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Horizontal lift of symmetric connections to the bundle of volume forms $\mathcal{V}$

Abstract. In this paper we present the horizontal lift of a symmetric affine connection with respect to another affine connection to the bundle of volume forms $\mathcal{V}$ and give formulas for its curvature tensor, Ricci tensor and the scalar curvature. Next, we give some properties of the horizontally lifted vector fields and certain infinitesimal transformations. At the end, we consider some substructures of a $F(3,1)$-structure on $\mathcal{V}$.

1. Introduction. Throughout the paper we assume that $i, k, \ldots = 1, 2, 3, \ldots, n$ and $\alpha, \beta, \ldots = 0, 1, 2, \ldots, n$. Moreover, the Einstein summation convention will be used with respect to these systems of indices.

Let M be an orientable n-dimensional manifold, $\mathcal{V}$ be the bundle of volume forms over M and let $\pi : \mathcal{V} \to M$ be a projection of the bundle. We consider two local charts (U, x^i) and $(U', x^{i'})$ on M, $U \cap U' \neq \emptyset$ and the volume form $\omega \in \mathcal{V}$. Assume that form ω is given by

$$\omega = v(x) dx^i \wedge \ldots \wedge dx^n,$$

in the local chart (U, x^i), where $v > 0$ is a smooth function and

$$\omega = v'(x') dx^{i'} \wedge \ldots \wedge dx^{n'}$$

in the chart $(U, x^{i'})$. Let functions $x^{i'} = x^{i'}(x)$ be orientation-preserving transition functions on manifold M. Then the transition functions on $\mathcal{V}$ are

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given by the following formulas

$$v' = \bar{\mathcal{I}} \cdot v, \quad x^{i'} = x^{i'}(x),$$

where $\bar{\mathcal{I}} = \det\left(\frac{\partial x^{i'}}{\partial x^j}\right)$ is the Jacobian of the map $x^{i'} = x^{i'}(x)$. Following Dhooghe ([3]), we introduce a new coordinate system $(x^0, x^1, \ldots, x^n)$ on $\mathcal{V}$, where $x^0 = \ln v$. Then the transition functions in the terms of these coordinate system are

$$x^{0'} = x^0(x) + \ln \bar{\mathcal{I}}(x)$$
$$x^{i'} = x^{i'}(x).$$

Let $\bar{\mathcal{J}}(x) = \ln \bar{\mathcal{I}}(x)$ and $\mathcal{J}(x') = \ln \mathcal{I}(x')$. Since $\mathcal{I} \cdot \bar{\mathcal{I}} = 1$, we have

$$\frac{\partial \mathcal{J}}{\partial x^{i'}} = -\frac{\partial \bar{\mathcal{J}}}{\partial x^j}\frac{\partial x^j}{\partial x^{i'}}$$

and

$$\frac{\partial \bar{\mathcal{J}}}{\partial x^i} = -\frac{\partial \mathcal{J}}{\partial x^{j'}}\frac{\partial x^{j'}}{\partial x^i}.$$

Then the Jacobi matrix of the transition functions on $\mathcal{V}$ has the following form

$$\begin{bmatrix} 1 & \dfrac{\partial \bar{\mathcal{J}}}{\partial x^i} \\ 0 & \dfrac{\partial x^{j'}}{\partial x^i} \end{bmatrix}.$$

For the further purposes we quote some theorems describing the properties of geometrical objects on the bundle of volume forms.

Theorem 1.1 ([3]). *Let $\nabla = (\Gamma^k_{ij})$ be a symmetric connection and $v = v^i \frac{\partial}{\partial x^i}$ be a vector field on a manifold M. Then*

$$\bar{v} = -v^i \Gamma^k_{ik} \frac{\partial}{\partial x^0} + v^i \frac{\partial}{\partial x^i}$$

is globally defined vector field on $\mathcal{V}$, which is called the horizontal lift of v.

Theorem 1.2 ([9]). *Let $\nabla = (\Gamma^k_{ij})$ be a symmetric connection and $g = (g_{ij})$ be a tensor of type $(0,2)$ on a manifold M. Then*

$$\bar{g} = \begin{bmatrix} 1 & \Gamma^k_{ik} \\ \Gamma^k_{ik} & g_{ij} + \Gamma^k_{ik}\Gamma^t_{jt} \end{bmatrix}$$

is globally defined $(0,2)$-tensor on $\mathcal{V}$, which is called the horizontal lift of g.

Theorem 1.3 ([9]). *Let $\nabla = (\Gamma^k_{ij})$ be a symmetric connection and g be a Riemannian metric on a manifold M. Then $\bar{g}$ is a Riemannian metric on $\mathcal{V}$ and*

$$(\bar{g})^{-1} = \begin{bmatrix} 1 + g^{ij}\Gamma^k_{ik}\Gamma^t_{jt} & -g^{ij}\Gamma^k_{ik} \\ -g^{ij}\Gamma^k_{ik} & g^{ij} \end{bmatrix}.$$

2. The horizontal lift of a symmetric connection. Curvatures of a horizontally lifted connection. At the beginning, we present a theorem on a horizontal lift of a symmetric connection with respect to another connection to the bundle of volume forms $\mathcal{V}$. Next, we give formulas for its curvature tensor, the Ricci tensor and the scalar curvature. In the sequel we will use the following conventions

$$\begin{aligned} \Gamma^k_{ij|m} &= \frac{\partial \Gamma^k_{ij}}{\partial x^m}, \\ \Gamma^k_{[ik|j]} &= \frac{1}{2}\left(\Gamma^k_{ik|j} - \Gamma^k_{jk|i}\right), \\ \Gamma^k_{[ik|jm]} &= \frac{1}{2}\left(\Gamma^k_{ik|jm} - \Gamma^k_{mk|ij}\right), \\ g_{jk|i} &= \frac{\partial g_{jk}}{\partial x^i}. \end{aligned}$$

Theorem 2.1. *Let $\nabla = (\Gamma^k_{ij})$ be a symmetric connection and $\nabla_1 = (\Phi^k_{ij})$ be a connection on M. Then an operator $\bar{\nabla}_1$ whose nonzero coefficients are given by*

$$\bar{\nabla}_{1_i}\frac{\partial}{\partial x^j} = \bar{\nabla}_{1_j}\frac{\partial}{\partial x^i} = \left(\Gamma^t_{it|j} - \Phi^r_{ij}\Gamma^t_{rt}\right)\frac{\partial}{\partial x^0} + \Phi^k_{ij}\frac{\partial}{\partial x^k}$$

is a linear connection on $\mathcal{V}$, which will be called the horizontal lift of the connection ∇_1 with respect to the connection ∇.

Proof. We are going to check that the coefficients $(\bar{\Phi}^\gamma_{\alpha\beta})$ of the connection $\bar{\nabla}_1$ satisfy the transformation rule of the connection. This transformation rule for the zero coefficients of the connection $\bar{\nabla}_1$ follows from simple calculations. For the nonzero coefficients we have

$$\begin{aligned} \bar{\Phi}^{k'}_{i'j'} &= \frac{\partial x^\alpha}{\partial x^{i'}}\frac{\partial x^\beta}{\partial x^{j'}}\frac{\partial x^{k'}}{\partial x^\gamma}\bar{\Phi}^\gamma_{\alpha\beta} + \frac{\partial x^{k'}}{\partial x^\alpha}\frac{\partial^2 x^\alpha}{\partial x^{i'}\partial x^{j'}} \\ &= \frac{\partial x^i}{\partial x^{i'}}\frac{\partial x^j}{\partial x^{j'}}\frac{\partial x^{k'}}{\partial x^k}\bar{\Phi}^k_{ij} + \frac{\partial x^{k'}}{\partial x^k}\frac{\partial^2 x^k}{\partial x^{i'}\partial x^{j'}} = \Phi^{k'}_{i'j'}. \end{aligned}$$

In the next part of the proof we use the following equality

$$\frac{\partial^2 x^{0'}}{\partial x^i \partial x^j} = \frac{\partial}{\partial x^i}\left(\frac{\partial \mathcal{J}}{\partial x^j}\right) = \frac{\partial}{\partial x^j}\left(\frac{\partial \mathcal{J}}{\partial x^i}\right).$$

