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Fixed points of periodic mappings in Hilbert spaces

Abstract. In this paper we give new estimates for the Lipschitz constants of n-periodic mappings in Hilbert spaces, in order to assure the existence of fixed points and retractions on the fixed point set.

1. Introduction. In order to assure the existence of fixed points for a continuous mapping on Banach spaces, we need to impose some conditions on the mapping or on the Banach space. We will deal with k-Lipschitzian mappings:

Definition 1.1. Let $T : C \to C$ be a mapping with C a nonempty, closed and convex subset of a Banach space X. T is called a Lipschitzian mapping if there is $k > 0$ such that

$$\|Tx - Ty\| \leq k\|x - y\|$$

holds for any $x, y \in C$ and we will write $T \in \mathscr{L}(k)$. If k_0 is the smallest number such that $T \in \mathscr{L}(k)$, we will write $T \in \mathscr{L}_0(k_0)$.

Definition 1.2. Let $T : C \to C$ where C is a nonempty, closed and convex subset of a Banach space X. If $T^n = I$, T is called an n-periodic mapping.

In 1981 K. Goebel and M. Koter, see [1, pp. 179–180], proved the following theorem which shows that the condition of periodicity for nonexpansive mappings is very strong:

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Theorem 1.3. *If C is a nonempty, closed and convex subset of a Banach space, then any nonexpansive n-periodic mapping $T : C \to C$ has a fixed point.*

This covers the case $k \le 1$, and thus we will study n-periodic and k-Lipschitzian mappings with $k > 1$.

Remark 1.4. If T is n-periodic, then $\mathrm{Fix}(T) = \mathrm{Fix}(T^{n-1})$. In fact, if $x \in \mathrm{Fix}(T)$, then it is clear that $x \in \mathrm{Fix}(T^{n-1})$ and if $x \in \mathrm{Fix}(T^{n-1})$, then $T(T^{n-1}x) = Tx$, that is, $x = Tx$.

Therefore, we will only consider n-periodic mappings $T \in \mathscr{L}_0(k)$ such that $T^{n-1} \in \mathscr{L}_0(p)$ with $p \ge k$ because if $p < k$, we will work with T^{n-1} instead of T.

Let us define the following number:

$$\gamma_n^X = \inf\{k : \exists (C \subset X,\ T : C \to C),\ T^n = I,\ T \in \mathscr{L}_0(k),\ \mathrm{Fix}(T) = \emptyset\},$$

where C is a nonempty, closed and convex subset of a Banach space X. In 1981 K. Goebel and M. Koter [1, pp. 179–180] showed that for any n, $\gamma_n^X > 1$.

In 1971 K. Goebel and E. Złotkiewicz [2] proved that if $k < 2$, then $\mathrm{Fix}(T) \neq \emptyset$ for 2-periodic and k-Lipschitzian mappings in general Banach spaces X, that is, $\gamma_2^X \ge 2$.

Furthermore, in 1986 M. Koter (see also [4]) proved that $\gamma_2^H \ge \sqrt{\pi^2 - 3} \approx 2.6209$ for Hilbert spaces H.

In 2005 J. Górnicki and K. Pupka [3] gave estimations of γ_n^X for $n \ge 3$ for any Banach space X, in particular $\gamma_3^X \ge 1.3821$, $\gamma_4^X \ge 1.2524$ and $\gamma_5^X \ge 1.1777$. These are the best estimations known nowadays for general Banach spaces; we will improve these estimations for Hilbert spaces.

2. Estimations of γ_n^H in Hilbert spaces. The following lemma gives conditions for the existence of fixed points and retractions on the fixed point set:

Lemma 2.1. *Let X be a complete metric space and $T : X \to X$ a continuous mapping. Suppose there are $u : X \to X$, $0 < A < 1$ and $B > 0$, such that for every $x \in X$:*

- **(i)** $\mathrm{d}(Tu(x), u(x)) \le A\, \mathrm{d}(Tx, x)$,
- **(ii)** $\mathrm{d}(u(x), x) \le B\, \mathrm{d}(Tx, x)$.

Then $\mathrm{Fix}(T) \neq \emptyset$.

If we define $R(x) = \lim_{n\to\infty} u^n(x)$ and u is a continuous mapping, then R is a retraction from X to $\mathrm{Fix}(T)$.

