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Inclusion properties of certain subclasses of analytic functions defined by generalized Sălăgean operator

Abstract. Let A denote the class of analytic functions with the normalization $f(0) = f'(0) - 1 = 0$ in the open unit disc $U = \{z : |z| < 1\}$. Set

$$f_\lambda^n(z) = z + \sum_{k=2}^{\infty} [1 + \lambda(k-1)]^n z^k \quad (n \in N_0;\ \lambda \geq 0;\ z \in U),$$

and define $f_{\lambda,\mu}^n$ in terms of the Hadamard product

$$f_\lambda^n(z) * f_{\lambda,\mu}^n = \frac{z}{(1-z)^\mu} \quad (\mu > 0;\ z \in U).$$

In this paper, we introduce several subclasses of analytic functions defined by means of the operator $I_{\lambda,\mu}^n : A \longrightarrow A$, given by

$$I_{\lambda,\mu}^n f(z) = f_{\lambda,\mu}^n(z) * f(z) \quad (f \in A;\ n \in N_{0;}\ \lambda \geq 0;\ \mu > 0).$$

Inclusion properties of these classes and the classes involving the generalized Libera integral operator are also considered.

1. Introduction. Let A denote the class of functions of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1.1}$$

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which are analytic in the open unit disc $U = \{z : |z| < 1\}$. If f and g are analytic in U, we say that f is subordinate to g, written $f \prec g$ or $f(z) \prec g(z)$, if there exists a Schwarz function $w(z)$, which (by definition) is analytic in U with $w(0) = 0$ and $|w(z)| < 1$ for all $z \in U$, such that $f(z) = g(w(z))$, $z \in U$. For $0 \leq \eta < 1$, we denote by $S^*(\eta)$, $K(\eta)$ and C the subclasses of A consisting of all analytic functions which are, respectively, starlike of order η, convex of order η and close-to-convex of order η in U (see, e.g., Srivastava and Owa [11]).

For $n \in N_0 = N \cup \{0\}$, where $N = \{1, 2, \dots\}$, $\lambda \geq 0$ and f given by (1.1), we consider the generalized Sălăgean operator defined as follows:

$$D_\lambda^n f(z) = z + \sum_{k=2}^{\infty} [1 + \lambda(k-1)]^n a_k z^k \quad (z \in U)\,. \tag{1.2}$$

The operator D_λ^n was introduced and studied by Al-Oboudi [1] which reduces to the Sălăgean differential operator [10] for $\lambda = 1$.

Let S be the class of all functions ϕ which are analytic and univalent in U and for which $\phi(U)$ is convex with $\phi(0) = 1$ and $\mathrm{Re}\{\phi(z)\} > 0$ $(z \in U)$. The Hadamard product (or convolution) $f * g$ of two analytic functions $f(z) = \sum_{k=0}^{\infty} a_k z^k$ and $g(z) = \sum_{k=0}^{\infty} b_k z^k$ is given by

$$(f * g)(z) = \sum_{k=0}^{\infty} a_k b_k z^k.$$

Making use of the principle of subordination between analytic functions, we introduce the subclasses $S^*(\eta; \phi)$, $K(\eta; \phi)$ and $C(\eta, \delta; \phi, \psi)$ of the class A for $0 \leq \eta,\ \delta < 1$ and $\phi, \psi \in S$ (cf., [3], [5] and [7]), which are defined by

$$S^*(\eta; \phi) = \left\{ f \in A : \frac{1}{1-\eta} \left(\frac{z f'(z)}{f(z)} - \eta \right) \prec \phi(z) \ \ (z \in U) \right\},$$

$$K(\eta; \Phi) = \left\{ f \in A : \frac{1}{1-\eta} \left(1 + \frac{z f''(z)}{f'(z)} - \eta \right) \prec \phi(z) \ \ (z \in U) \right\}$$

and

$$C(\eta, \delta; \phi, \psi) = \left\{ f \in A : \exists g \in S^*(\eta; \phi) \text{ s. t. } \frac{1}{1-\delta} \left(\frac{z f'(z)}{g(z)} - \delta \right) \prec \psi(z) \right.$$
$$\left. (z \in U) \right\}.$$