We receive from above identity

$$\frac{\partial^2 x^d}{\partial x^{a'}\partial x^{r'}}\frac{\partial^2 x^{r'}}{\partial x^j \partial x^d}\frac{\partial x^{a'}}{\partial x^i} = \frac{\partial^2 x^d}{\partial x^{a'}\partial x^{r'}}\frac{\partial^2 x^{r'}}{\partial x^i \partial x^d}\frac{\partial x^{a'}}{\partial x^j}.$$

For the coefficients $(\bar{\Phi}^{0'}_{i'j'})$ we have

$$\begin{aligned}
\bar{\Phi}^{0'}_{i'j'} &= \frac{\partial x^{\alpha}}{\partial x^{i'}}\frac{\partial x^{\beta}}{\partial x^{j'}}\frac{\partial x^{0'}}{\partial x^{\gamma}}\bar{\Phi}^{\gamma}_{\alpha\beta} + \frac{\partial x^{0'}}{\partial x^{\alpha}}\frac{\partial^2 x^{\alpha}}{\partial x^{i'}\partial x^{j'}} \\
&= \frac{\partial x^{a}}{\partial x^{i'}}\frac{\partial x^{b}}{\partial x^{j'}}\bar{\Phi}^{0}_{ab} + \frac{\partial x^{a}}{\partial x^{i'}}\frac{\partial x^{b}}{\partial x^{j'}}\frac{\partial \mathcal{J}}{\partial x^{p}}\bar{\Phi}^{p}_{ab} + \frac{\partial^2 x^{0}}{\partial x^{i'}\partial x^{j'}} + \frac{\partial \mathcal{J}}{\partial x^{p}}\frac{\partial^2 x^{p}}{\partial x^{i'}\partial x^{j'}} \\
&= \frac{\partial x^{a}}{\partial x^{i'}}\frac{\partial x^{b}}{\partial x^{j'}}\left[\Gamma^{t}_{at|b} - \Phi^{r}_{ab}\Gamma^{t}_{rt}\right] + \frac{\partial x^{a}}{\partial x^{i'}}\frac{\partial x^{b}}{\partial x^{j'}}\frac{\partial \mathcal{J}}{\partial x^{p}}\Phi^{p}_{ab} \\
&\quad + \frac{\partial^2 x^{0}}{\partial x^{i'}\partial x^{j'}} + \frac{\partial \bar{\mathcal{J}}}{\partial x^{p}}\frac{\partial^2 x^{p}}{\partial x^{i'}\partial x^{j'}} \\
&= \frac{\partial x^{a}}{\partial x^{i'}}\frac{\partial x^{b}}{\partial x^{j'}}\frac{\partial}{\partial x^{b}}\left[\frac{\partial x^{c'}}{\partial x^{a}}\Gamma^{t'}_{c't'} + \frac{\partial \bar{\mathcal{J}}}{\partial x^{a}}\right] \\
&\quad - \frac{\partial x^{a}}{\partial x^{i'}}\frac{\partial x^{b}}{\partial x^{j'}}\left[\frac{\partial x^{d'}}{\partial x^{a}}\frac{\partial x^{e'}}{\partial x^{b}}\frac{\partial x^{r}}{\partial x^{f'}}\Phi^{f'}_{d'e'} + \frac{\partial x^{r}}{\partial x^{e'}}\frac{\partial^2 x^{e'}}{\partial x^{a}\partial x^{b}}\right]\cdot\left[\frac{\partial x^{c'}}{\partial x^{r}}\Gamma^{t'}_{c't'} + \frac{\partial \bar{\mathcal{J}}}{\partial x^{r}}\right] \\
&\quad + \frac{\partial x^{a}}{\partial x^{i'}}\frac{\partial x^{b}}{\partial x^{j'}}\frac{\partial \bar{\mathcal{J}}}{\partial x^{p}}\left[\frac{\partial x^{d'}}{\partial x^{a}}\frac{\partial x^{e'}}{\partial x^{b}}\frac{\partial x^{p}}{\partial x^{f'}}\Phi^{f'}_{d'e'} + \frac{\partial x^{p}}{\partial x^{e'}}\frac{\partial^2 x^{e'}}{\partial x^{a}\partial x^{b}}\right] \\
&\quad + \frac{\partial^2 x^{0}}{\partial x^{i}\partial x^{j}} + \frac{\partial \bar{\mathcal{J}}}{\partial x^{p}}\frac{\partial^2 x^{p}}{\partial x^{i'}\partial x^{j'}} \\
&= \frac{\partial^2 x^{c'}}{\partial x^{a}\partial x^{b}}\frac{\partial x^{a}}{\partial x^{i'}}\frac{\partial x^{b}}{\partial x^{j'}}\Gamma^{t'}_{c't'} + \Gamma^{t'}_{i't'|j} + \frac{\partial^2 \bar{\mathcal{J}}}{\partial x^{a}\partial x^{b}}\frac{\partial x^{a}}{\partial x^{i'}}\frac{\partial x^{b}}{\partial x^{j'}} - \Phi^{c'}_{i'j'}\Gamma^{t'}_{c't'} \\
&\quad + \frac{\partial \mathcal{J}}{\partial x^{f'}}\Phi^{f'}_{i'j'} - \frac{\partial^2 x^{c'}}{\partial x^{a}\partial x^{b}}\frac{\partial x^{a}}{\partial x^{i'}}\frac{\partial x^{b}}{\partial x^{j'}}\Gamma^{t'}_{c't'} + \frac{\partial \mathcal{J}}{\partial x^{e'}}\frac{\partial^2 x^{e'}}{\partial x^{a}\partial x^{b}}\frac{\partial x^{a}}{\partial x^{i'}}\frac{\partial x^{b}}{\partial x^{j'}} \\
&\quad - \frac{\partial \mathcal{J}}{\partial x^{f'}}\Phi^{f'}_{i'j'} - \frac{\partial \mathcal{J}}{\partial x^{e'}}\frac{\partial^2 x^{e'}}{\partial x^{a}\partial x^{b}}\frac{\partial x^{a}}{\partial x^{i'}}\frac{\partial x^{b}}{\partial x^{j'}} + \frac{\partial^2 \mathcal{J}}{\partial x^{i'}\partial x^{j'}} + \frac{\partial \bar{\mathcal{J}}}{\partial x^{p}}\frac{\partial^2 x^{p}}{\partial x^{i'}\partial x^{j'}} \\
&= \Gamma^{t'}_{i't'|j'} - \Phi^{c'}_{i'j'}\Gamma^{t'}_{c't'} + \frac{\partial^2 \bar{\mathcal{J}}}{\partial x^{a}\partial x^{b}}\frac{\partial x^{a}}{\partial x^{i'}}\frac{\partial x^{b}}{\partial x^{j'}} + \frac{\partial^2 \mathcal{J}}{\partial x^{i'}\partial x^{j'}} + \frac{\partial \bar{\mathcal{J}}}{\partial x^{p}}\frac{\partial^2 x^{p}}{\partial x^{i'}\partial x^{j'}}.
\end{aligned}$$

Now, we have

$$\frac{\partial^2 \bar{\mathcal{J}}}{\partial x^{a}\partial x^{b}}\frac{\partial x^{a}}{\partial x^{i'}}\frac{\partial x^{b}}{\partial x^{j'}} = \frac{\partial^2 x^{e}}{\partial x^{p'}\partial x^{i'}}\frac{\partial^2 x^{p'}}{\partial x^{e}\partial x^{b}}\frac{\partial x^{b}}{\partial x^{j'}} + \frac{\partial^3 x^{p'}}{\partial x^{e}\partial x^{b}\partial x^{a}}\frac{\partial x^{e}}{\partial x^{p'}}\frac{\partial x^{a}}{\partial x^{i'}}\frac{\partial x^{b}}{\partial x^{j'}}$$

and

$$\begin{aligned}
\frac{\partial^3 x^{p'}}{\partial x^{e}\partial x^{a}\partial x^{b}}\frac{\partial x^{e}}{\partial x^{p'}}\frac{\partial x^{a}}{\partial x^{i'}}\frac{\partial x^{b}}{\partial x^{j'}} &= -2\frac{\partial^2 x^{p'}}{\partial x^{e}\partial x^{b}}\frac{\partial^2 x^{e}}{\partial x^{p'}\partial x^{i'}}\frac{\partial x^{b}}{\partial x^{j'}} - \frac{\partial^3 x^{e}}{\partial x^{p'}\partial x^{i'}\partial x^{j'}}\frac{\partial x^{p'}}{\partial x^{e}} \\
&\quad - \frac{\partial \mathcal{J}}{\partial x^{r'}}\frac{\partial^2 x^{r'}}{\partial x^{a}\partial x^{b}}\frac{\partial x^{a}}{\partial x^{i'}}\frac{\partial x^{b}}{\partial x^{j'}}, \\
\frac{\partial^2 \mathcal{J}}{\partial x^{i'}\partial x^{j'}} &= \frac{\partial^2 x^{e'}}{\partial x^{a}\partial x^{b}}\frac{\partial x^{b}}{\partial x^{i'}}\frac{\partial^2 x^{a}}{\partial x^{e'}\partial x^{j'}} + \frac{\partial^3 x^{a}}{\partial x^{e'}\partial x^{j'}\partial x^{i'}}\frac{\partial x^{e'}}{\partial x^{a}}.
\end{aligned}$$

