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An extension of the univalence criterion for a family of integral operators

Abstract. The main object of the present paper is to extend the univalence condition for a family of integral operators. Relevant connections of some of the results obtained in this paper with those in earlier works are also provided.

1. Introduction and preliminaries. Let $\mathcal{A}$ denote the class of functions f normalized by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disk $\mathcal{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$ and satisfy the conditions $f(0) = f'(0) - 1 = 0$.

Consider $\mathcal{S} = \{f \in \mathcal{A} : f \text{ is a univalent function in } \mathcal{U}\}$.

A function $f(z) \in \mathcal{A}$ is said to be a member of the class $\mathcal{B}(\gamma)$ if and only if

$$\left| \frac{z^2 f'(z)}{f^2(z)} - 1 \right| < 1 - \gamma, \quad 0 \le \gamma < 1. \tag{1.2}$$

Recently, Frasin and Darus (see [6]) defined and studied the class $\mathcal{B}(\gamma)$. In his paper Frasin (see [4]) obtained some results for functions belonging

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to this class and also he showed that if $f(z) \in \mathcal{B}(\gamma)$ then $f(z)$ satisfies the following inequality

$$\left|\frac{zf''(z)}{f'(z)}\right| \leq \frac{(1-\gamma)(2+|z|)}{1-|z|} \quad (z \in \mathcal{U}). \tag{1.3}$$

For $\gamma = 0$ the class $\mathcal{B}(0) = \mathcal{T}$ was studied by Ozaki and Nunokawa (see [8]).

We denote by $\mathcal{W}$ the class of functions w which are analytic in $\mathcal{U}$ satisfying the conditions $|w(z)| < 1$ and $w(0) = w'(0) = 0$ for all $z \in \mathcal{U}$.

Now, by Schwarz's lemma, it follows that

$$|w(z)| < |z|. \tag{1.4}$$

In [7], we see that if $w(z) \in \mathcal{W}$, then $w(z)$ satisfies

$$|w'(z)| \leq \frac{1-|w(z)|^2}{1-|z|^2} \quad (z \in \mathcal{U}). \tag{1.5}$$

In [11], N. Seenivasagan and D. Breaz considered the following family of integral operators $\mathcal{F}_{\alpha_1,\alpha_2,\ldots,\alpha_n,\beta}(z)$ defined as follows

$$\mathcal{F}_{\alpha_1,\alpha_2,\ldots,\alpha_n,\beta}(z) := \left(\beta \int_0^z t^{\beta-1} \prod_{i=1}^{n} \left(\frac{f_i(t)}{t}\right)^{\frac{1}{\alpha_i}} dt\right)^{\frac{1}{\beta}}, \tag{1.6}$$

where $f_i \in \mathcal{A}$, $f_i''(0) = 0$ and $\alpha_1, \alpha_2, \ldots, \alpha_n, \beta \in \mathbb{C}$ for all $i \in \{1, 2, \ldots, n\}$.

When $\alpha_i = \alpha$ for all $i \in \{1, 2, \ldots, n\}$, $\mathcal{F}_{\alpha_1,\alpha_2,\ldots,\alpha_n,\beta}(z)$ becomes the integral operator $\mathcal{F}_{\alpha,\beta}(z)$ considered in (see [1]).

We begin by recalling each of the following theorems dealing with univalence criterion, which will be required in our present paper.

In [10], Pascu proved the following theorem.

Theorem 1 (Pascu [10]). *Let β be a complex number with* $\mathrm{Re}(\beta) > 0$ *and* $f \in \mathcal{A}$. *If*

$$\frac{1-|z|^{2\,\mathrm{Re}(\beta)}}{\mathrm{Re}(\beta)} \left|\frac{zf''(z)}{f'(z)}\right| \leq 1,$$

for $z \in \mathcal{U}$, then the function

$$F_\beta(z) := \left(\beta \int_0^z t^{\beta-1} f'(t)\,dt\right)^{\frac{1}{\beta}}$$

belongs to $\mathcal{S}$.

In [9], Pascu and Pescar obtained the next result.