We note that, for special choices for the functions ϕ and ψ involved in these definitions, we can obtain the well-known subclasses of A. For example, we have

$$S^*\left(\eta; \frac{1+z}{1-z}\right) = S^*(\eta),\ K\left(\eta; \frac{1+z}{1-z}\right) = K(\eta)$$

and

$$C\left(0,0;\frac{1+z}{1-z},\frac{1+z}{1-z}\right)=C.$$

Setting

$$f_{\lambda}^{n}\left(z\right)=z+\sum_{k=2}^{\infty}[1+\lambda\left(k-1\right)]^{n}z^{k}\quad\left(n\in N_0;\ \lambda\geq 0\right),$$

we define the function $f_{\lambda,\mu}^{n}$ in terms of the Hadamard product by

$$f_{\lambda}^{n}\left(z\right)*f_{\lambda,\mu}^{n}\left(z\right)=\frac{z}{(1-z)^{\mu}}\quad\left(\mu>0;\ z\in U\right). \tag{1.3}$$

We now introduce the operator $I_{\lambda,\mu}^{n}:A\longrightarrow A$, which is defined here by

$$\begin{gathered}I_{\lambda,\mu}^{n}f(z)=f_{\lambda,\mu}^{n}\left(z\right)*f(z)=z+\sum_{k=2}^{\infty}\tfrac{(\mu)_{k-1}}{(k-1)![1+\lambda(k-1)]^{n}}a_kz^k\\ (f\in A;\ n\in N_0;\ \lambda\geq 0;\ \mu>0),\end{gathered} \tag{1.4}$$

where $(\theta)_k$ is the Pochhammer symbol defined, in terms of the Gamma function $\lceil$, by

$$(\theta)_k=\frac{\lceil(\theta+k)}{\lceil(\theta)}=\begin{cases}1 & (k=0,\ \theta\in C\backslash\{0\}),\\ \theta\left(\theta+1\right)\ldots\left(\theta+k-1\right) & (k\in N,\ \theta\in C).\end{cases}$$

We note that $I_{1,2}^{1}f(z)=f(z)$ and $I_{0,2}^{0}f(z)=zf'(z)$.

From (1.4), we obtain the following relations:

$$\lambda z(I_{\lambda,\mu}^{n+1}f(z))'=I_{\lambda,\mu}^{n}f(z)-(1-\lambda)I_{\lambda,\mu}^{n+1}f(z)\quad(\lambda>0) \tag{1.5}$$

and

$$z(I_{\lambda,\mu}^{n}f(z))'=\mu I_{\lambda,\mu+1}^{n}f(z)-(\mu-1)I_{\lambda,\mu}^{n}f(z). \tag{1.6}$$

Next, by using the operator $I_{\lambda,\mu}^{n}$, we introduce the following classes of analytic functions for ϕ,ψ:

$$S_{\lambda,\mu}^{n}\left(\eta;\phi\right)=\left\{f\in A:I_{\lambda,\mu}^{n}f(z)\in S^{*}\left(\eta;\phi\right)\right\},$$

$$K_{\lambda,\mu}^{n}\left(\eta;\phi\right)=\left\{f\in A:I_{\lambda,\mu}^{n}f(z)\in K\left(\eta;\phi\right)\right\}$$

and

$$C_{\lambda,\mu}^{n}(\eta,\delta;\phi,\psi)=\left\{f\in A:I_{\lambda,\mu}^{n}f(z)\in C\left(\eta,\delta;\phi,\psi\right)\right\}.$$

We also note that

$$f(z)\in K_{\lambda,\mu}^{n}(\eta;\phi)\Longleftrightarrow zf'(z)\in S_{\lambda,\mu}^{n}\left(\eta;\phi\right). \tag{1.7}$$