Moreover,

$$0=\frac{\partial}{\partial x^b}\left[\frac{\partial x^{r'}}{\partial x^a}\frac{\partial x^a}{\partial x^{i'}}\right]=\frac{\partial^2 x^{r'}}{\partial x^a\partial x^b}\frac{\partial x^a}{\partial x^{i'}}+\frac{\partial x^{r'}}{\partial x^a}\frac{\partial^2 x^a}{\partial x^{i'}\partial x^{p'}}\frac{\partial x^{p'}}{\partial x^b}$$

and hence

$$\begin{aligned}-\frac{\partial \mathcal{J}}{\partial x^{r'}}\frac{\partial^2 x^{r'}}{\partial x^a\partial x^b}\frac{\partial x^a}{\partial x^{i'}}\frac{\partial x^b}{\partial x^{j'}}&=\frac{\partial \mathcal{J}}{\partial x^{r'}}\frac{\partial x^{r'}}{\partial x^a}\frac{\partial x^2 x^a}{\partial x^{i'}\partial x^{p'}}\frac{\partial x^{p'}}{\partial x^b}\frac{\partial x^b}{\partial x^{j'}}\\&\quad-\frac{\partial \mathcal{J}}{\partial x^{r'}}\frac{\partial^2 x^{r'}}{\partial x^a\partial x^b}\frac{\partial x^a}{\partial x^{i'}}\frac{\partial x^b}{\partial x^{j'}}\\&=-\frac{\partial \bar{\mathcal{J}}}{\partial x^a}\frac{\partial^2 x^a}{\partial x^{i'}\partial x^{j'}}.\end{aligned}$$

Using the above formulas, we get

$$\begin{aligned}\bar{\Phi}^{0'}_{i'j'}&=\Gamma^{t'}_{i't'|j'}-\Phi^{c'}_{i'j'}\Gamma^{t'}_{c't'}+\frac{\partial^2 x^e}{\partial x^{p'}\partial x^{i'}}\frac{\partial^2 x^{p'}}{\partial x^e\partial x^b}\frac{\partial x^b}{\partial x^{j'}}-2\frac{\partial^2 x^{p'}}{\partial x^e\partial x^b}\frac{\partial^2 x^e}{\partial x^{p'}\partial x^{i'}}\frac{\partial x^b}{\partial x^{j'}}\\&\quad-\frac{\partial^3 x^e}{\partial x^{p'}\partial x^{i'}\partial x^{j'}}\frac{\partial x^{p'}}{\partial x^e}-\frac{\partial \mathcal{J}}{\partial x^{r'}}\frac{\partial^2 x^{r'}}{\partial x^a\partial x^b}\frac{\partial x^a}{\partial x^{i'}}\frac{\partial x^b}{\partial x^{j'}}+\frac{\partial^2 x^{e'}}{\partial x^a\partial x^b}\frac{\partial x^b}{\partial x^{i'}}\frac{\partial^2 x^a}{\partial x^{e'}\partial x^{j'}}\\&\quad+\frac{\partial^3 x^a}{\partial x^{e'}\partial x^{i'}\partial x^{j'}}\frac{\partial x^{e'}}{\partial x^a}+\frac{\partial \bar{\mathcal{J}}}{\partial x^p}\frac{\partial^2 x^p}{\partial x^{i'}\partial x^{j'}}\\&=\Gamma^{t'}_{i't'|j'}-\Phi^{c'}_{i'j'}\Gamma^{t'}_{c't'}-\frac{\partial \mathcal{J}}{\partial x^{r'}}\frac{\partial^2 x^{r'}}{\partial x^a\partial x^b}\frac{\partial x^a}{\partial x^{i'}}\frac{\partial x^b}{\partial x^{j'}}+\frac{\partial \bar{\mathcal{J}}}{\partial x^p}\frac{\partial^2 x^p}{\partial x^{i'}\partial x^{j'}}\\&=\Gamma^{t'}_{i't'|j'}-\Phi^{c'}_{i'j'}\Gamma^{t'}_{c't'}-\frac{\partial \mathcal{J}}{\partial x^{r'}}\frac{\partial x^2 x^{r'}}{\partial x^a\partial x^b}\frac{\partial x^a}{\partial x^{i'}}\frac{\partial x^b}{\partial x^{j'}}+\frac{\partial \bar{\mathcal{J}}}{\partial x^p}\frac{\partial^2 x^p}{\partial x^{i'}\partial x^{j'}}\\&=\Gamma^{t'}_{i't'|j'}-\Phi^{c'}_{i'j'}\Gamma^{t'}_{c't'}-\frac{\partial \bar{\mathcal{J}}}{\partial x^p}\frac{\partial^2 x^p}{\partial x^{i'}\partial x^{j'}}+\frac{\partial \bar{\mathcal{J}}}{\partial x^p}\frac{\partial^2 x^p}{\partial x^{i'}\partial x^{j'}}\\&=\Gamma^{t'}_{i't'|j'}-\Phi^{c'}_{i'j'}\Gamma^{t'}_{c't'}.\end{aligned}$$

□

Now, we give formulas for coefficients of a curvature tensor, a Ricci tensor and a scalar curvature for the horizontally lifted connection $\bar{\nabla}_1$. Let R be the curvature tensor of a connection ∇ on a manifold M. The coefficients (R^s_{ijk}) of R are expressed in terms of the connection $\nabla=(\Gamma^k_{ij})$ by the formula ([1])

$$R^s_{ikj}=\Gamma^l_{ik}\Gamma^s_{jl}-\Gamma^l_{jk}\Gamma^s_{il}+\Gamma^s_{ik|j}-\Gamma^s_{jk|i}.$$

Theorem 2.2. *Let $(\bar{R}^\gamma_{\alpha\beta})$ be the coefficients of the curvature tensor $\bar{R}$ of the horizontal lift of the connection $\nabla_1=(\Phi^k_{ij})$ with respect to the symmetric connection $\nabla=(\Gamma^k_{ij})$. Then nonzero coefficients of the tensor $\bar{R}$ are given by the following formulas*

$$\bar{R}^s_{ikj}=R^s_{ikj},$$

$$\bar{R}^0_{ikj}=-\Gamma^t_{rt}R^r_{ikj}+2\Phi^d_{ik}\Gamma^t_{[jt|d]}+2\Phi^d_{jk}\Gamma^t_{[dt|i]}+2\Gamma^t_{[it|kj]},$$

where $R=(R^s_{ijk})$ is the curvature tensor of the connection ∇_1 on M.