If additionally $u \in \mathscr{L}(p)$:

- **(a)** *If $p < 1$, then T has a unique fixed point.*
- **(b)** *If $p = 1$, then R is a nonexpansive mapping.*

(c) *If $p > 1$ and $D = \operatorname{diam}(X) < \infty$, then R is a Hölder continuous retraction from X to* $\operatorname{Fix}(T)$.

Proof. Górnicki in [3] proved that if (i) and (ii) hold and $x \in X$, the sequence $\{u^n(x)\}_{n=1}^{\infty}$ converges to a fixed point of T. Furthermore, for every $m, n \in \mathbb{N}$,

$$\mathrm{d}(u^{n+m}(x), u^n(x)) \leq BA^n \frac{1}{1-A} \mathrm{d}(Tx, x). \tag{2.1}$$

If $x \in \operatorname{Fix}(T)$, $Rx = x$, since clearly $\operatorname{Fix}(u) = \operatorname{Fix}(T)$. Thus, in order to prove that R is a retraction, we only need to show that R is a continuous mapping.

Let $E_L = \{x \in X : \mathrm{d}(x, Tx) < L\}$. Then $X = \bigcup_L E_L$. For $x \in E_L$, by (2.1) we have

$$\mathrm{d}(R(x), u^n(x)) \leq \frac{BA^n}{1-A} \mathrm{d}(Tx, x) < \frac{LBA^n}{1-A}.$$

Since the last inequality does not depend on x, and since $A < 1$, u^n converges uniformly to R on E_L, and hence R is continuous in X.

(a) If $p < 1$, then u is a contraction and has a unique fixed point, hence T has a unique fixed point.

(b) If $p = 1$, $\mathrm{d}(Rx, Ry) = \lim\limits_{n\to\infty} \mathrm{d}(u^n(x), u^n(y)) \leq \mathrm{d}(x, y)$.

(c) Let $p > 1$ and $D = \operatorname{diam}(X)$. For any $n \in \mathbb{N}$ and any $x, y \in X$ we have

$$\begin{aligned}
\mathrm{d}(Rx, Ry) &\leq \mathrm{d}(Rx, T^n x) + \mathrm{d}(T^n x, T^n y) + \mathrm{d}(T^n y, Ry) \\
&\leq BA^n \frac{1}{1-A} (\mathrm{d}(Tx, x) + \mathrm{d}(Ty, y)) + p^n \mathrm{d}(x, y) \\
&\leq \frac{2DB}{1-A} A^n + p^n \mathrm{d}(x, y) = E(n).
\end{aligned}$$

Let us define $n_0 \in \mathbb{N}$ as follows:

$$n_0 = \min\left\{ n \in \mathbb{N} : \mathrm{d}(x, y) p^n \geq \frac{2DBA^n}{1-A} \right\}.$$

If $\mathrm{d}(x, y) \geq \frac{2DB}{1-A}$, then we have $n_0 = 0$, and $\mathrm{d}(Rx, Ry) \leq 2\,\mathrm{d}(x, y)$. Suppose that we have $\mathrm{d}(x, y) < \frac{2DB}{1-A}$, then $n_0 > 0$. With

$$s_0 = \frac{\ln\left(\frac{2DB}{(1-A)\,\mathrm{d}(x,y)}\right)}{\ln(p) + \ln(1/A)},$$

we have the equality

$$\mathrm{d}(x, y) p^{s_0} = \frac{2DB}{1-A} A^{s_0},$$

hence $n_0 - 1 < s_0 \leq n_0$ and there is $0 \leq r_0 < 1$ such that $n_0 = s_0 + r_0$. In consequence we have

$$n_0 = \frac{\ln\left(\frac{2DB}{(1-A)\,\mathrm{d}(x,y)} e^{(\ln(p)+\ln(1/A))r_0}\right)}{\ln(p) + \ln(1/A)}$$

and

$$\begin{aligned} E(n_0) &= \mathrm{d}(x,y)p^{n_0} + A^{n_0}\frac{2DB}{1-A} \\ &\leq 2\,\mathrm{d}(x,y)p^{n_0} \\ &\leq 2\left(\frac{2DB}{1-A}\right)^{\frac{\ln(p)}{\ln(p)+\ln(1/A)}} p\,\mathrm{d}(x,y)^{\frac{\ln(1/A)}{\ln(p)+\ln(1/A)}}. \end{aligned} \qquad \square$$

The following lemma is a generalization of the parallelogram law for Hilbert spaces, and we will use it throughout this paper:

Lemma 2.2. *Let H be a Hilbert space and let $n \in \mathbb{N}$ and $a_i \in [0,1]$ for $i = 1, \ldots, n$, such that $\sum_{i=1}^n a_i = 1$. If $x_i \in H$ for $i = 1, \ldots, n$, then*

$$\left\| \sum_{i=1}^n a_i x_i \right\|^2 = \sum_{i=1}^n a_i \|x_i\|^2 - \sum_{1 \leq i < j \leq n} a_i a_j \|x_i - x_j\|^2.$$

Proposition 2.3. *Let $n \in \mathbb{N}$ and $T : C \to C$ be n-periodic and k-Lipschitzian mapping, where C is a nonempty, closed and convex subset of a Hilbert space. Let $a_i > 0$ for $i = 1, \ldots, n$, such that $\sum_{i=1}^n a_i = 1$. Let us define $a_0 = a_n$. If for $x \in C$ we define*

$$z = \sum_{i=1}^n a_i T^i x,$$

then we have $\|z - x\| \leq \sum_{i=1}^{n-1} a_i \|T^i x - x\|$ and

$$\|z - Tz\|^2 \leq \sum_{0 \leq j < i \leq n-1} F(k, a_j, a_{j+1}, a_i, a_{i+1}) \|T^j x - T^i x\|^2,$$

where $F(k, x, y, u, w) = k^2(yu + xw - xu) - xu$.

Proof. With a_i, x and z as above, the first inequality is trivial. Now let $L = \sum_{0\le j<i\le n-1} a_i a_j \|T^i x - T^j x\|^2$. By the previous lemma we have

$$\begin{aligned}
\|z - Tz\|^2 &= \left\| \sum_{i=1}^{n} a_i (T^i x - Tz) \right\|^2 \\
&= \sum_{i=1}^{n} a_i \|T^i x - Tz\|^2 - \sum_{0\le j<i\le n-1} a_i a_j \|T^i x - T^j x\|^2 \\
&\le k^2 \sum_{s=1}^{n} a_s \|z - T^{s-1} x\|^2 - L \\
&= k^2 \sum_{s=1}^{n} a_s \left\| \sum_{i=1}^{n} a_i (T^i x - T^{s-1} x) \right\|^2 - L \\
&= k^2 \sum_{s=1}^{n} a_s \left(\sum_{i=1}^{n} a_i \|T^i x - T^{s-1} x\|^2 \right) - (k^2+1)L.
\end{aligned}$$

The first term of the last expression is equal to

$$\begin{aligned}
&k^2 \sum_{s=1}^{n} a_s \left(\sum_{i=1}^{n} a_i \|T^i x - T^{s-1} x\|^2 \right) = k^2 \sum_{s=0}^{n-1} a_{s+1} \left(\sum_{i=1}^{n} a_i \|T^i x - T^s x\|^2 \right) \\
&= k^2 \sum_{s=1}^{n-1} a_{s+1} \left(\sum_{i=1}^{n} a_i \|T^i x - T^s x\|^2 \right) + k^2 a_1 \sum_{i=1}^{n} a_i \|T^i x - x\|^2 \\
&= k^2 \sum_{s=1}^{n-1} a_{s+1} \left(\sum_{i=1}^{n-1} a_i \|T^i x - T^s x\|^2 \right) + k^2 a_1 \sum_{i=1}^{n} a_i \|T^i x - x\|^2 \\
&\quad + k^2 \sum_{s=1}^{n-1} a_{s+1} a_n \|x - T^s x\|^2 \\
&= k^2 \sum_{1\le s<i\le n-1} (a_{s+1} a_i + a_s a_{i+1}) \|T^i x - T^s x\|^2 + k^2 a_1 \sum_{i=1}^{n} a_i \|T^i x - x\|^2 \\
&\quad + k^2 \sum_{s=1}^{n-1} a_{s+1} a_n \|x - T^s x\|^2 \\
&= k^2 \sum_{0\le s<i\le n-1} (a_{s+1} a_i + a_s a_{i+1}) \|T^i x - T^s x\|^2.
\end{aligned}$$

Hence

$$\begin{aligned}\|z-Tz\|^2 &\leq \sum_{0\leq j<i\leq n-1} [k^2(a_{j+1}a_i + a_j a_{i+1} - a_j a_i) - a_j a_i]\|T^j x - T^i x\|^2 \\ &= \sum_{0\leq j<i\leq n-1} F(k, a_j, a_{j+1}, a_i, a_{i+1})\|T^j x - T^i x\|^2. \qquad \square\end{aligned}$$