In particular, we set

$$S_{\lambda,\mu}^{n}\left(\eta;\left(\frac{1+Az}{1+Bz}\right)^{\alpha}\right)=S_{\lambda,\mu}^{n}(\eta;A,B;\alpha)\quad(0<\alpha\leq 1;\ -1\leq B<A\leq 1)$$

and

$$K_{\lambda,\mu}^{n}\left(\eta;\left(\frac{1+Az}{1+Bz}\right)^{\alpha}\right)=K_{\lambda,\mu}^{n}(\eta;A,B;\alpha)\quad(0<\alpha\leq 1;\ -1\leq B<A\leq 1).$$

We note that for $\lambda=1$ in the above classes, we obtain the following classes $S_{\mu}^{n}(\eta;\phi)$, $K_{\mu}^{n}(\eta;\phi)$ and $C_{\mu}^{n}(\eta,\delta;\phi,\psi)$.

In this paper, we investigate several inclusion properties of the classes $S_{\lambda,\mu}^{n}(\eta;\phi)$, $K_{\lambda,\mu}^{n}(\eta;\phi)$ and $C_{\lambda,\mu}^{n}(\eta,\delta;\phi,\psi)$ associated with the operator $I_{\lambda,\mu}^{n}$. Some applications involving these and other classes of integral operators are also considered.

2. Inclusion properties involving the operator $\boldsymbol{I_{\lambda,\mu}^{n}}$. The following lemmas will be required in our investigation.

Lemma 1 ([4]). *Let ϕ be convex univalent in U with $\phi(0)=1$ and* $\operatorname{Re}\{\mu\phi(z)+\nu\}>0$ *$(\mu,\nu\in C)$. If p is analytic in U with $p(0)=1$, then*

$$p(z)+\frac{zp'(z)}{\mu p(z)+\nu}\prec\phi(z)\quad(z\in U)$$

implies that

$$p(z)\prec\phi(z)\quad(z\in U).$$

Lemma 2 ([8]). *Let ϕ be convex univalent in U and w be analytic in U with* $\operatorname{Re}\{w(z)\}\geq 0$. *If p is analytic in U and $p(0)=\phi(0)$, then*

$$p(z)+w(z)zp'(z)\prec\phi(z)\quad(z\in U)$$

implies that

$$p(z)\prec\phi(z)\quad(z\in U).$$

At first, with the help of Lemma 1, we obtain the following theorem.

Theorem 1. *Let $n\in N_0$, $\lambda>0$, $\mu\geq 1$ and* $\operatorname{Re}\{(1-\eta)\phi(z)+\frac{1}{\lambda}-1+\eta\}>0$. *Then we have*

$$S_{\lambda,\mu+1}^{n}(\eta;\phi)\subset S_{\lambda,\mu}^{n}(\eta;\phi)\subset S_{\lambda,\mu}^{n+1}(\eta;\phi)$$

$(0\leq\eta<1;\ \phi\in S)$.

Proof. First of all, we will show that

$$S_{\lambda,\mu+1}^{n}(\eta;\phi)\subset S_{\lambda,\mu}^{n}(\eta;\phi).$$

Let $f\in S_{\lambda,\mu+1}^{n}(\eta;\phi)$ and put

$$p(z)=\frac{1}{1-\eta}\left(\frac{z\left(I_{\lambda,\mu}^{n}f(z)\right)'}{I_{\lambda,\mu}^{n}f(z)}-\eta\right),\tag{2.1}$$

where $p(z)$ is analytic in U with $p(0) = 1$. Using the identity (1.6) in (2.1), we obtain

$$\mu \frac{I^n_{\lambda,\mu+1} f(z)}{I^n_{\lambda,\mu} f(z)} = (1-\eta)p(z) + \mu - 1 + \eta. \tag{2.2}$$

Differentiating (2.2) logarithmically with respect to z and multiplying by z, we obtain

$$\frac{1}{1-\eta}\left(\frac{z\left(I^n_{\lambda,\mu+1} f(z)\right)'}{I^n_{\lambda,\mu+1} f(z)} - \eta\right) = p(z) + \frac{zp'(z)}{(1-\eta)p(z) + \mu - 1 + \eta} \tag{2.3}$$

$(z \in U)$. Applying Lemma 1 to (2.3), we see that $p(z) \prec \phi(z)$, that is, $f \in S^n_{\lambda,\mu}(\eta;\phi)$.

