doi: 10.2478/v10062-010-0007-9

ANNALES
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LUBLIN – POLONIA

VOL. LXIV, NO. 1, 2010 SECTIO A 81–91

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On the order of starlikeness and convexity of complex harmonic functions with a two-parameter coefficient condition

Abstract. The article of J. Clunie and T. Sheil-Small [3], published in 1984, intensified the investigations of complex functions harmonic in the unit disc Δ. In particular, many papers about some classes of complex mappings with the coefficient conditions have been published. Consideration of this type was undertaken in the period 1998–2004 by Y. Avci and E. Złotkiewicz [2], A. Ganczar [5], Z. J. Jakubowski, G. Adamczyk, A. Łazińska and A. Sibelska [1], [8], [7], H. Silverman [12] and J. M. Jahangiri [6], among others. This work continues the investigations described in [7]. Our results relate primarily to the order of starlikeness and convexity of functions of the aforementioned classes.

1. Introduction.

Let

$$\Delta := \{z \in \mathbb{C} : |z| < 1\},$$
$$\Delta_r := \{z \in \mathbb{C} : |z| < r\}, \quad r > 0,$$
$$A := \{(\alpha, p) \in \mathbb{R}^2 : 0 \le \alpha \le 1,\ p > 0\},$$
$$U_n(\alpha, p) := \alpha\, n^p + (1-\alpha) n^{p+1}, \quad n = 2, 3, \dots, \quad (\alpha, p) \in A.$$

In [7] the following classes of complex harmonic functions have been introduced:

2000 *Mathematics Subject Classification.* 30C45, 31A05.

Key words and phrases. Complex harmonic functions, analytic conditions, convexity of order β, starlikeness of order β.

Definition 1. Let (α, p) be a fixed pair of parameters in the set A. By $\mathcal{HS}(\alpha, p)$ we denote the class of the functions f of the form:

$$\begin{aligned} &f(z) = h(z) + \overline{g(z)}, \\ &h(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad g(z) = \sum_{n=1}^{\infty} b_n z^n, \quad z \in \Delta, \quad |b_1| < 1, \end{aligned} \tag{1.1}$$

satisfying the condition

$$|b_1| + \sum_{n=2}^{\infty} U_n(\alpha, p)\big(|a_n| + |b_n|\big) \leq 1. \tag{1.2}$$

Definition 2. Let $(\alpha, p) \in A$ be fixed. Let us denote

$$\mathcal{HS}^0(\alpha, p) := \{f \in \mathcal{HS}(\alpha, p) : b_1 = 0\}.$$

The classes $\mathcal{HS}(1,1)$, $\mathcal{HS}^0(1,1)$, $\mathcal{HS}(1,2)$, $\mathcal{HS}^0(1,2)$ were investigated in the paper [2]. The results contained in [5] refer to the classes $\mathcal{HS}(1,p)$, $\mathcal{HS}^0(1,p)$, $p > 0$ and in [8] the classes $\mathcal{HS}(\alpha,1)$, $\mathcal{HS}^0(\alpha,1)$ for $\alpha \in [0,1]$ were considered.

In this paper we will show results related to the order of starlikeness and convexity of functions which belong to the aforementioned classes.

Recall the definition of complex harmonic functions starlike (convex) and starlike (convex) of the order β, $\beta \in [0,1)$.

Definition 3. A univalent and sense-preserving complex harmonic function f of the form (1.1) is called starlike with respect to the origin (starlike) in Δ, if $f(\Delta)$ is a domain starlike with respect to the origin.

Definition 4. A univalent and sense-preserving complex harmonic function f of the form (1.1) is called convex in Δ if $f(\Delta)$ is a convex domain.