Applying the last result we have the following:

Proposition 2.4. *Let C be a nonempty, closed and convex subset of a Hilbert space H and $T : C \to C$, $T \in \mathscr{L}(k)$ be an n-periodic mapping, with $n \geq 3$. For $x \in C$ let us define*

$$u = \frac{1}{n}\left(x + Tx + \ldots + T^{n-1}x\right),$$

then $u \in \mathscr{L}\left(\frac{1+k+\ldots+k^{n-1}}{n}\right)$ and

$$\begin{aligned}\|u-Tu\|^2 &\leq \frac{1}{n^2}\left[(k^2-1)k^{2(n-1)} + \sum_{j=2}^{n-1}(k^{2j}-1)\left(\frac{k^{n-j}-1}{k-1}\right)^2\right]\|x-Tx\|^2 \\ &= A(k)\|x-Tx\|^2.\end{aligned}$$

Thus, if $A(k) < 1$, then $\mathrm{Fix}(T) \neq \emptyset$ *and* $\mathrm{Fix}(T)$ *is a retract of C. If $k = 1$,* $\mathrm{Fix}(T)$ *is a nonexpansive retract of C and if $k > 1$ with C bounded,* $\mathrm{Fix}(T)$ *is a Hölder continuous retract of C.*

Proof. From Proposition 2.3, taking $a_i = \frac{1}{n}$ we get

$$\|u-Tu\|^2 \leq \sum_{0\leq j<i\leq n-1} \frac{k^2-1}{n^2}\|T^j x - T^i x\|^2. \tag{2.2}$$

Since for $j < i$, $\|T^j x - T^i x\| \leq k^{i-j}\|x - T^{i-j}\|$, for $j < n-1$,

$$\|x-T^j x\| \leq \sum_{i=0}^{j-1}\|T^i x - T^{i+1}x\| \leq \sum_{i=0}^{j-1} k^i\|x-Tx\| = \frac{k^j-1}{k-1}\|x-Tx\| \tag{2.3}$$

and

$$\|x-T^{n-1}x\| = \|T^n x - T^{n-1}x\| \leq k^{n-1}\|Tx - x\|, \tag{2.4}$$

we get

$$
\begin{aligned}
\|u - Tu\|^2 &\le \frac{k^2-1}{n^2} \sum_{j=0}^{n-2} \sum_{i=j+1}^{n-1} \|T^j x - T^i x\|^2 \\
&\le \frac{k^2-1}{n^2} \sum_{j=0}^{n-2} k^{2j} \sum_{i=1}^{n-j-1} \|x - T^i x\|^2 \\
&= \frac{k^2-1}{n^2} \sum_{i=1}^{n-1} \left(\sum_{j=0}^{n-i-1} k^{2j} \right) \|x - T^i x\|^2 \\
&= \sum_{i=1}^{n-1} \frac{k^{2(n-i)}-1}{n^2} \|x - T^i x\|^2.
\end{aligned}
$$

Thus

$$
\begin{aligned}
\|u - Tu\|^2 &\le \frac{1}{n^2} \sum_{j=1}^{n-1} (k^{2(n-j)} - 1) \|x - T^j x\|^2 \\
&= \frac{1}{n^2} \sum_{j=1}^{n-2} (k^{2(n-j)} - 1) \|x - T^j x\|^2 + \frac{1}{n^2}(k^2-1)\|x - T^{n-1}x\|^2 \qquad (2.5) \\
&\le \frac{1}{n^2} \left[(k^2-1)k^{2(n-1)} + \sum_{j=2}^{n-1} (k^{2j}-1) \left(\frac{k^{n-j}-1}{k-1} \right)^2 \right] \|x - Tx\|^2.
\end{aligned}
$$

By Lemma 2.1, if $A(k) < 1$, $\mathrm{Fix}(T) \neq \emptyset$ and is a retract of C. Also if $k = 1$, $\mathrm{Fix}(T)$ is a nonexpansive retract of C and if $k > 1$ and C is bounded, $\mathrm{Fix}(T)$ is a Hölder continuous retract of C. □

Since for fixed n, $\lim_{k \to 1} A(k) = 0$, there is $k > 1$ such that $A(k) < 1$, and this is another proof that for Hilbert spaces H, $\gamma_n^H > 1$.