Theorem 2 (Pascu and Pescar [9]). *Let β and μ be complex numbers, and $g \in \mathcal{S}$. If $\mathrm{Re}(\beta) > 0$ and $|\mu| \leq \min\left\{\frac{1}{2}\mathrm{Re}(\beta); \frac{1}{4}\right\}$, then the function*

$$\mathcal{G}_{\beta,\mu}(z) := \left(\beta \int_0^z t^{\beta-1}\left(\frac{g(t)}{t}\right)^{\mu} dt\right)^{\frac{1}{\beta}}$$

belongs to $\mathcal{S}$.

Note that Theorem 2 includes the special case of Pascu and Pescar's theorem (see [9]) when $\mathrm{Re}(\alpha) = \mathrm{Re}(\beta)$.

In the present paper, we propose to investigate further univalence condition involving the general a family of integral operators defined by (1.6).

2. Main results. In this section, we first state an inclusion for $f(z) \in \mathcal{B}(\gamma)$, then we give the main univalence condition involving the general integral operator given by (1.6).

Theorem 3. *If $f(z) \in \mathcal{B}(\gamma)$, then the inequality is satisfied*

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leq \frac{(1-\gamma)(1+|z|)}{1-|z|} \tag{2.1}$$

for all $z \in \mathcal{U}$.

Proof. Let $f(z) \in \mathcal{B}(\gamma)$. Then we have

$$\frac{z^2 f'(z)}{f^2(z)} = 1 + (1-\gamma)w(z), \tag{2.2}$$

where $w(z) \in \mathcal{W}$. By applying the logarithmic differentiation, we obtain from (2.2) that

$$\frac{zf''(z)}{f'(z)} = \frac{(1-\gamma)zw'(z)}{1+(1-\gamma)w(z)} + \frac{2zf'(z)}{f(z)} - 2$$

and

$$\frac{zf'(z)}{f(z)} - 1 = \frac{1}{2}\left(\frac{zf''(z)}{f'(z)} - \frac{(1-\gamma)zw'(z)}{1+(1-\gamma)w(z)}\right),$$

thereby, it follows that

$$\begin{aligned}\left|\frac{zf'(z)}{f(z)} - 1\right| &= \left|\frac{1}{2}\left(\frac{zf''(z)}{f'(z)} - \frac{(1-\gamma)zw'(z)}{1+(1-\gamma)w(z)}\right)\right| \\ &\leq \frac{1}{2}\left(\left|\frac{zf''(z)}{f'(z)}\right| + \left|\frac{(1-\gamma)zw'(z)}{1+(1-\gamma)w(z)}\right|\right) \\ &\leq \frac{1}{2}\left(\left|\frac{zf''(z)}{f'(z)}\right| + \frac{(1-\gamma)|z||w'(z)|}{1-(1-\gamma)|w(z)|}\right).\end{aligned}$$

From (1.3) and (1.5), we have

$$\left|\frac{zf'(z)}{f(z)}-1\right| \le \frac{1}{2}\left(\frac{(1-\gamma)(2+|z|)}{1-|z|}+\frac{(1-\gamma)\,|z|}{1-(1-\gamma)\,|w(z)|}\frac{1-|w(z)|^2}{1-|z|^2}\right) \tag{2.3}$$

and for $0 \le \gamma < 1$, it is easy to show that

$$\frac{1-|w(z)|}{1-(1-\gamma)\,|w(z)|} \le 1 \quad (z \in \mathcal{U}). \tag{2.4}$$

From (1.4), (2.3) and (2.4), we obtain that

$$\left|\frac{zf'(z)}{f(z)}-1\right| \le \frac{(1-\gamma)(1+|z|)}{1-|z|}. \tag{2.5}$$

This evidently completes the proof of Theorem 3. □

Next we prove the following main theorem.

Theorem 4. *Let $f_i(z) \in \mathcal{B}(\gamma)$ for $i \in \{1,2,\ldots,n\}$. Let β be a complex number with* $\mathrm{Re}(\beta) > 0$. *If*

$$\sum_{i=1}^{n}\frac{1}{|\alpha_i|} \le \min\left\{\frac{1}{2(1-\gamma)}\operatorname{Re}\beta;\ \frac{1}{4(1-\gamma)}\right\} \tag{2.6}$$

for all $z \in \mathcal{U}$, then the function

$$\mathcal{F}_{\alpha_1,\alpha_2,\ldots,\alpha_n,\beta}(z) := \left(\beta\int_0^z t^{\beta-1}\prod_{i=1}^{n}\left(\frac{f_i(t)}{t}\right)^{\frac{1}{\alpha_i}}dt\right)^{\frac{1}{\beta}}$$

belongs to $\mathcal{S}$.