Proof. From the definition of the curvature tensor we have

$$\bar{R}^{\tau}_{\beta\gamma\delta} = \bar{\Phi}^{\alpha}_{\beta\gamma}\bar{\Phi}^{\tau}_{\delta\alpha} - \bar{\Phi}^{\alpha}_{\delta\gamma}\bar{\Phi}^{\tau}_{\beta\alpha} + \bar{\Phi}^{\tau}_{\beta\gamma|\delta} - \bar{\Phi}^{\tau}_{\delta\gamma|\beta}.$$

The statements

$$\bar{R}^{\beta}_{000} = \bar{R}^{\beta}_{i00} = \bar{R}^{\beta}_{0k0} = \bar{R}^{\beta}_{00j} = \bar{R}^{\beta}_{0kj} = \bar{R}^{\beta}_{i0j} = \bar{R}^{\beta}_{ik0} = 0$$

and

$$\bar{R}^{s}_{ikj} = R^{s}_{ikj}$$

follows from the definition of the curvature tensor, Theorem 2.1 and simple calculations. Next, we have

$$\begin{aligned}
\bar{R}^{0}_{ikj} &= \bar{\Phi}^{\alpha}_{ik}\bar{\Phi}^{0}_{j\alpha} - \bar{\Phi}^{\alpha}_{jk}\bar{\Phi}^{0}_{i\alpha} + \bar{\Phi}^{0}_{ik|j} - \bar{\Phi}^{0}_{jk|i} \\
&= \Phi^{d}_{ik}\left(\Gamma^{t}_{jt|d} - \Phi^{r}_{jd}\Gamma^{t}_{rt}\right) - \Phi^{d}_{jk}\left(\Gamma^{t}_{it|d} - \Phi^{r}_{id}\Gamma^{t}_{rt}\right) + \frac{\partial}{\partial x^{j}}\left(\Gamma^{t}_{it|k} - \Phi^{r}_{ik}\Gamma^{t}_{rt}\right) \\
&\quad - \frac{\partial}{\partial x^{i}}\left(\Gamma^{t}_{jt|k} - \Phi^{d}_{jk}\Gamma^{r}_{rt}\right) \\
&= -\Gamma^{t}_{rt}\left(\Phi^{d}_{ik}\Phi^{r}_{jd} - \Phi^{d}_{jk}\Phi^{r}_{id} + \Phi^{r}_{ik|j} - \Phi^{d}_{jk|i}\right) + \Phi^{d}_{ik}\left(\Gamma^{t}_{jt|d} - \Gamma^{t}_{dt|j}\right) \\
&\quad + \Phi^{d}_{jk}\left(\Gamma^{t}_{dt|i} - \Gamma^{t}_{it|d}\right) + \Gamma^{t}_{it|kj} - \Gamma^{t}_{jt|ki} \\
&= -\Gamma^{t}_{rt}R^{r}_{ikj} + 2\Phi^{d}_{ik}\Gamma^{t}_{[jt|d]} + 2\Phi^{d}_{jk}\Gamma^{t}_{[dt|i]} + 2\Gamma^{t}_{[it|kj]}.
\end{aligned}$$

□

We recall that the linear connection ∇ is locally volume preserving at $p \in M$, if there exists a neighbourhood U of p in M and a volume form $\omega \in U$ such that $\nabla\omega = 0$ ([4]). The integrability conditions for existence of such a local volume forms in local coordinates (U, x^{i}) on a neighbourhood of $p \in M$ are given by

$$\Gamma^{k}_{ik|j} = \Gamma^{k}_{jk|i},$$

where (Γ^{k}_{ij}) are the coefficients of the connection ∇ ([13]). The connection is called globally volume preserving, if such a volume form exists on M. Moreover, it is well known that the Riemannian connection on the Riemannian manifold M is locally volume preserving.

Corollary 2.1. *Let ∇ be a Riemannian connection on a Riemannian manifold (M, g) and ∇_1 be a connection on the manifold M. Then the nonzero coefficients of the curvature tensor $\bar{R} = \left(\bar{R}^{\delta}_{\alpha\beta\gamma}\right)$ of the horizontally lifted connection $\bar{\nabla}_1$ with respect to the connection $\nabla = (\Gamma^{k}_{ij})$ are given by*

$$\begin{aligned}
\bar{R}^{s}_{ikj} &= R^{s}_{ikj}, \\
\bar{R}^{0}_{ikj} &= -\Gamma^{t}_{rt}R^{r}_{ikj},
\end{aligned}$$

where $R = (R^{s}_{ijk})$ is the curvature tensor of the connection ∇_1 on M.

For a Ricci tensor and a scalar curvature of the horizontally lifted connection we have the following theorems.

Theorem 2.3. *Let ∇ be a symmetric connection, ∇_1 be a connection on the manifold M and $(\bar{R}_{\alpha\beta})$ be the coefficients of a Ricci tensor $\bar{R}$ of the horizontally lifted connection $\bar{\nabla}_1$ with respect to connection ∇. Then the nonzero coefficients of $\bar{R}$ are given by the formulas*

$$\bar{R}_{ik} = R_{ik},$$

where (R_{ik}) are the coefficients of Ricci tensor R of the connection ∇_1 on the manifold M.

Theorem 2.4. *Let g be a Riemannian metric on the manifold M and let $\bar{g}$ be a horizontally lifted Riemannian metric on $\mathcal{V}$. If $\bar{K}$ is a scalar curvature of the horizontally lifted connection $\bar{\nabla}_1$ with respect to the symmetric connection ∇, then*

$$\bar{K} = \frac{n-1}{n+1}K,$$

where K is a scalar curvature of the connection ∇ on M.

Proof. From the definition of the scalar curvature, Theorem 2.3 and formula for $(\bar{g}^{\alpha\beta})$ we have

$$\bar{K} = \frac{1}{n(n+1)}\bar{R}_{\alpha\beta}\bar{g}^{\alpha\beta} = \frac{1}{n(n+1)}R_{ik}g^{ik} = \frac{n-1}{n+1}K. \qquad \square$$

Let π be a non-singular tensor field of type $(1,1)$ on the manifold M. Rompała in [12] described some properties of π-conjugate connection with respect to a given connection. Now, we prove that if we have a π-conjugate connection ∇_2 with respect to a connection ∇_1 on the manifold M, then the horizontally lifted connection $\bar{\nabla}_2$ is $\bar{\pi}$-conjugate with respect to the horizontally lifted connection $\bar{\nabla}_1$ on $\mathcal{V}$, where $\bar{\pi}$ is the horizontal lift of π.

Definition 2.1. Let $\nabla = (\Gamma^k_{ij})$ be the linear connection and let π be a non-singular tensor field of the type $(0,2)$ on M. The connection $\nabla^* = (G^i_{ks})$ which is given by

$$G^i_{ks} = \pi^{pi}\nabla_k\pi_{ps} + \Gamma^i_{ks}$$

is said to be a π-conjugate connection with respect to the connection ∇.

For the horizontally lifted connections $\bar{\nabla}_1$ and $\bar{\nabla}_2$ with respect to the connection ∇ and the horizontally lifted tensor field $\bar{\pi}$ of type $(0,2)$ we have the following theorem.

Theorem 2.5. *Let ∇_2 be a π-conjugate connection with respect to a connection ∇_1 on manifold M. Let $\bar{\nabla}_1$ and $\bar{\nabla}_2$ be horizontally lifted connections with respect to a connection ∇ on $\mathcal{V}$. Then $\bar{\nabla}_2$ is a $\bar{\pi}$-conjugate connection with respect to a horizontally lifted connection $\bar{\nabla}_1$, where $\bar{\pi}$ is horizontal lift of π with respect to a connection ∇.*

Proof. We are going to show the nonzero coefficients $(\bar{G}^{\gamma}_{\alpha\beta})$ of the connection $\bar{\nabla}_2$ are given by the formulas

$$\bar{G}^k_{ij} = G^k_{ij},$$
$$\bar{G}^0_{ij} = \Gamma^t_{it|j} - G^r_{ij}\Gamma^t_{rt}.$$