Remark 1. It is known that in order to prove starlikeness of the image of the disc Δ by a univalent sense-preserving mapping f it is sufficient to prove starlikeness of $f(\Delta_r)$ for every $r \in (0,1)$, i.e. to show that for any $r \in (0,1)$ we have

$$\frac{\partial}{\partial\theta}\Big(\arg f(re^{i\theta})\Big) > 0, \quad \theta \in [0, 2\pi].$$

By analogy, in order to prove convexity of the image of the disc Δ by a univalent sense-preserving mapping f, it is sufficient to prove convexity of $f(\Delta_r)$ for every $r \in (0,1)$, i.e. to show that for any $r \in (0,1)$ we have

$$\frac{\partial}{\partial\theta}\Big(\arg \frac{\partial}{\partial\theta}(f(re^{i\theta}))\Big) > 0, \quad \theta \in [0, 2\pi].$$

Definition 5. Let $\beta \in [0,1)$. A univalent and sense-preserving complex harmonic function f of the form (1.1) is called starlike of the order β with respect to the origin in Δ if for any $r \in (0,1)$, we have

$$\frac{\partial}{\partial\theta}\Big(\arg f(re^{i\theta})\Big) > \beta, \quad \theta \in [0, 2\pi]. \tag{1.3}$$

Definition 6. Let $\beta \in [0,1)$. A univalent and sense-preserving complex harmonic function f of the form (1.1) is called convex of the order β in Δ if for any $r \in (0,1)$, we have

$$\frac{\partial}{\partial\theta}\Big(\arg\frac{\partial}{\partial\theta}(f(re^{i\theta}))\Big) > \beta, \quad \theta \in [0,2\pi]. \tag{1.4}$$

Remark 2. Definitions 5 and 6 are analogues of the appropriate classical definitions for the normalized holomorphic functions in Δ. In [9], among other things, the definition of a starlike (convex) function f of the form $f(z) = z + a_2z^2 + \dots$, $z \in \Delta$ is stated. The subclass of the class of starlike (convex) functions satisfying the condition $\mathrm{Re}\{zf'(z)/f(z)\} > \beta$ ($\mathrm{Re}\{1 + zf''(z)/f'(z)\} > \beta$), $z \in \Delta$, $0 < \beta < 1$ is called the class of functions starlike (convex) of the order β.

In his paper B. Pinchuk ([10]) noted that in 1936 M. S. Robertson ([11]) introduced more restrictive definitions of the classes of holomorphic functions starlike (convex) of the order β. Namely, holomorphic, normalized starlike (convex) functions were called by him starlike (convex) functions of the order β, $\beta \in [0,1)$, if they satisfy the condition $\mathrm{Re}\{zf'(z)/f(z)\} \ge \beta$, $z \in \Delta$ ($\mathrm{Re}\{1 + zf''(z)/f'(z)\} \ge \beta$, $z \in \Delta$) and for any sufficiently small $\epsilon > 0$ there exists a point $z_0 \in \overline{\Delta}$ such that $\mathrm{Re}\{z_0f'(z_0)/f(z_0)\} < \beta + \epsilon$ ($\mathrm{Re}\{1 + z_0f''(z_0)/f'(z_0)\} < \beta + \epsilon$). The second part of Robertson's definition is overlooked by his followers.

In this paper we use generalization of the classical definition of the functions starlike (convex) of the order β, for complex harmonic functions.

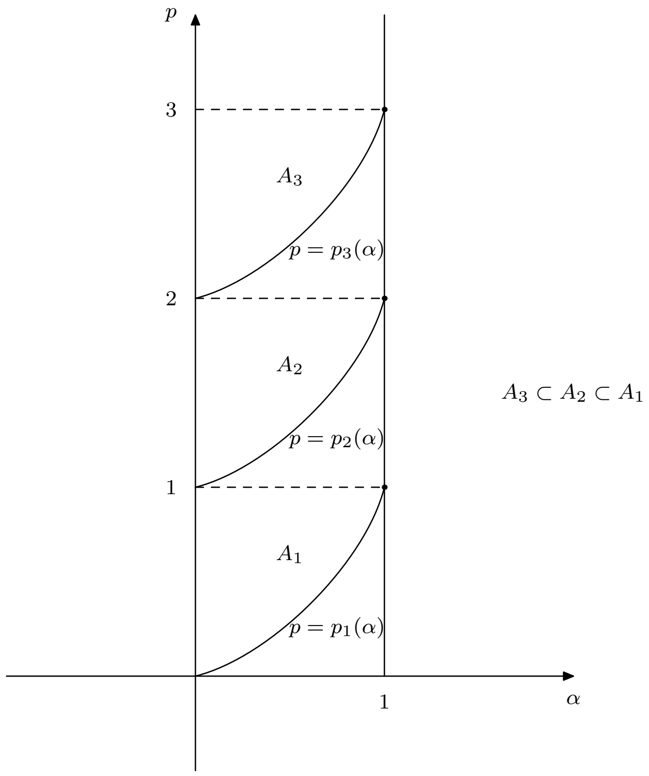
It is worth remembering that the property of starlikeness (convexity) is hereditary for functions holomorphic in Δ, but for complex harmonic functions it need not be so [4].