In 2000, M. Koter [5] gave the following estimations: $\gamma_3^H \ge 1.3666$, $\gamma_4^H \ge 1.1962$ and $\gamma_5^H \ge 1.0849$. But her procedure cannot be applied in order to estimate γ_n^H if $n > 6$. In 2005 J. Górnicki and K. Pupka [3] gave the following estimations in general Banach spaces: $\gamma_3^X \ge 1.3821$, $\gamma_4^X \ge 1.2524$, $\gamma_5^X \ge 1.1777$ and $\gamma_6^X \ge 1.1329$.

Applying (2.5) for $n = 3$, if k satisfies the inequality

$$
\frac{1}{9}\big((k^2-1)k^4 + (k^4-1)\big) < 1,
$$

then we have a fixed point. Thus $\gamma_3^H \ge 1.4678$; similarly we get $\gamma_4^H \ge 1.2905$.

For $n \ge 5$, $j = 1$ and $i = n-1$, the estimate in (2.2) improves if we take

$$
\|Tx - T^{n-1}x\| \le \|x - Tx\| + \|x - T^{n-1}x\| \le (1 + k^{n-1})\|x - Tx\| , \qquad (2.6)
$$

from this $\gamma_5^H \ge 1.1986$.

For $n \geq 6$ we shall also take the following estimations: if $j = 0$, $i = n-2$,

$$\begin{aligned} \|x - T^{n-2}x\| &\leq \|x - T^{n-1}x\| + \|T^{n-2}x - T^{n-1}x\| \\ &\leq (k^{n-2} + k^{n-1})\|x - Tx\|, \end{aligned} \tag{2.7}$$

and if $j = 2$, $i = n-1$,

$$\|T^2x - T^{n-1}x\| \leq \|x - T^2x\| + \|x - T^{n-1}\| \leq (1 + k + k^{n-1})\|x - Tx\|. \tag{2.8}$$

With this we get $\gamma_6^H \geq 1.15$.

In the case above we considered $a_i = 1/n$ because the calculations are straightforward, but we can choose other convex combinations in order to get better estimations of γ_n^H.

Proposition 2.5. *Let H be a Hilbert space. Then $\gamma_3^H \geq 1.5549$, $\gamma_4^H \geq 1.3267$, $\gamma_5^H \geq 1.2152$, and $\gamma_6^H \geq 1.1562$.*

Proof. Let F be as in Proposition 2.3. We will only take the case in which $F(k, a_j, a_{j+1}, a_i, a_{i+1}) \geq 0$ for $1 \leq j < i \leq n-1$.

If $n = 3$, we checked by numerical computation that this case gives us the solution with the greatest possible value of k. For n larger than 3, we do not know if this case gives us the best estimate, but it is easier to compute.

For $n = 3$, let $z = a_1Tx + a_2T^2x + a_3x$, where $a_1 + a_2 + a_3 = 1$, with $a_i \geq 0$. By Proposition 2.3 we have

$$\begin{aligned} &\|z - Tz\|^2 \leq F(k, a_3, a_1, a_1, a_2)\|x - Tx\|^2 \\ &\quad + F(k, a_3, a_1, a_2, a_3)\|x - T^2x\|^2 + F(k, a_1, a_2, a_2, a_3)\|Tx - T^2x\|^2 \\ &\leq F(k, a_3, a_1, a_1, a_2)\|x - Tx\|^2 + k^4F(k, a_3, a_1, a_2, a_3)\|x - Tx\|^2 \\ &\quad + k^2F(k, a_1, a_2, a_2, a_3)\|x - Tx\|^2 = B(k, a_1, a_2, a_3)\|x - Tx\|^2. \end{aligned} \tag{2.9}$$

Using differential calculus techniques, we conclude that the solution of the equation $B(k, a_1, a_2, a_3) = 1$ with the optimal value of k is the following: $k = 1.5549978175686$, $a_1 = 0.22027175125$, $a_2 = 0.44334559817$ and $a_3 = 0.33638265058$.