From Definition 2.2 and definition of the covariant derivative of a tensor field of type $(0,2)$ we have

$$\bar{G}^{\tau}_{\gamma\varrho} = \bar{\pi}^{\beta\tau}(\bar{\pi}_{\beta\varrho|\gamma} - \bar{\Phi}^{\alpha}_{\beta\gamma}\bar{\pi}_{\alpha\varrho} - \bar{\Phi}^{\alpha}_{\varrho\gamma}\bar{\pi}_{\beta\alpha}) + \bar{\Phi}^{\tau}_{\gamma\varrho},$$

where $(\bar{\Phi}^{\gamma}_{\alpha\beta})$ are the coefficients of the connection $\bar{\nabla}_1$ and $\bar{\pi} = (\bar{\pi}_{\alpha\beta})$. For the nonzero coefficients we get

$$\begin{aligned}
&\bar{\pi}^{\beta 0}(\bar{\pi}_{\beta s|k} - \bar{\Phi}^{\alpha}_{\beta k}\bar{\pi}_{\alpha s} - \bar{\Phi}^{\alpha}_{sk}\bar{\pi}_{\beta\alpha}) + \bar{\Phi}^0_{ks}\\
&= -\pi^{db}\Gamma^v_{bv}\left[\pi_{ds|k} - \Phi^r_{dk}\pi_{rs} - \Phi^r_{sk}\pi_{dr}\right] + \Gamma^t_{kt|s} - \Phi^r_{ks}\Gamma^u_{ru}\\
&= \Gamma^t_{kt|s} - \Gamma^v_{bv}G^b_{ks} = \bar{G}^0_{ks},\\
&\bar{\pi}^{\beta i}(\bar{\pi}_{\beta s|k} - \bar{\Phi}^{\alpha}_{\beta k}\bar{\pi}_{\alpha s} - \bar{\Phi}^{\alpha}_{sk}\bar{\pi}_{\beta\alpha}) + \bar{\Phi}^i_{ks}\\
&= \bar{\pi}^{0i}(\Gamma^t_{st|k} - \Gamma^t_{st|k} - \Phi^r_{sk}\Gamma^u_{ru} - \Phi^r_{sk}\Gamma^t_{rt})\\
&\quad + \pi^{di}\Big[\pi_{ds|k} + \Gamma^t_{dt|k}\Gamma^u_{su} + \Gamma^t_{dt}\Gamma^u_{su|k} - \Gamma^t_{dt|k}\Gamma^u_{su} + \Phi^r_{dk}\Gamma^t_{rt}\Gamma^u_{su}\\
&\quad -\Phi^r_{dk}(\pi_{rs} + \Gamma^t_{rt}\Gamma^u_{su}) - \Gamma^t_{st|k}\Gamma^u_{du} + \Phi^r_{sk}\Gamma^t_{rt}\Gamma^u_{du} - \Phi^r_{sk}(\pi_{dr} + \Gamma^t_{dt}\Gamma^u_{ru})\Big] + \Phi^i_{ks}\\
&= \pi^{di}(\pi_{ds|k} - \Phi^r_{dk}\pi_{rs} - \Phi^r_{sk}\pi_{dr}) + \Phi^i_{ks} = G^i_{ks} = \bar{G}^i_{ks}.
\end{aligned}$$

□

3. Some properties of a horizontally lifted vector field. Dhooghe in [3] described the horizontal lift of a vector field to the bundle of volume forms $\mathcal{V}$. In this chapter we give some properties of such horizontally lifted vector fields. Let us consider when a horizontally lifted vector field is a Killing field on $\mathcal{V}$. We have

Theorem 3.1. *Let (M,g) be a Riemannian manifold. If X is a vector field and ∇ is a symmetric, locally volume preserving connection on M, then the horizontally lifted vector field $\bar{X}$ is a Killing field on $(\mathcal{V},\bar{g})$ if and only if X is a Killing field on M.*

Proof. Let $\nabla = (\Gamma^k_{ij})$. Then X is a Killing field on the manifold (M,g) if and only if $\mathcal{L}_X g = 0$, where $\mathcal{L}$ is a Lie derivative of the Riemannian metric g ([7]). Let X be the Killing field on the manifold (M,g). From the assumption that the ∇ is the symmetric, locally volume preserving connection and from the formula of the Lie derivative of the metric $\bar{g}$ with respect to $\bar{X}$ we have

$$\mathcal{L}_{\bar{X}}\bar{g}_{00} = (-\Gamma^k_{jk}X^k)_{|0} + (-\Gamma^k_{jk}X^k)_{|0} = 0,$$
$$\mathcal{L}_{\bar{X}}\bar{g}_{0b} = \mathcal{L}_{\bar{X}}\bar{g}_{b0} = X^e(\Gamma^t_{bt|e} - \Gamma^t_{et|b}) = 0,$$

$$\begin{aligned}
\mathcal{L}_{\bar{X}}\bar{g}_{bc} &= \bar{X}^{\alpha}\bar{g}_{bc|\alpha} + \bar{g}_{\alpha c}\bar{X}^{\alpha}_{|b} + \bar{g}_{b\alpha}\bar{X}^{\alpha}_{|c} \\
&= X^{d}\left(g_{bc} + \Gamma^{t}_{bt}\Gamma^{u}_{cu}\right)_{|c} + \Gamma^{t}_{ct}\left(-X^{d}\Gamma^{u}_{du}\right)_{|b} \\
&\quad + \left(g_{dc} + \Gamma^{u}_{du}\Gamma^{t}_{ct}\right)X^{d}_{|b} + \Gamma^{t}_{bt}\left(-X^{d}\Gamma^{u}_{du}\right)_{|c} + \left(g_{bd} + \Gamma^{u}_{bu}\Gamma^{t}_{dt}\right)X^{d}_{|c} \\
&= X^{d}g_{bc|d} + X^{d}\Gamma^{t}_{bt|d}\Gamma^{u}_{cu} + X^{d}\Gamma^{t}_{bt}\Gamma^{u}_{cu|d} = \mathcal{L}_{X}g_{bc} = 0.
\end{aligned}$$

We have $\mathcal{L}_{\bar{X}}\bar{g} = 0$ so $\bar{X}$ is the Killing field on $\mathcal{V}$.

Let $\bar{X}$ be the Killing on the bundle of volume forms $\mathcal{V}$. Then we have $\mathcal{L}_{\bar{X}}\bar{g} = 0$ and from the first part of the proof we get $\mathcal{L}_{X}g = 0$, so X is the Killing field on the manifold (M, g). □

Yamauchi in [15] studied certain types of an infinitesimal transformations on tangent bundles. Now, we show that the horizontally lifted vector field $\bar{X}$ is an infinitesimal affine transformation of horizontally lifted connection on $\mathcal{V}$ if and only if the vector field X is an infinitesimal affine transformation of a connection on the manifold M.

Theorem 3.2. *Let $\bar{\nabla}_1$ be the horizontal lift of the connection ∇_1 with respect to the symmetric connection ∇ on M and let $\bar{X}$ be the horizontal lift of the vector field X on $\mathcal{V}$. Then $\bar{X}$ is an infinitesimal affine transformation of the horizontally lifted connection $\bar{\nabla}_1$ if and only if X is an infinitesimal affine transformation of the connection ∇_1 on M.*

Proof. Let X be the infinitesimal affine transformation ([15]) of the connection $\nabla_1 = (\Phi^{k}_{ij})$ on M. Then we have

$$\mathcal{L}_{X}\Phi^{k}_{ij} = 0.$$

For the horizontally lifted connection $\bar{\nabla}_1 = (\bar{\Phi}^{\gamma}_{\alpha\beta})$ and the horizontally lifted vector field $\bar{X}$ we have

$$\begin{aligned}
&\mathcal{L}_{\bar{X}}\bar{\Phi}^{\chi}_{00} = \mathcal{L}_{\bar{X}}\bar{\Phi}^{\chi}_{0l} = \mathcal{L}_{\bar{X}}\bar{\Phi}^{\chi}_{m0} = 0, \\
&\mathcal{L}_{\bar{X}}\bar{\Phi}^{h}_{ml} = \mathcal{L}_{X}\Phi^{h}_{ml} = 0, \\
&\mathcal{L}_{\bar{X}}\bar{\Phi}^{0}_{ml} = -\Gamma^{t}_{dt}\mathcal{L}_{X}\Phi^{d}_{ml} = 0.
\end{aligned}$$

Thus, $\bar{X}$ is the infinitesimal affine transformation of the connection $\bar{\nabla}_1$ on $\mathcal{V}$.

On the other hand, let $\bar{X}$ be the infinitesimal affine transformation of the connection $\bar{\nabla}_1$ on $\mathcal{V}$. Then from the first part of this proof we get

$$\mathcal{L}_{X}\Phi^{k}_{ij} = 0$$

and X is the infinitesimal affine transformation of the connection ∇_1 on the manifold M. □

Now, we will examine a problem of a fibre-preserving infinitesimal transformation on the bundle of volume forms $\mathcal{V}$. We have the following theorem.