To prove the second part, let $f \in S^n_{\lambda,\mu}(\eta;\phi)$ and put

$$h(z) = \frac{1}{1-\eta}\left(\frac{z\left(I^{n+1}_{\lambda,\mu} f(z)\right)'}{I^{n+1}_{\lambda,\mu} f(z)} - \eta\right),$$

where h is analytic in U with $h(0) = 1$. Then, by using the arguments similar to these detailed above with (1.5), it follows that $h \prec \phi \ (z \in U)$, which implies that $f \in S^{n+1}_{\lambda,\mu}(\eta;\phi)$. This completes the proof of Theorem 1. □

Theorem 2. *Let $n \in N_0$, $\lambda > 0$ and $\mu \geq 1$. Then we have*

$$K^n_{\lambda,\mu+1}(\eta;\phi) \subset K^n_{\lambda,\mu}(\eta;\phi) \subset K^{n+1}_{\lambda,\mu}(\eta;\phi)$$

$(0 \leq \eta < 1;\ \phi \in S)$.

Proof. Applying (1.7) and Theorem 1, we observe that

$$\begin{aligned}
f(z) \in K^n_{\lambda,\mu+1}(\eta;\phi) &\iff I^n_{\lambda,\mu+1} f(z) \in K(\eta;\phi)\\
&\iff z(I^n_{\lambda,\mu+1} f(z))' \in S^*(\eta;\phi)\\
&\iff I^n_{\lambda,\mu+1}(zf'(z)) \in S^*(\eta;\phi)\\
&\iff zf'(z) \in S^n_{\lambda,\mu+1}(\eta;\phi)\\
&\implies zf'(z) \in S^n_{\lambda,\mu}(\eta;\phi)\\
&\iff I^n_{\lambda,\mu}(zf'(z)) \in S^*(\eta;\phi)\\
&\iff z(I^n_{\lambda,\mu} f(z))' \in S^*(\eta;\phi)\\
&\iff I^n_{\lambda,\mu} f(z) \in K(\eta;\phi)\\
&\iff f(z) \in K^n_{\lambda,\mu}(\eta;\phi)
\end{aligned}$$

and

$$\begin{aligned}
f(z) \in K^n_{\lambda,\mu}(\eta;\phi) &\iff zf'(z) \in S^n_{\lambda,\mu}(\eta;\phi)\\
&\implies zf'(z) \in S^{n+1}_{\lambda,\mu}(\eta;\phi)\\
&\iff z(I^{n+1}_{\lambda,\mu} f(z))' \in S^*(\eta;\phi)\\
&\iff I^{n+1}_{\lambda,\mu} f(z) \in K(\eta;\phi)\\
&\iff f(z) \in K^{n+1}_{\lambda,\mu}(\eta;\phi),
\end{aligned}$$

which evidently proves the theorem. □

Remark. Taking

$$\phi(z) = \left(\frac{1+Az}{1+Bz}\right)^{\alpha} \quad (-1 \le B < A \le 1;\ 0 < \alpha \le 1;\ z \in U)$$

in Theorems 1 and 2, we have the following corollary.

Corollary 1. *Let $n \in N_0$, $\lambda > 0$ and $\mu \ge 1$. Then we have*

$$S^{n}_{\lambda,\mu+1}(\eta;\ A,B;\ \alpha) \subset S^{n}_{\lambda,\mu}(\eta;\ A,B;\ \alpha) \subset S^{n+1}_{\lambda,\mu}(\eta;\ A,B;\ \alpha)$$

($0 \le \eta < 1$; $-1 \le B < A \le 1$; $0 < \alpha \le 1$), and

$$K^{n}_{\lambda,\mu+1}(\eta;\ A,B;\ \alpha) \subset K^{n}_{\lambda,\mu}(\eta;\ A,B;\ \alpha) \subset K^{n+1}_{\lambda,\mu}(\eta;\ A,B;\ \alpha)$$

($0 \le \eta < 1$; $-1 \le B < A \le 1$; $0 < \alpha \le 1$).