Let $n = 4$, by Proposition 2.3 and using the estimations of the terms $\|T^jx - T^ix\|$ as in the proof of Proposition 2.4, we have to solve

$$\begin{aligned} G(k, x, y, z, w) = F(k, a_4, a_1, a_1, a_2) &+ (1 + k)^2F(k, a_4, a_1, a_2, a_3) \\ &+ k^6F(k, a_4, a_1, a_3, a_4) + k^2F(k, a_1, a_2, a_2, a_3) \\ &+ k^2(1 + k)^2F(k, a_1, a_2, a_3, a_4) \\ &+ k^4F(k, a_2, a_3, a_3, a_4) - 1 = 0. \end{aligned}$$

The following optimal solution was found numerically:

$$\begin{aligned}
k &= 1.326774364525014,\\
a_1 &= 0.242229079187726,\\
a_2 &= 0.239942791859123,\\
a_3 &= 0.328255853776722,\\
a_4 &= 0.189572275176429.
\end{aligned}$$

Similarly for $n = 5$ and using the estimate (2.6), we need to solve

$$\begin{aligned}
&F(k, a_5, a_1, a_1, a_2) + (1+k)^2 F(k, a_5, a_1, a_2, a_3)\\
&+ k^6(1+k)^2 F(k, a_5, a_1, a_3, a_4) + k^8 F(k, a_5, a_1, a_4, a_5)\\
&+ k^2 F(k, a_1, a_2, a_2, a_3) + k^2(1+k)^2 F(k, a_1, a_2, a_3, a_4)\\
&+ (1+k^4)^2 F(k, a_1, a_2, a_4, a_5) + k^4 F(k, a_2, a_3, a_3, a_4)\\
&+ k^4(1+k)^2 F(k, a_2, a_3, a_4, a_5) + k^6 F(k, a_3, a_4, a_4, a_5) - 1 = 0.
\end{aligned}$$

We found the optimal solution: $k = 1.215236$, $a_1 = 0.14448498$, $a_2 = 0.23632485$, $a_3 = 0.24363867$, $a_4 = 0.20374357$ and $a_5 = 0.17180793$.

For $n = 6$ using (2.6), (2.7) and (2.8), we obtain the equation to solve

$$\begin{aligned}
&F(k, a_6, a_1, a_1, a_2) + (1+k)^2 F(k, a_6, a_1, a_2, a_3)\\
&+ (1+k+k^2)^2 F(k, a_6, a_1, a_3, a_4) + (k^5+k^4)^2 F(k, a_6, a_1, a_4, a_5)\\
&+ k^{10} F(k, a_6, a_1, a_5, a_6) + k^2 F(k, a_1, a_2, a_2, a_3)\\
&+ k^2(1+k)^2 F(k, a_1, a_2, a_3, a_4) + k^2(1+k+k^2)^2 F(k, a_1, a_2, a_4, a_5)\\
&+ (1+k^5)^2 F(k, a_1, a_2, a_5, a_6) + k^4 F(k, a_2, a_3, a_3, a_4)\\
&+ k^4(1+k)^2 F(k, a_2, a_3, a_4, a_5) + (1+k+k^5)^2 F(k, a_2, a_3, a_5, a_6)\\
&+ k^6 F(k, a_3, a_4, a_4, a_5) + k^6(1+k)^2 F(k, a_3, a_4, a_5, a_6)\\
&+ k^8 F(k, a_4, a_5, a_5, a_6) - 1 = 0.
\end{aligned}$$

We get the following optimal solution: $k = 1.1562$, $a_1 = 0.15958598$, $a_2 = 0.15893532$, $a_3 = 0.17823298$, $a_4 = 0.19267723$, $a_5 = 0.15822986$ and $a_6 = 0.15233863$. □

3. $T \in \mathscr{L}(k) \cap \mathscr{U}(p)$ with $p < k^{n-1}$. In Proposition 2.4 we used $T^j \in \mathscr{L}(k^j)$ in order to calculate the best estimation of γ_n^H. In fact, there are n-periodic functions such that for each $j = 1, \ldots, n-1$, $T^j \in \mathscr{L}_0(k^j)$ that is $T^j \notin \mathscr{L}(p)$ for $p < k^j$,

Example 3.1. Let $X = c_0^n(\mathbb{R})$, $C = \{(x_1, \ldots, x_n) \in X : x_i \geq 0,\ i = 1, \ldots, n\}$ and $k > 1$. We define $T : C \to C$ as follows:

$$T(x_1, \ldots, x_n) = \left(kx_2, kx_3, \ldots, kx_n, \frac{x_1}{k^{n-1}}\right).$$

We have $T^n = I$ and, in fact, for each $j = 1, \dots, n-1$, $T^j \in \mathscr{L}(k^j)$ but $T^j \notin \mathscr{L}(p)$ for $p < k^j$.

