is complete. □

Moreover, we have

Theorem 3. *Assume that $f \in H(\mathbb{B})$, $2n < p < \infty$, $r \in (0,1)$. Then*

$$f \in B_p \;\Leftrightarrow\; \int_{E^c(a,r)} |\nabla f(z)| \frac{dv(z)}{(1 - |z|^2)^{2n-1}} = (\mathcal{M}f)(a) \in L^p(\mathbb{B}, d\tau).$$

Proof. By subharmonicity of $\left|\frac{\partial f}{\partial z_i}\right|$ we have

$$\left|\frac{\partial f}{\partial z_i}(z)\right| \le \int_{\mathbb{B}} \left|\frac{\partial f}{\partial z_i}(z+\delta w)\right| dv(w) = \frac{1}{\delta^{2n}} \int_{B(z,\delta)} \left|\frac{\partial f}{\partial z_i}(w)\right| dv(w)$$

for $z \in \mathbb{B}$ and $0 \le \delta < 1 - |z|$. Thus for $r \in (0,1)$,

$$\begin{aligned}\left|\frac{\partial f}{\partial z_i}(z)\right| &\le \frac{2^{2n}}{r^{2n}(1-|z|^2)^{2n}} \int_{B(z,\frac{r}{2}(1-|z|^2))} \left|\frac{\partial f}{\partial z_i}(w)\right| dv(w) \\ &\le C \int_{B(z,\frac{r}{2}(1-|z|^2))} \left|\frac{\partial f}{\partial z_i}(w)\right| \frac{dv(w)}{(1-|w|^2)^{2n}}.\end{aligned}$$

Consequently,

$$|\nabla f(z)|(1-|z|^2) \le C \int_{B(z,\frac{r}{2}(1-|z|^2))} |\nabla f(w)| \frac{dv(w)}{(1-|w|^2)^{2n-1}},$$

which proves the implication "⇒".

Now, let $f \in B_p$. Then $\omega_r(f) \in L^p(\mathbb{B}, d\tau)$ by Theorem A. It follows from the proof of Theorem 2 that

$$|\nabla f(z)|(1-|z|^2) \le C\omega_r^c(f)(z) \le C\omega_r(f)(z).$$

Hence

$$\begin{aligned}&\int_{\mathbb{B}} (\mathcal{M}f)^p(a)d\tau(a) \\ &\quad = \int_{\mathbb{B}} \left(\int_{E^c(a,r)} |\nabla f(z)|(1-|z|^2) \frac{dv(z)}{(1-|z|^2)^{2n}} \right)^p d\tau(a) \\ &\quad \le C \int_{\mathbb{B}} \left(\int_{E^c(a,r)} \omega_r(f)(z) \frac{dv(z)}{(1-|z|^2)^{2n}} \right)^p d\tau(a) \\ &\quad = C \int_{\mathbb{B}} \left(\int_{E^c(a,r)} \left(\sup_{w \in E(z,r)} |f(z)-f(w)| \right) \frac{dv(z)}{(1-|z|^2)^{2n}} \right)^p d\tau(a).\end{aligned}$$

To complete the proof, we apply the following triangle inequalities for the pseudo-hyperbolic metric $\rho(z,a) = |\varphi_a(z)|$ (see, e.g. [2])

$$(12) \qquad \frac{|\rho(z,a)-\rho(a,w)|}{1-\rho(z,a)\rho(a,w)} \le \rho(z,w) \le \frac{\rho(z,a)+\rho(a,w)}{1+\rho(z,a)\rho(a,w)}, \quad z,w,a \in \mathbb{B}.$$

This inequality implies that if $w \in E(a,r)$ and $a \in E(z,r)$, then $w \in E(z, 2r/(1+r^2))$. Consequently, using inclusion (10),

$$\begin{aligned}
&\int_{E^c(a,r)} \left(\sup_{w\in E(z,r)} |f(z)-f(w)| \right) \frac{dv(z)}{(1-|z|^2)^{2n}} \\
&\quad \le C \sup_{z\in E^c(a,r)} \left(\sup_{w\in E(z,r)} (|f(z)-f(a)| + |f(w)-f(a)|) \right) \\
&\quad \le C \sup_{z\in E(a,\frac{2r}{1+r^2})} |f(z)-f(a)| + \sup_{w\in E(a,\frac{2r}{1+r^2})} |f(a)-f(w)| \\
&\quad \le 2C \sup_{w\in E(a,\frac{2r}{1+r^2})} |f(a)-f(w)| \\
&\quad = 2C\omega_{\frac{2r}{1+r^2}}(f)(a) \in L^p(\mathbb{B}, d\tau). \qquad \square
\end{aligned}$$