Next, by using Lemma 2, we obtain the following inclusion relation for the class $C^{n}_{\lambda,\mu}(\eta,\delta;\phi,\psi)$.

Theorem 3. *Let $n \in N_0$, $\lambda > 0$ and $\mu \ge 1$. Then we have*

$$C^{n}_{\lambda,\mu+1}(\eta,\delta;\phi,\psi) \subset C^{n}_{\lambda,\mu}(\eta,\delta;\phi,\psi) \subset C^{n+1}_{\lambda,\mu}(\eta,\delta;\phi,\psi)$$

($0 \le \eta,\ \delta < 1$; $\phi,\psi \in S$).

Proof. We begin by proving that

$$C^{n}_{\lambda,\mu+1}(\eta,\delta;\phi,\psi) \subset C^{n}_{\lambda,\mu}(\eta,\delta;\phi,\psi).$$

Let $f \in C^{n}_{\lambda,\mu+1}(\eta,\delta;\phi,\psi)$. Then, in view of the definition of the class $C^{n}_{\lambda,\mu+1}(\eta,\delta;\phi,\psi)$, there exists a function $g \in S^{n}_{\lambda,\mu+1}(\eta;\phi)$ such that

$$\frac{1}{1-\delta}\left(\frac{z\left(I^{n}_{\lambda,\mu+1}f(z)\right)'}{I^{n}_{\lambda,\mu+1}g(z)} - \delta\right) \prec \psi(z) \quad (z \in U).$$

Now let

$$p(z) = \frac{1}{1-\delta}\left(\frac{z\left(I^{n}_{\lambda,\mu}f(z)\right)'}{I^{n}_{\lambda,\mu}g(z)} - \delta\right),$$

where $p(z)$ is analytic in U with $p(0) = 1$. Using (1.6), we have

$$[(1-\delta)p(z)+\delta]I^{n}_{\lambda,\mu}g(z) + (\mu-1)I^{n}_{\lambda,\mu}f(z) = \mu I^{n}_{\lambda,\mu+1}f(z). \tag{2.4}$$

Differentiating (2.4) with respect to z and multiplying by z, we obtain

$$\begin{aligned}(1-\delta)zp'(z)I^{n}_{\lambda,\mu}g(z) + [(1-\delta)p(z)+\delta]z(I^{n}_{\lambda,\mu}g(z))' \\ = \mu z(I^{n}_{\lambda,\mu+1}f(z))' - (\mu-1)\, z(I^{n}_{\lambda,\mu}f(z))'.\end{aligned} \tag{2.5}$$

Since $g(z) \in S^n_{\lambda,\mu+1}(\eta;\phi)$, by Theorem 1, $g \in S^n_{\lambda,\mu}(\eta;\phi)$. Let

$$q(z) = \frac{1}{1-\eta}\left(\frac{z\left(I^n_{\lambda,\mu}g(z)\right)'}{I^n_{\lambda,\mu}g(z)} - \eta\right).$$

Then, using (1.6) once again, we have

$$\mu\frac{I^n_{\lambda,\mu+1}g(z)}{I^n_{\lambda,\mu}g(z)} = (1-\eta)q(z) + \mu - 1 + \eta. \tag{2.6}$$

From (2.5) and (2.6), we obtain

$$\frac{1}{1-\delta}\left(\frac{z\left(I^n_{\lambda,\mu+1}f(z)\right)'}{I^n_{\lambda,\mu+1}g(z)} - \delta\right) = p(z) + \frac{zp'(z)}{(1-\eta)\,q(z) + \mu - 1 + \eta}.$$

Since $0 \le \eta < 1,\ \mu \ge 1$ and $q(z) \prec \phi(z)\ (z \in U)$, we have

$$\mathrm{Re}\{(1-\eta)q(z) + \mu - 1 + \eta\} > 0 \quad (z \in U).$$

Hence, applying Lemma 2, we can show that $p(z) \prec \psi(z)$, so that $f \in C^n_{\lambda,\mu}(\eta,\delta;\phi,\psi)$.