In this case we have $T \in \mathscr{U}_0(k^{n-1})$ according to the following definition:

Definition 3.2. Let $T : C \to C$ be a mapping, where C is a nonempty, closed and convex subset of a Banach space. We will say that T is uniformly Lipschitzian if there is $k > 0$ such that for every j and $x, y \in C$,

$$\|T^j x - T^j y\| \leq k\|x - y\|.$$

We will write $T \in \mathscr{U}(k)$. If

$$k = \min\left\{l : \|T^j x - T^j y\| \leq l\|x - y\|,\ j \in \mathbb{N},\ x, y \in C\right\},$$

we will write $T \in \mathscr{U}_0(k)$.

However, there are also cases such that T is n-periodic, $T \in \mathscr{L}_0(k)$ and $T \in \mathscr{U}_0(p)$ with $p < k^{n-1}$. For these functions we could improve the estimations considered in Proposition 2.4.

The extreme case is when $T^j \in \mathscr{L}_0(k)$ for $1 \leq j \leq n-1$, that is, $T \in \mathscr{U}_0(k)$. The next example shows that such functions exist.

Example 3.3. Let $X = c_0^n(\mathbb{R})$, $C = \{(x_1, \dots, x_n) \in X : x_i \geq 0,\ i = 1, \dots, n\}$ and $k > 1$. We define $T : C \to C$ as follows:

$$T(x_1, \dots, x_n) = \left(x_2, \dots, x_{n-1}, kx_n, \frac{x_1}{k}\right).$$

We have $T^n = I$ and for each $j = 1, \dots, n-1$, $T^j \in \mathscr{L}(k)$.

For this reason, we will introduce the following definition: let X be a Banach space, we define

$$\tilde{\gamma}_n^X = \inf\{p : \exists(C \subset X, T : C \to C),\ T^n = I,\ T \in \mathscr{U}_0(p),\ \mathrm{Fix}(T) = \emptyset\},$$

where C is a nonempty, closed and convex subset of the Banach space X. It is clear that $\tilde{\gamma}_n^X \geq \gamma_n^X$, since $T \in \mathscr{L}_0(k)$ implies $T \in \mathscr{U}_0(p)$ with $p \geq k$.

As before we want to estimate $\tilde{\gamma}_n^H$.

Proposition 3.4. *Let H be a Hilbert space. Then $\tilde{\gamma}_3^H \geq 1.6047$, $\tilde{\gamma}_4^H \geq 1.3867$, $\tilde{\gamma}_5^H \geq 1.2958$ and $\tilde{\gamma}_6^H \geq 1.2181$.*

Proof. Let $T \in \mathscr{U}(p)$ and $T^n = I$. By Proposition 2.3, if we take $z = \sum_{i=1}^n a_i T^i x$, then we have

$$\|z - Tz\|^2 \leq \sum_{0 \leq j < i \leq n-1} \left[p^2(a_{j+1}a_i + a_j a_{i+1} - a_j a_i) - a_j a_i\right] \|T^j x - T^i x\|^2.$$

Let $n \geq 3$ and $d = \|x - Tx\|$. We will use the estimates:

(1) $\|x - T^j x\| \leq \min\{((j-1)p + 1)d, (n-j)pd\}$,
(2) $\|Tx - T^j x\| \leq \min\{(j-1)pd, ((n-j)p + 1)d\}$,
(3) if $i > 1$ and $i+j < n$, $\|T^i x - T^{i+j} x\| \leq \min\{((n-j-1)p+1)d, pjd\}$.

As in Proposition 2.5 we only take $F(p, a_j, a_{j+1}, a_i, a_{i+1}) \geq 0$ for $0 \leq j < i \leq n-1$. For $n = 3$ we know that this is the best possibility but for $n > 3$ we do not know if this is the case.

Thus, for $n = 3$ we have to solve the equation

$$F(p, a_3, a_1, a_1, a_2) + p^2 F(p, a_3, a_1, a_2, a_3) + p^2 F(p, a_1, a_2, a_2, a_3) = 1.$$

The solution with the optimal value of p is: $p = 1.6047$, $a_1 = 0.4278208$, $a_2 = 0.34664038$ and $a_3 = 0.22553882$.