Theorem 3.3. *Let $\bar{X}$ be the horizontal lift of the vector field X to the bundle of volume forms $\mathcal{V}$. Then $\bar{X}$ is a fibre-preserving infinitesimal transformation on $\mathcal{V}$.*

Proof. Since the fibres of the bundle $\mathcal{V}$ form a trivial foliation we have ([8]) that the horizontally lifted vector field is the fibre-preserving infinitesimal transformation on $\mathcal{V}$ if and only if the coordinates $\bar{X}^1, \dots, \bar{X}^n$ of the vector field $\bar{X} = \bar{X}^i \dfrac{\partial}{\partial x^i} + \bar{X}^0 \dfrac{\partial}{\partial x^0}$ depends only of the coordinates $(x^1, x^2, \dots, x^n)$. From the above and Theorem 1.1 we get that the horizontally lifted vector field $\bar{X}$ is the fibre-preserving infinitesimal transformation on the bundle $\mathcal{V}$. □

At the end of this chapter we show that a horizontally lifted vector field is not a projective infinitesimal transformation of a horizontally lifted connection and is not a conformal infinitesimal transformation on $\mathcal{V}$ with respect to horizontally lifted Riemannian metric. In the next part of the paper we will use the following theorem.

Theorem 3.4. *Let ∇ be a locally volume preserving connection on a Riemannian manifold (M, g). If there exists a nonzero function ϱ such that $\mathcal{L}_X g = 2\varrho g$, then $\mathcal{L}_{\bar{X}} \bar{g}_{\alpha 0} = 0$ and $\mathcal{L}_{\bar{X}} \bar{g}_{bc} = 2\varrho^{\mathcal{V}} g_{bc}$, where $\bar{X}$ is the horizontal lift of the vector field X on a Riemannian manifold $(\mathcal{V}, \bar{g})$ and $\varrho^{\mathcal{V}} = \varrho \circ \pi$.*

Proof. It follows from definition of the Lie derivatives of the Riemannian metric and simple calculations. □

Using the above theorem we get the following

Theorem 3.5. *Let X be a vector field and let g be a Riemannian metric on M. Then the horizontally lifted vector field $\bar{X}$ is never a conformal infinitesimal transformation on $\mathcal{V}$.*

Proof. Let the horizontally lifted vector field $\bar{X}$ be the conformal infinitesimal transformation ([15]) on $\mathcal{V}$. Then there exist the nonzero function f on $\mathcal{V}$ such that

$$\mathcal{L}_{\bar{X}} \bar{g} = 2f\bar{g}.$$

From Theorem 3.4 we have that $\mathcal{L}_{\bar{X}} \bar{g}_{00} = 0$ and $\bar{g}_{00} = 1$. So $f = 0$ and $\bar{X}$ is never the conformal infinitesimal transformation on $\mathcal{V}$. □

Theorem 3.6. *Let ∇_1, ∇ be symmetric connections on a connected manifold M. If $\bar{X}$ denotes the horizontal lift of the vector field X to the bundle of volume forms $\mathcal{V}$ and $\bar{\nabla}_1$ denotes the horizontal lift of the connection ∇_1 with respect to the connection ∇, then $\bar{X}$ is never the infinitesimal projective transformation on $\mathcal{V}$.*

Proof. Let $\bar{X}$ be the infinitesimal projective transformation ([15]) on $\mathcal{V}$. Then exist a nonzero 1-form ϕ such that

$$\mathcal{L}_{\bar{X}}\bar{\Phi}^{\gamma}_{\alpha\beta} = \delta^{\gamma}_{\alpha}\phi_{\beta} + \delta^{\gamma}_{\beta}\phi_{\alpha}.$$

From the proof of Theorem 3.2 we have

$$\mathcal{L}_{\bar{X}}\bar{\Phi}^{k}_{0m} = \mathcal{L}_{\bar{X}}\bar{\Phi}^{0}_{0m} = \mathcal{L}_{\bar{X}}\bar{\Phi}^{k}_{00} = \mathcal{L}_{\bar{X}}\bar{\Phi}^{0}_{00} = 0.$$

On the other hand, we have

$$\begin{aligned}\mathcal{L}_{\bar{X}}\bar{\Phi}^{k}_{m0} &= \delta^{k}_{m}\phi_{0},\\ \mathcal{L}_{\bar{X}}\bar{\Phi}^{0}_{m0} &= \phi_{m},\\ \mathcal{L}_{\bar{X}}\bar{\Phi}^{0}_{00} &= 2\phi_{0}.\end{aligned}$$

Hence we get

$$\begin{aligned}\delta^{k}_{m}\phi_{0} &= 0,\\ \phi_{m} &= 0,\\ \phi_{0} + \phi_{m} &= 0,\end{aligned}$$

so, $\phi = 0$ and $\bar{X}$ is never the infinitesimal projective transformation on the bundle of volume forms $\mathcal{V}$. □

4. $\bar{F}(3,1)$-structures on $\mathcal{V}$. In this chapter we consider some tensor structures on $\mathcal{V}$ which depend of a tensor of type $(1,1)$. In [9] authors defined a horizontal lift of a tensor field of type $(1,1)$ to the bundle of volume forms $\mathcal{V}$.

Theorem 4.1 ([9]). *Let $F = (F^{j}_{i})$ be a tensor field of type $(1,1)$ and let $\nabla = (\Gamma^{k}_{ij})$ be a linear connection on manifold M. Then*

$$\bar{F} = \begin{bmatrix} 1 & -F^{t}_{j}\Gamma^{k}_{tk} + \Gamma^{k}_{jk} \\ 0 & F^{i}_{j} \end{bmatrix}$$

is a tensor field of type $(1,1)$ on the bundle of volume forms, which is called a horizontal lift of the tensor field F.

A tensor field of type $(1,1)$ defines some interesting structures on manifolds. Now, we recall definition of a $F(k,(-1)^{k+1})$-structure.

Theorem 4.2 ([2]). *Let F be a nonzero tensor field of type $(1,1)$ on an n-dimensional manifold M such that*

$$F^{k} - (-1)^{k+1}F = 0$$

and

$$F^{m} - (-1)^{m+1}F \neq 0$$

for $1 < m < k$, where k is a fixed positive integer greater than 2. Such a structure is called $F(k,(-1)^{k+1})$-structure on M.

We have the following theorem for $F(k,(-1)^{k+1})$-structure on the manifold M.

Theorem 4.3 ([2]). *Let F be a tensor field of type $(1,1)$ which defines a $F(k,(-1)^{k+1})$-structure on a manifold M. The $F(k,(-1)^{k+1})$-structure is integrable if and only if $N_F(X,Y)=0$ for any vector fields X, Y on the manifold M, where N_F is the Nijenhuis tensor field of the tensor F.*

Applying Definition 4.1 and Theorem 4.2 to the horizontally lifted tensor field $\bar{F}$ on $\mathcal{V}$ we obtain the following theorems.

Theorem 4.4. *Let F be a tensor field of type $(1,1)$ which define the $F(3,1)$-structure on a manifold M and let ∇ be a linear connection on M. Then the horizontally lifted tensor field $\bar{F}$ defines the $\bar{F}(3,1)$-structure on the bundle of volume forms $\mathcal{V}$.*

Theorem 4.5. *Let F be a tensor field of type $(1,1)$ which define the $F(3,1)$-structure on manifold the M and let ∇ be a linear volume-preserving connection on M. Then the horizontally lifted tensor field $\bar{F}$ defines an integrable $\bar{F}(3,1)$-structure on $\mathcal{V}$ if and only if the $F(3,1)$-structure is integrable on M.*

In the next part of this chapter we consider some special substructures of F-structures on a manifold M. The authors in [14] defined some substructures of differential manifolds with F-structures by using a covariant derivative and a Lie derivative. Now, we recall necessary definitions and theorems.

Definition 4.1 ([14]). Let F be a tensor field of type $(1,1)$ on a manifold M. The manifold M is called $F(3,\varepsilon)$-manifold, if

$$F^3 = \varepsilon F,$$

where $\varepsilon = \pm 1$.

Let

$$L = \varepsilon F^2$$

and

$$A = 1 - \varepsilon F^2.$$

From [6] we know that on $F(3,\varepsilon)$-manifold there always exists a Riemannian metric g satisfying a condition

$$g(X,Y) = g(FX,FY) + g(AX,Y).$$

This metric is called the Ishihara–Yano metric. Let G be a tensor field of type $(0,2)$ defined by the form

$$G(X,Y) = g(FX,Y).$$

Let ∇ be a metric connection of the Ishihara–Yano metric g on the manifold M. Then we have the following definitions.