Let $\alpha \in [0,1]$ and

$$\begin{aligned} p_1(\alpha) &\coloneqq 1 - \log_2(2-\alpha),\\ p_2(\alpha) &\coloneqq 1 + p_1(\alpha),\\ p_3(\alpha) &\coloneqq 1 + p_2(\alpha), \quad \log_2 1 = 0. \end{aligned}$$

Let us denote

$$\begin{aligned} A_1 &\coloneqq \{(\alpha,p) \in A \,:\, p \ge p_1(\alpha)\},\\ A_2 &\coloneqq \{(\alpha,p) \in A \,:\, p \ge p_2(\alpha)\},\\ A_3 &\coloneqq \{(\alpha,p) \in A \,:\, p \ge p_3(\alpha)\}. \end{aligned}$$



In [7] the following theorems are proved:

Theorem 1. *Let $(\alpha, p) \in A_1$. Then $\mathcal{HS}(\alpha, p)$ is the class of univalent and sense-preserving functions in Δ.*

A_1 is the largest set, in which every function $f \in \mathcal{HS}(\alpha, p)$, $(\alpha, p) \in A$ is univalent in Δ.

Theorem 2. *If $(\alpha, p) \in A_1$, then the functions of the class $\mathcal{HS}^0(\alpha, p)$ are starlike in Δ. If $(\alpha, p) \in A_2$, then the functions of the class $\mathcal{HS}^0(\alpha, p)$ are convex in Δ.*

Remark 3. The sets A_1 and A_2 are the largest subsets of A, in which every function $f \in \mathcal{HS}^0(\alpha, p)$ is starlike, convex in Δ, respectively.

2. Main results. In view of Theorem 2 it seems natural to ask a question about the order of starlikeness of functions of the class $\mathcal{HS}^0(\alpha, p)$, $(\alpha, p) \in A_1$ and about the order of convexity of functions of the class $\mathcal{HS}^0(\alpha, p)$, $(\alpha, p) \in A_2$. The next theorems solve this problem.

Theorem 3. *Let $(\alpha, p) \in A_1$. If $f \in \mathcal{HS}^0(\alpha, p)$, then f is a univalent sense-preserving function starlike of the order $\beta^*(\alpha, p) := \frac{U_2(\alpha,p)-2}{U_2(\alpha,p)+1}$ in Δ.*

Proof. The univalence and sense-preservation of the function f in Δ are guaranteed by Theorem 1.

Using a method similar to previously used in [2], J. M. Jahangiri ([6]) has proved that if a function f of the form (1.1) satisfies the condition

$$\sum_{n=1}^{\infty}\left(\frac{n-\beta}{1-\beta}|a_n|+\frac{n+\beta}{1-\beta}|b_n|\right)\le 2,\quad 0\le\beta<1,$$

then for every $r\in(0,1)$, $\theta\in[0,1]$, the condition (1.3) holds, so f is starlike of the order β in Δ.

If f belongs to $\mathcal{HS}^0(\alpha,p)$, the aforementioned coefficient condition takes the form

$$\sum_{n=2}^{\infty}\left(\frac{n-\beta}{1-\beta}|a_n|+\frac{n+\beta}{1-\beta}|b_n|\right)\le 1,\quad 0\le\beta<1. \tag{2.1}$$

It is obvious that for any $\beta\in[0,1)$ we have $\frac{n-\beta}{1-\beta}\le\frac{n+\beta}{1-\beta}$, $n=2,3,\ldots$. Therefore, it is sufficient to show that β^* is the largest constant such that for every fixed $(\alpha,p)\in A_1$, we have $\beta^*\in[0,1)$ and, due to the condition (1.2), $\frac{n+\beta^*}{1-\beta^*}\le U_n(\alpha,p)$ for any $n=2,3,\ldots$.