We remark that the last theorem is equivalent to the statement that a function f holomorphic on $\mathbb{B}$ is in B_p if and only if the integral mean of f at a given by

$$(Mf)(a) = \frac{1}{v(E^c(a,r))} \int_{E^c(a,r)} |\nabla f(z)|(1-|z|^2)dv(z)$$

is in $L^p(\mathbb{B}, d\tau)$.

Let us define

$$(\mathcal{H}f)(a) = \int_{E^c(a,r)} \frac{|f(z)-f(a)|}{|z-a|}(1-|z|^2)^{\frac{1}{2}}(1-|a|^2)^{\frac{1}{2}} \frac{dv(z)}{(1-|z|^2)^{2n}}.$$

Our last theorem refers to Holland–Walsh characterization of the Bloch space.

Theorem 4. *Let f be a holomorphic function in $\mathbb{B}$ and $r \in (0,1)$. Then the following statements are equivalent*

(i) $f \in B_p$,
(ii) $\mathcal{H}f \in L^p(\mathbb{B}, d\tau)$,
(iii) $\displaystyle\int_{\mathbb{B}} \int_{E^c(a,r)} \frac{|f(z)-f(a)|^p}{|z-a|^p}(1-|z|^2)^{\frac{p}{2}}(1-|a|^2)^{\frac{p}{2}} \frac{dv(z)}{(1-|z|^2)^{2n}} d\tau(a) < \infty.$

Proof. (i)⇒(ii) Suppose $f \in B_p$. Using the invariance of the measure $\frac{dv(z)}{(1-|z|^2)^{2n}}$ under the map $\varphi_a^c(z)$, we obtain

$$\int_{E^c(a,r)} \frac{|f(z)-f(a)|}{|z-a|}(1-|z|^2)^{\frac{1}{2}}(1-|a|^2)^{\frac{1}{2}}\frac{dv(z)}{(1-|z|^2)^{2n}}$$
$$\leq \int_{E^c(a,r)} \left(\sup_{z\in E^c(a,r)} |f(z)-f(a)|\right) \frac{(1-|z|^2)^{\frac{1}{2}}(1-|a|^2)^{\frac{1}{2}}}{|z-a|}\frac{dv(z)}{(1-|z|^2)^{2n}}$$
$$= \omega_r^c(f)(a) \int_{E^c(a,r)} \frac{\sqrt{1-|\varphi_a^c(z)|^2}}{|\varphi_a^c(z)|}\frac{dv(z)}{(1-|z|^2)^{2n}}$$
$$= \omega_r^c(f)(a) \int_{E^c(0,r)} \frac{\sqrt{1-|z|^2}}{|z|}\frac{dv(z)}{(1-|z|^2)^{2n}} = C\omega_r^c(f)(a) \in L^p(\mathbb{B}, d\tau).$$

(ii)⇒(iii) It is enough to apply the Jensen inequality.
(iii)⇒(i) From (8) we see that if $|\varphi_a^c(z)| < r$, then

$$\frac{(1-|z|^2)^{\frac{1}{2}}(1-|a|^2)^{\frac{1}{2}}}{|z-a|} = \frac{\sqrt{1-|\varphi_a^c(z)|^2}}{|\varphi_a^c(z)|} \geq \frac{\sqrt{1-r^2}}{r}.$$

This and (11) imply

$$|\nabla f(a)|^p(1-|a|^2)^p$$
$$\leq C\int_{E^c(a,r)} \frac{|f(z)-f(a)|^p}{|z-a|^p}(1-|z|^2)^{\frac{p}{2}}(1-|a|^2)^{\frac{p}{2}}\frac{dv(z)}{(1-|z|^2)^{2n}},$$

which proves the implication. □

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