Definition 4.2 ([14]). Let M be $F(3,\varepsilon)$-manifold. Then the F-structure is called FK-structure if and only if

$$\nabla_{FX}(F) = 0,$$

FAK-structure if and only if

$$dS(FX, FY, FZ) = 0,$$

FNK-structure if and only if

$$\nabla_{FX}(G)(FY, FZ) - \varepsilon\nabla_{FY}(G)(FX, FZ) = 0,$$

FQK-structure if and only if

$$\begin{aligned}&2\nabla_{FX}(G)(FY, FZ) + (1-\varepsilon)\nabla_{F^2X}(G)(F^2Y, FZ)\\&= (1+\varepsilon)\left[\nabla_{F^2Z}(G)(FX, F^2Y) - \nabla_{F^2Y}(G)(FZ, F^2X)\right],\end{aligned}$$

FH-structure if and only if

$$N(FX, FY) = 0,$$

where S is a 1-form defined by $S(X,Y) = -S(Y,X) = g(FX,Y)$, N is a Nijenhuis tensor of F, X, Y, Z are vector fields on the manifold M and $G(X,Y) = g(FX,Y)$.

The authors in [14] studied a problem of inclusion of these substructures and they gave the following theorems:

Theorem 4.6 ([14]). *Let M be a $F(3,\varepsilon)$-manifold with a FK, FAK, FNK, FQK, FH-structure. Then we have*

$$FK \begin{matrix}\subseteq FAK\\ \subseteq FNK\end{matrix}\Bigg\} \subseteq FQK.$$

Theorem 4.7 ([14]). *Let F be a tensor field of type $(1,1)$ which defines a $F(3,\varepsilon)$-structure on a manifold M. Then the $F(3,\varepsilon)$-structure is a FH-structure, if*

$$\nabla_{FX}(F)(FY) = F\nabla_X(F)(FY),$$

where ∇ is the Levi-Civita connection of an Ishihara–Yano metric g.

Let F be a tensor field of type $(1,1)$ on a manifold M which gives a $F(3,\varepsilon)$-structure and let g be a Ishihara–Yano metric on the manifold M. It is easy to check that a horizontally lifted Riemannian metric $\bar{g}$ on $\mathcal{V}$ is Ishihara–Yano metric on $\mathcal{V}$, thus we have the following theorem for $\bar{g}$:

Theorem 4.8 ([5]). *Let g be a Riemannian metric on manifold M and let $\bar{g}$ be the horizontal lift of the Riemannian metric g to the bundle of volume forms $\mathcal{V}$. Then the nonzero coefficients of a Levi-Civita connection $\widetilde{\nabla}$ for the horizontally lifted Riemannian metric $\bar{g}$ are given by the formulas*

$$\widetilde{\nabla}_n \frac{\partial}{\partial x^m} = \widetilde{\nabla}_m \frac{\partial}{\partial x^n} = \left(\Gamma^t_{mt|n} - \Gamma^t_{mn}\Gamma^k_{tk}\right)\frac{\partial}{\partial x^0} + \Gamma^s_{mn}\frac{\partial}{\partial x^s},$$

where $\nabla = (\Gamma^k_{ij})$ is the Levi-Civita connection on M.

Now, we consider a problem of particular substructures of $\bar{F}(3,1)$-structure on the bundle of volume forms $\mathcal{V}$.

Theorem 4.9. *Let F be a tensor field of type $(1,1)$ which defines a $F(3,\varepsilon)$-structure and ∇ be the Levi-Civita connection on a Riemannian manifold (M,g). Let $\bar{g}$ be the horizontally lifted Riemannian metric and let $\widetilde{\nabla}$ be the Levi-Civita connection on the Riemannian manifold $(\mathcal{V},\bar{g})$. Then the horizontal lift $\bar{F}$ of the tensor field F defines the $\bar{F}QK$-structure on $\mathcal{V}$ if and only if the tensor field F defines the FQK-structure on M.*

Proof. First we determine coefficients of a tensor $\bar{G} = (\bar{g}_{\alpha\beta}\bar{F}^\alpha_\gamma)$ on the bundle on volume forms $\mathcal{V}$ and we get

$$\begin{aligned}
\bar{G}_{00} &= \bar{g}_{\alpha 0}\bar{F}^\alpha_0 = 1,\\
\bar{G}_{i0} &= \bar{G}_{\alpha 0}\bar{F}^\alpha_i = \Gamma^k_{ik},\\
\bar{G}_{0j} &= \bar{g}_{\alpha j}\bar{F}^\alpha_0 = \Gamma^k_{jk},\\
\bar{G}_{ij} &= \bar{g}_{\alpha j}\bar{F}^\alpha_i = g_{dj}F^d_i + \Gamma^k_{ik}\Gamma^u_{ju}.
\end{aligned}$$

In our case $\varepsilon = 1$ and the condition from Definition 4.3 takes the form

$$\nabla_{FX}(G)(FY,FZ) = \nabla_{F^2Z}(G)(FX,F^2Y) - \nabla_{F^2Y}(G)(FZ,F^2X).$$

Let the tensor field F define the FQK-structure on the manifold M. We want to show that then on $\mathcal{V}$ the condition

$$\nabla_{\bar{F}X}(\bar{G})(\bar{F}Y,\bar{F}Z) = \nabla_{\bar{F}^2Z}(\bar{G})(\bar{F}X,\bar{F}^2Y) - \nabla_{\bar{F}^2Y}(\bar{G})(\bar{F}Z,\bar{F}^2X)$$

is true, where $\bar{F}$ is the horizontal lift of the tensor F nad X, Y, Z are the vector fields on $\mathcal{V}$. For the nonzero terms of the left side of the above formula we get

$$\begin{aligned}
&\bar{F}^\alpha_b(\nabla_\alpha \bar{G}_{\delta\gamma})\bar{F}^\delta_p\bar{F}^\gamma_c = \bar{F}^d_b(\nabla_d\bar{G}_{0\gamma})\bar{F}^0_p\bar{F}^\gamma_c + \bar{F}^d_b(\nabla_d\bar{G}_{r\gamma})\bar{F}^r_p\bar{F}^\gamma_c\\
&= F^d_b(\nabla_d\bar{G}_{ru})F^r_pF^u_c = F^d_b\left(\bar{G}_{ru|d} - \bar{\Gamma}^\alpha_{rd}\bar{G}_{\alpha u} - \bar{\Gamma}^\alpha_{ud}\bar{G}_{\alpha r}\right)F^r_pF^u_c\\
&= F^d_b\left(g_{au|d}F^a_r + g_{au}F^a_{r|d} - g_{au}F^a_v\Gamma^v_{rd} - g_{av}F^a_r\Gamma^v_{ud}\right)F^r_pF^u_c\\
&= F^d_b\left(\nabla_d G_{ru}\right)F^r_pF^u_c.
\end{aligned}$$

For the terms of the right side of the condition from Definition 4.3 we have

$$\bar{H}^\alpha_c(\nabla_\alpha \bar{G}_{\delta\gamma})\bar{F}^\delta_b \bar{H}^\gamma_p - \bar{H}^\alpha_p(\nabla_\alpha \bar{G}_{\delta\gamma})F^\delta_c \bar{H}^\gamma_b = \bar{H}^d_c(\nabla_d \bar{G}_{ru})\bar{F}^r_b \bar{H}^u_p$$
$$- \bar{H}^d_p(\nabla_d \bar{G}_{ru})\bar{F}^r_c \bar{H}^u_b = H^d_c(\nabla_d G_{ru})F^r_a H^u_b - H^d_p(\nabla_d G_{ru})F^r_c H^u_b,$$

where

$$\bar{H} = \bar{F}^2 = \begin{bmatrix} 1 & -F^t_i F^i_a \Gamma^k_{tk} + \Gamma^u_{au} \\ 0 & F^j_i F^i_a \end{bmatrix} = \begin{bmatrix} 1 & -F^t_i F^i_a \Gamma^k_{tk} + \Gamma^u_{au} \\ 0 & H^j_a \end{bmatrix} = \begin{bmatrix} \bar{H}^0_0 & \bar{H}^0_a \\ \bar{H}^j_0 & \bar{H}^j_a \end{bmatrix}.$$

So, on the bundle of volume forms we have

$$F^d_b\,(\nabla_d G_{ru})\,F^r_p F^u_c = H^d_c(\nabla_d G_{ru})F^r_a H^u_b - H^d_p(\nabla_d G_{ru})F^r_c H^u_b$$

and from the assumption, that F defines the FQK-structure on the manifold M, we have that $\bar{F}$ defines $\bar{F}QK$-structure on $\mathcal{V}$.