For the second part, by using the arguments similar to these detailed above with (1.5), we obtain

$$C^n_{\lambda,\mu}(\eta,\delta;\phi,\psi) \subset C^{n+1}_{\lambda,\mu}(\eta,\delta;\phi,\psi).$$

This completes the proof of Theorem 3. □

3. Inclusion properties involving the integral operator F_c. In this section, we consider the generalized Libera integral operator F_c (see [2], [6] and [9]) defined by

$$F_c(f) = F_c(f)(z) = \frac{c+1}{z^c}\int_0^z t^{c-1}f(t)dt \quad (f \in A;\ c > -1). \tag{3.1}$$

We first prove the following theorem.

Theorem 4. *Let $c \ge 0,\ n \in N_0,\ \lambda > 0$ and $\mu > 0$. If $f \in S^n_{\lambda,\mu}(\eta;\phi)$ $(0 \le \eta < 1;\ \phi \in S)$, then we have $F_c(f) \in S^n_{\lambda,\mu}(\eta;\phi)$ $(0 \le \eta < 1;\ \phi \in S)$.*

Proof. Let $f \in S^n_{\lambda,\mu}(\eta;\phi)$ and put

$$p(z) = \frac{1}{1-\eta}\left(\frac{z\left(I^n_{\lambda,\mu}F_c\,(f)\,(z)\right)'}{I^n_{\lambda,\mu}F_c\,(f)\,(z)} - \eta\right), \tag{3.2}$$

where $p(z)$ is analytic in U with $p(0) = 1$. From (3.1), we have

$$z(I^n_{\lambda,\mu}F_c(f)(z))' = (c+1)I^n_{\lambda,\mu}f(z) - cI^n_{\lambda,\mu}F_c(f)(z). \tag{3.3}$$

Then, by using (3.2) and (3.3), we have

$$(c+1)\frac{I^n_{\lambda,\mu}f(z)}{I^n_{\lambda,\mu}F_c(f)(z)} = (1-\eta)p(z) + c + \eta. \tag{3.4}$$

Differentiating (3.4) logarithmically with respect to z and multiplying by z, we obtain

$$p(z) + \frac{zp'(z)}{(1-\eta)p(z)+c+\eta} = \frac{1}{1-\eta}\left(\frac{z(I^n_{\lambda,\mu}f(z))'}{I^n_{\lambda,\mu}f(z)} - \eta\right) \quad (z \in U).$$

Hence from Lemma 1, we conclude that $p(z) \prec \phi(z)$ $(z \in U)$, which implies $F_c(f) \in S^n_{\lambda,\mu}(\eta;\phi)$. □

Next, we derive an inclusion property involving F_c, which is given by the following theorem.

Theorem 5. *Let $c \geq 0$, $\lambda > 0$, $n \in N_0$ and $\mu > 0$. If $f \in K^n_{\lambda,\mu}(\eta;\phi)$ $(0 \leq \eta < 1;\ \phi \in S)$, then we have*

$$F_c(f) \in K^n_{\lambda,\mu}(\eta;\phi) \quad (0 \leq \eta < 1;\ \phi \in S).$$

Proof. By applying Theorem 4, we have

$$\begin{aligned} f(z) \in K^n_{\lambda,\mu}(\eta;\phi) &\iff zf'(z) \in S^n_{\lambda,\mu}(\eta;\phi) \\ &\implies F_c(zf'(z)) \in S^n_{\lambda,\mu}(\eta;\phi) \\ &\iff z(F_c(f)(z))' \in S^n_{\lambda,\mu}(\eta;\phi) \\ &\iff F_c(f)(z) \in K^n_{\lambda,\mu}(\eta;\phi) \end{aligned}$$

which proves Theorem 5. □

From Theorems 4 and 5, we have the following corollary.