For $n = 4$ we have to solve

$$\begin{aligned}
&F(p, a_4, a_1, a_1, a_2) + (1+p)^2 F(p, a_4, a_1, a_2, a_3) \\
&+ p^2 F(p, a_4, a_1, a_3, a_4) + p^2 F(p, a_1, a_2, a_2, a_3) \\
&+ (1+p)^2 F(p, a_1, a_2, a_3, a_4) + p^2 F(p, a_2, a_3, a_3, a_4) - 1 = 0.
\end{aligned}$$

The optimal solution is: $p = 1.3867$, $a_1 = 0.30095499$, $a_2 = 0.23635124$, $a_3 = 0.2667267$ and $a_4 = 0.19596707$.

For $n = 5$ we have to solve the equation

$$\begin{aligned}
&F(p, a_5, a_1, a_1, a_2) + (1+p)^2 F(p, a_5, a_1, a_2, a_3) \\
&+ 4p^2 F(p, a_5, a_1, a_3, a_4) + p^2 F(p, a_5, a_1, a_4, a_5) \\
&+ p^2 F(p, a_1, a_2, a_2, a_3) + 4p^2 F(p, a_1, a_2, a_3, a_4) \\
&+ (1+p)^2 F(p, a_1, a_2, a_4, a_5) + p^2 F(p, a_2, a_3, a_3, a_4) \\
&+ 4p^2 F(p, a_2, a_3, a_4, a_5) + p^2 F(p, a_3, a_4, a_4, a_5) - 1 = 0.
\end{aligned}$$

The optimal solution is: $p = 1.2958$, $a_1 = 0.20310133$, $a_2 = 0.19687386$, $a_3 = 0.24013125$, $a_4 = 0.15037377$ and $a_5 = 0.20951979$.

Finally, for $n = 6$ we have the equation

$$\begin{aligned}
&F(p, a_6, a_1, a_1, a_2) + (1+p)^2 F(p, a_6, a_1, a_2, a_3) \\
&+ (1+2p)^2 F(p, a_6, a_1, a_3, a_4) + 4p^2 F(p, a_6, a_1, a_4, a_5) \\
&+ p^2 F(p, a_6, a_1, a_5, a_6) + p^2 F(p, a_1, a_2, a_2, a_3) \\
&+ 4p^2 F(p, a_1, a_2, a_3, a_4) + (1+2p)^2 F(p, a_1, a_2, a_4, a_5) \\
&+ (1+p)^2 F(p, a_1, a_2, a_5, a_6) + p^2 F(p, a_2, a_3, a_3, a_4) \\
&+ 4p^2 F(p, a_2, a_3, a_4, a_5) + (1+2p)^2 F(p, a_2, a_3, a_5, a_6) \\
&+ p^2 F(p, a_3, a_4, a_4, a_5) + 4p^2 F(p, a_3, a_4, a_5, a_6) \\
&+ p^2 F(p, a_4, a_5, a_5, a_6) - 1 = 0
\end{aligned}$$

and the optimal solution is: $p = 1.2181$, $a_1 = 0.1682856$, $a_2 = 0.14103694$, $a_3 = 0.19292656$, $a_4 = 0.16166393$, $a_5 = 0.13527042$ and $a_6 = 0.20081655$. $\square$

Summing up, we present the following two tables:

$T \in \mathscr{L}(k)$					
	M. Koter, 2000	Our method $a_i = 1/n$	Our method $\sum a_i = 1$		Górnicki and Pupka, 2005
γ_3^H	1.3666	1.4678	**1.5549**	γ_3^X	1.3821
γ_4^H	1.1962	1.2905	**1.3267**	γ_4^X	1.2524
γ_5^H	1.0849	1.1986	**1.2152**	γ_5^X	1.1777
γ_6^H	1.0228	1.15	**1.1562**	γ_6^X	1.1329

$T \in \mathscr{U}_0(p)$				
	M. Koter, 2000	Our method $\sum a_i = 1$		Górnicki and Pupka, 2005
$\tilde{\gamma}_3^H$	1.5447	**1.6047**	$\tilde{\gamma}_3^X$	1.4558
$\tilde{\gamma}_4^H$	1.2418	**1.3867**	$\tilde{\gamma}_4^X$	1.2917
$\tilde{\gamma}_5^H$	1.1429	**1.2958**	$\tilde{\gamma}_5^X$	1.2001
$\tilde{\gamma}_6^H$	1.0277	**1.2181**	$\tilde{\gamma}_6^X$	1.1482

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