Proof. Define function

$$h(z) = \int_0^z \prod_{i=1}^{n}\left(\frac{f_i(t)}{t}\right)^{\frac{1}{\alpha_i}}dt.$$

We have $h(0) = h'(0) - 1 = 0$. Also, a simple computation yields

$$h'(z) = \prod_{i=1}^{n}\left(\frac{f_i(z)}{z}\right)^{\frac{1}{\alpha_i}}. \tag{2.7}$$

Making use of logarithmic differentiation in (2.7), we obtain

$$\frac{zh''(z)}{h'(z)} = \sum_{i=1}^{n}\frac{1}{\alpha_i}\left(\frac{zf_i'(z)}{f_i(z)}-1\right). \tag{2.8}$$

We thus have from (2.8) that

$$\left|\frac{zh''(z)}{h'(z)}\right| \le \sum_{i=1}^{n}\frac{1}{|\alpha_i|}\left|\frac{zf_i'(z)}{f_i(z)}-1\right|.$$

By using the Theorem 3, we get the inequality

$$\left|\frac{zh''(z)}{h'(z)}\right| \leq \sum_{i=1}^{n} \frac{1}{|\alpha_i|} \frac{(1-\gamma)(1+|z|)}{1-|z|}. \tag{2.9}$$

From (2.9), we obtain

$$\begin{aligned}\frac{1-|z|^{2\operatorname{Re}(\beta)}}{\operatorname{Re}(\beta)}\left|\frac{zh''(z)}{h'(z)}\right| &\leq \frac{1-|z|^{2\operatorname{Re}(\beta)}}{\operatorname{Re}(\beta)} \frac{(1-\gamma)(1+|z|)}{1-|z|} \sum_{i=1}^{n} \frac{1}{|\alpha_i|} \\ &\leq \frac{1-|z|^{2\operatorname{Re}(\beta)}}{1-|z|} \frac{2(1-\gamma)}{\operatorname{Re}(\beta)} \sum_{i=1}^{n} \frac{1}{|\alpha_i|}\end{aligned} \tag{2.10}$$

for all $z \in \mathcal{U}$.

Let us denote $|z| = x$, $x \in [0,1)$, $\operatorname{Re}(\beta) = a > 0$ and $\psi(x) = \frac{1-x^{2a}}{1-x}$. It is easy to prove that

$$\psi(x) \leq \begin{cases} 1, & \text{if } 0 < a < \frac{1}{2} \\ 2a, & \text{if } \frac{1}{2} < a < \infty. \end{cases} \tag{2.11}$$

From (2.6), (2.10) and (2.11), we have

$$\begin{aligned}\frac{1-|z|^{2\operatorname{Re}(\beta)}}{\operatorname{Re}(\beta)}\left|\frac{zh''(z)}{h'(z)}\right| &\leq \begin{cases} \frac{2(1-\gamma)}{\operatorname{Re}(\beta)} \sum\limits_{i=1}^{n} \frac{1}{|\alpha_i|}, & \text{if } 0 < \operatorname{Re}(\beta) < \frac{1}{2} \\ 4(1-\gamma) \sum\limits_{i=1}^{n} \frac{1}{|\alpha_i|}, & \text{if } \frac{1}{2} < \operatorname{Re}(\beta) < \infty \end{cases} \\ &\leq 1\end{aligned}$$

for all $z \in \mathcal{U}$.

Finally, by applying Theorem 1, we conclude that the function $\mathcal{F}_{\alpha_1,\alpha_2,\ldots,\alpha_n,\beta}(z)$ defined by (1.6) is in the function class $\mathcal{S}$. This evidently completes the proof of Theorem 4. □

3. Some applications of Theorem 4. In this section, we give some results of Theorem 4.

First of all, upon setting $\alpha_i = \alpha$, for all $i \in \{1, 2, \ldots, n\}$ in Theorem 4, we immediately arrive at the following application of Theorem 4.