Corollary 2. *Let $c \geq 0$, $\lambda > 0$, $n \in N_0$ and $\mu > 0$. If $f(z)$ belongs to the class $S^n_{\lambda,\mu}(\eta;A,B;\alpha)$ (or $K^n_{\lambda,\mu}(\eta;A,B;\alpha)$) $(0 \leq \eta < 1;\ -1 \leq B < A \leq 1;\ 0 < \alpha \leq 1)$, then $F_c(f)$ belongs to the class $S^n_{\lambda,\mu}(\eta;A,B;\alpha)$ (or $K^n_{\lambda,\mu}(\eta;A,B;\alpha)$) $(0 \leq \eta < 1;\ -1 \leq B < A \leq 1;\ 0 < \alpha \leq 1)$.*

Finally, we prove the following theorem.

Theorem 6. *Let $c \geq 0$, $\lambda > 0$, $n \in N_0$ and $\mu > 0$. If $f \in C^n_{\lambda,\mu}(\eta,\delta;\phi,\psi)$ $(0 \leq \eta,\ \delta < 1;\ \phi,\psi \in S)$, then we have $F_c(f) \in C^n_{\lambda,\mu}(\eta,\delta;\phi,\psi)$ $(0 \leq \eta,\ \delta < 1;\ \phi,\psi \in S)$.*

Proof. Let $f \in C^n_{\lambda,\mu}(\eta, \delta; \phi, \psi)$. Then, in view of the definition of the class $C^n_{\lambda,\mu}(\eta, \delta; \phi, \psi)$, there exists a function $g \in S^n_{\lambda,\mu}(\eta; \phi)$ such that

$$\frac{1}{1-\delta}\left(\frac{z(I^n_{\lambda,\mu}f(z))'}{I^n_{\lambda,\mu}g(z)} - \delta\right) \prec \psi(z) \quad (z \in U). \tag{3.5}$$

Thus, we put

$$p(z) = \frac{1}{1-\delta}\left(\frac{z\left(I^n_{\lambda,\mu}F_c(f)(z)\right)'}{I^n_{\lambda,\mu}F_c(g)(z)} - \delta\right),$$

where $p(z)$ is analytic in U with $p(0) = 1$. Since $g(z) \in S^n_{\lambda,\mu}(\eta; \phi)$, we see from Theorem 4 that $F_c(g) \in S^n_{\lambda,\mu}(\eta; \phi)$. Using (3.3), we have

$$[(1-\delta)p(z) + \delta]I^n_{\lambda,\mu}F_c(g)(z) + cI^n_{\lambda,\mu}F_c(f)(z) = (c+1)I^n_{\lambda,\mu}f(z). \tag{3.6}$$

Differentiating (3.6) with respect to z and multiplying by z, we obtain

$$(c+1)\frac{z(I^n_{\lambda,\mu}f(z))'}{I^n_{\lambda,\mu}F_c(g)(z)} = [(1-\delta)p(z) + \delta][(1-\eta)q(z) + c + \eta] + (1-\delta)zp'(z),$$

where

$$q(z) = \frac{1}{1-\eta}\left(\frac{z\left(I^n_{\lambda,\mu}F_c(g)(z)\right)'}{I^n_{\lambda,\mu}F_c(g)(z)} - \eta\right).$$

Hence, we have

$$\frac{1}{1-\delta}\left(\frac{z\left(I^n_{\lambda,\mu}f(z)\right)'}{I^n_{\lambda,\mu}g(z)} - \delta\right) = p(z) + \frac{zp'(z)}{(1-\eta)q(z) + c + \eta}.$$

The remaining part of the proof in Theorem 6 is similar to that of Theorem 3 and so we omit it. □

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