On the other hand, let $\bar{F}$ defines the $\bar{F}QK$-structure on $\mathcal{V}$, then we have

$$\nabla_{\bar{F}X}(\bar{G})(\bar{F}Y, \bar{F}Z) = \nabla_{\bar{F}^2 Z}(\bar{G})(\bar{F}X, \bar{F}^2 Y) - \nabla_{\bar{F}^2 Y}(\bar{G})(\bar{F}Z, \bar{F}^2 X).$$

From the calculations from the first part of this proof we get

$$F^d_b\,(\nabla_d G_{ru})\,F^r_p F^u_c = H^d_c(\nabla_d G_{ru})F^r_a H^u_b - H^d_p(\nabla_d G_{ru})F^r_c H^u_b.$$

So, F defines the FQK-structure on the manifold M. □

From the above theorem and Theorem 4.4 we get

Corollary 4.1. *Let F be a tensor field of type $(1,1)$ which defines a $F(3,\varepsilon)$-structure and ∇ be the Levi-Civita connection on a Riemannian manifold (M,g). Let $\bar{g}$ be a horizontally lifted Riemannian metric and let $\widetilde{\nabla}$ be the Levi-Civita connection on the Riemannian manifold $(\mathcal{V},\bar{g})$. Then a horizontally lifted tensor field $\bar{F}$ defines the $\bar{F}K$, $\bar{F}AK$, $\bar{F}NK$-structure on $\mathcal{V}$ if and only if the tensor F defines the FK, FAK, FNK-structure on the M, respectively.*

At the end of this paper we give a theorem on a $\bar{F}H$-structure on the bundle of volume forms.

Theorem 4.10. *Let F be a tensor field of type $(1,1)$ which defines a $F(3,\varepsilon)$-structure and ∇ be the Levi-Civita connection on a Riemannian manifold (M,g). Let $\bar{g}$ be a horizontally lifted Riemannian metric and let $\widetilde{\nabla}$ be the Levi-Civita connection on a Riemannian manifold $(\mathcal{V},\bar{g})$. Then a horizontally lifted tensor field $\bar{F}$ defines a $\bar{F}H$-structure on $\mathcal{V}$ if and only if the tensor field F defines a FH-structure on M.*

Proof. Let F define the FH-structure on the manifold M. Then from the Theorem 4.5 we have

$$F^t_i(\nabla_t F^j_a)F^a_s = F^j_t(\nabla_i F^t_a)F^a_s.$$

We prove that on the bundle of volume forms $\mathcal{V}$ we get

$$\widetilde{\nabla}_{\bar{F}X}(\bar{F})(\bar{F}Y) = \bar{F}\widetilde{\nabla}_X(\bar{F})(\bar{F}Y),$$

for all vector fields X, Y, Z on $\mathcal{V}$. For the nonzero terms of the left side of the above formula we get

$$\bar{F}_a^\alpha(\widetilde{\nabla}_\alpha \bar{F}_\beta^0)\bar{F}_s^\beta = -F_a^l\Gamma_{tk}^k(\nabla_l F_j^t)F_s^j,$$
$$\bar{F}_a^\alpha(\widetilde{\nabla}_\alpha \bar{F}_\beta^i)\bar{F}_s^\beta = F_a^l(\nabla_l F_j^i)F_s^j.$$

The terms of the right side corresponding to the nonzero terms of the left side, are equal to

$$(\widetilde{\nabla}_l \bar{F}_\beta^\alpha)\bar{F}_\alpha^0\bar{F}_s^\beta = -F_i^t F_s^j\Gamma_{tk}^k(\nabla_l F_j^i),$$
$$(\widetilde{\nabla}_l \bar{F}_\beta^\alpha)\bar{F}_\alpha^k\bar{F}_s^\beta = (\widetilde{\nabla}_l \bar{F}_\beta^i)\bar{F}_i^k\bar{F}_s^j = (\nabla_l F_j^i)F_i^k F_s^j.$$

From the assumption that F defines the FH-structure on the manifold M, we have

$$\bar{F}_a^\alpha(\widetilde{\nabla}_\alpha \bar{F}_\beta^0)\bar{F}_s^\beta = -F_a^l\Gamma_{tk}^k(\nabla_l F_j^t)F_s^j = -F_i^t F_s^j\Gamma_{tk}^k(\nabla_l F_j^i) = (\widetilde{\nabla}_l \bar{F}_\beta^\alpha)\bar{F}_\alpha^0\bar{F}_s^\beta,$$
$$\bar{F}_a^\alpha(\widetilde{\nabla}_\alpha \bar{F}_\beta^i)\bar{F}_s^\beta = F_a^l(\nabla_l F_j^i)F_s^j = (\nabla_l F_j^i)F_i^k F_s^j = (\widetilde{\nabla}_l \bar{F}_\beta^\alpha)\bar{F}_\alpha^k\bar{F}_s^\beta$$

and the horizontally lifted tensor field $\bar{F}$ defines the $\bar{F}H$-structure on $\mathcal{V}$.

On the other hand, let $\bar{F}$ define the $\bar{F}H$-structure on $\mathcal{V}$. Then from the first part of the proof we have

$$F_c^d\Gamma_{uk}^k(\nabla_d F_r^u)F_b^r = F_r^u\Gamma_{uk}^k(\nabla_c F_d^r)F_b^d,$$
$$F_c^d(\nabla_d F_r^a)F_b^r = F_d^a(\nabla_c F_r^d)F_b^r.$$

Thus the tensor field F defines the FH-structure on the manifold M. □

References

[1] do Carmo, M. P., *Riemannian Geometry*, Graduate Texts in Mathematics 166, Birkhäuser Boston Inc., Boston Ma., 1992.

[2] Das, L. S., *Complete lifts of a structure satisfying* $F^K - (-1)^{K+1} = 0$, Internat. J. Math. Math. Sci. **15** (1992), 803–808.

[3] Dhooghe, P. F., *The T. Y. Thomas construction of projectively related manifolds*, Geom. Dedicata, **55** (1995), 221–235.

[4] Dhooghe, P. F., Van Vlierden A., *Projective geometry on the bundle of volume forms*, J. Geom. **62** (1998), 66–83.

[5] Gąsior, A., *Curvatures for horizontal lift of Riemannian metric*, Ann. Univ. Mariae Curie-Skłodowska Sect. A **60** (2006), 17–21.

[6] Ishihara, S., Yano, K., *Structure defined by* f *satisfying* $f^3 + f = 0$, Proc. U.S.-Japan Seminar in Differential Geometry (Kyoto, 1965) pp. 153–166 Nippon Hyoronsha, Tokyo, 1966.

[7] Kobayashi. S., Nomizu, K., *Foundations of Differential Geometry*, John Wiley & Sons, New York–London, 1969.

[8] Molino, P., *Riemannian Foliations*, Progression Mathematics, 73, Birkhäuser Boston Inc., Boston Ma., 1988.

[9] Miernowski A., Mozgawa W., *Horizontal Lift to the Bundle of Volume Forms*, Ann. Univ. Mariae Curie-Skłodowska Sect. A **57** (2003), 69–75.

[10] Norden, A. P., *Spaces with Affine Connection*, Izdat. Nauka, Moscow, 1976 (Russian).

[11] Radziszewski, K., π*-geodesics and lines of shadow*, Colloq. Math. **26** (1972), 157–163.

[12] Rompała, W., *Liftings of π-conjugate connections*, Ann. Univ. Mariae Curie-Skłodowska Sect. A **32** (1978), 109–126.
[13] Schouten, J. A., *Ricci-Calculus*, 2nd ed., Springer-Verlag, Berlin, Göttingen, Heidelberg, 1954.
[14] Singh, K. D., Singh, R., *Some $f(3,\varepsilon)$-structure manifolds*, Demonstratio Math. **10** (1977), 637–645.
[15] Yamauchi, K., *On Riemannian manifolds admitting infinitesimal projective transformations*, Hokkaido Math. J. **16** (1987), 115–125.
[16] Yano, K., *On structure defined by tensor field f of type $(1,1)$ satisfying $f^3+f=0$*, Tensor (N.S) **14** (1963), 99–109.

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