Corollary 1. *Let $f_i(z) \in \mathcal{B}(\gamma)$ for $i \in \{1, 2, \ldots, n\}$. Let β be a complex number with $\operatorname{Re}(\beta) > 0$. If*

$$\frac{1}{|\alpha|} \leq \min\left\{\frac{1}{2n(1-\gamma)} \operatorname{Re}\beta; \frac{1}{4n(1-\gamma)}\right\} \tag{3.1}$$

holds for all $z \in \mathcal{U}$, then the function

$$\mathcal{F}_{\alpha,\beta}(z) := \left(\beta \int_0^z t^{\beta-1} \prod_{i=1}^{n} \left(\frac{f_i(t)}{t}\right)^{\frac{1}{\alpha}} dt\right)^{\frac{1}{\beta}}$$

belongs to $\mathcal{S}$.

Next we set $n = 1$ in Theorem 4, we thus obtain the following interesting consequence of Theorem 4.

Corollary 2. *Let the functions* $f(z) \in \mathcal{B}(\gamma)$. *Let* β *be a complex number with* $\operatorname{Re}\beta > 0$. *If*

$$\frac{1}{|\alpha|} \leq \min\left\{\frac{1}{2(1-\gamma)}\operatorname{Re}\beta; \frac{1}{4(1-\gamma)}\right\} \tag{3.2}$$

holds for all $z \in \mathcal{U}$, *then the function*

$$\mathcal{G}_{\beta,\alpha}(z) := \left(\beta \int_0^z t^{\beta-1}\left(\frac{f(t)}{t}\right)^{\frac{1}{\alpha}} dt\right)^{\frac{1}{\beta}}$$

belongs to $\mathcal{S}$.

Remark 1.

(i) Corollary 2 provides an extension of Theorem 2 due to Pascu and Pescar (see [9]).
(ii) If we set $\gamma = 0$, $n = 1$ and $\frac{1}{\alpha} = \mu$ in Theorem 4, we obtain Theorem 2 due to Pascu and Pescar (see [9]).
(iii) If we put $\gamma = 0$, $\beta = 1$ and α instead of $\frac{1}{\alpha}$ in Corollary 2, we arrive at the result by Kim and Merkes (see [5]).

Remark 2. Some authors gave similar univalence conditions by using bounded functions $f(z) \in \mathcal{A}$ in their papers, see the works (for example Breaz et al. (see [2]), Breaz et al. (see [3])). We note that the functions $f \in \mathcal{A}$ do not have to be bounded.

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References

[1] Breaz, D., Breaz, N., *The univalent conditions for an integral operator on the classes* $\mathcal{S}(p)$ *and* $\mathcal{T}_2$, J. Approx. Theory Appl. **1**(2) (2005), 93–98.
[2] Breaz, D., Breaz, N., *Sufficient univalence conditions for analytic functions*, J. Inequal. Appl. **2007** (2007), Article ID 86493, 5 pp.
[3] Breaz, D., Breaz, N. and Srivastava, H. M., *An extension of the univalent condition for a family of integral operators*, Appl. Math. Lett. **22** (2009), 41–44.
[4] Frasin, B. A., *A note on certain analytic and univalent functions*, Southeast Asian Bull. Math. **28**(5) (2004), 829–836.
[5] Kim, Y. J., Merkes, E. P., *On an integral of powers of a spirallike function*, Kyungpook Math. J. **12** (1972), 249–252.
[6] Frasin, B. A., Darus, M., *On certain analytic univalent functions*, Int. J. Math. Math. Sci. **25**(5) (2001), 305–310.
[7] Nehari, Z., *Conformal Mapping*, Dover Publications, Inc., New York, 1975.

[8] Ozaki, S., Nunokawa, M., *The Schwarzian derivative and univalent functions*, Proc. Amer. Math. Soc. **33** (1972), 392–394.

[9] Pascu, N. N., Pescar, V., *On the integral operators of Kim–Merkes and of Pfaltzgraff*, Mathematica (Cluj) **32**(55), no. 2 (1990) 185–192.

[10] Pascu, N., *An improvement of Becker's univalence criterion*, Proceedings of the Commemorative Session Simion Stoilow (Braşov, 1987), 43–98, Univ. Braşov, Braşov, 1987.

[11] Seenivasagan, N., Breaz, D., *Certain sufficient conditions for univalence*, Gen. Math. **15**(4) (2007), 7–15.

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