We have $0\le\beta^*(\alpha,p)<1$ if $(\alpha,p)\in A_1$. Indeed, the inequality $p\ge p_1(\alpha)$, $\alpha\in[0,1]$ is equivalent to $2^p(2-\alpha)\ge 2$, i.e. $U_2(\alpha,p)\ge 2$, therefore $\beta^*(\alpha,p)\ge 0$, $(\alpha,p)\in A_1$. The upper estimation is immediate.

The inequality $\frac{n+\beta^*}{1-\beta^*}\le U_n(\alpha,p)$, $n=2,3,\ldots$ is equivalent to the inequality

$$\beta^*(\alpha,p)\le\frac{U_n(\alpha,p)-n}{U_n(\alpha,p)+1},\quad n=2,3,\ldots,\quad(\alpha,p)\in A_1. \tag{2.2}$$

Let us consider the function of the form

$$t(x)=t(\alpha,p;x):=\frac{x^p(\alpha+(1-\alpha)x)-x}{x^p(\alpha+(1-\alpha)x)+1},$$

$x\ge 2$, $(\alpha,p)\in A_1$. We will prove that $\min_{x\ge 2}t(x)=t(2)$. We have

$$t'(x)=\frac{x^{p+1}p(1-\alpha)+x^p(1-2\alpha+p)+x^{p-1}p\alpha-1}{[x^p(\alpha+(1-\alpha)x)+1]^2},\quad x\ge 2.$$

Let us denote $s(x):=x^{p+1}p(1-\alpha)+x^p(1-2\alpha+p)+x^{p-1}p\alpha-1$, $x\ge 2$, $(\alpha,p)\in A_1$. Then we have $s'(x)=px^{p-2}\big[(p+1)(1-\alpha)x^2+(1-2\alpha+p)x+\alpha(p-1)\big]$, $x\ge 2$.

Let us consider three cases:

1) Let $\alpha\in[0,1]$ and $p\ge 1$. Then $s'(x)>0$, $x\ge 2$, therefore s is an increasing function in $(2,+\infty)$. Moreover, $s(2)\ge 0$.

Indeed, $s(2)=2^{p-1}(2-\alpha)(3p+4)-6\cdot 2^{p-1}-1\ge 2^{p-1}(3p-2)-1$. Let us note that $3p-2\ge 2^{1-p}$ for $p\ge 1$ and $2^{p-1}(3p-2)\ge 1$. Hence $s(x)\ge 0$,

$x \geq 2$, therefore $t'(x) \geq 0$, $x \geq 2$ and consequently $\min_{x\geq 2} t(x) = t(2) = \beta^*(\alpha, p)$, $\alpha \in [0,1]$, $p \geq 1$.

2) Let now $\alpha \in (0,1)$ and $p_1(\alpha) \leq p < 1$. We will investigate the quadratic equation $(p+1)(1-\alpha)x^2 + (1-2\alpha+p)x + \alpha(p-1)$, $x \geq 2$ with zeros $x_1 = -1$, $x_2 = \frac{\alpha(1-p)}{(1-\alpha)(p+1)} \leq 2$. The last inequality is equivalent to $p \geq \frac{3\alpha-2}{2-\alpha}$ and $\chi(\alpha) := \frac{3\alpha-2}{2-\alpha} - p_1(\alpha) < 0$, $\alpha \in (0,1)$. Hence $s'(x) \geq 0$, $x \geq 2$.

In this case we need to show that $s(2) \geq 0$, $\alpha \in (0,1)$, $p_1(\alpha) \leq p < 1$. We have

$$s'_p(2) = 2^{p-1}\big\{(2-\alpha)[(3p+4)\ln 2 + 3] - 6\ln 2\big\}.$$

The inequality $(2-\alpha)[(3p+4)\ln 2 + 3] - 6\ln 2 \geq 0$ is equivalent to $p \geq \frac{2}{2-\alpha} - \frac{3+4\ln 2}{3\ln 2}$.

Let us denote $\psi(\alpha) := \frac{2}{2-\alpha} - \frac{3+4\ln 2}{3\ln 2}$, $\alpha \in (0,1)$. We have $\psi(\alpha) < 0$ for $\alpha \in (0,1)$, so $\psi(\alpha) < p_1(\alpha) \leq p$, $\alpha \in (0,1)$, i.e. s is an increasing function of variable p in $(p_1(\alpha), 1)$, $\alpha \in (0,1)$. Moreover, $s(2)|_{p=p_1(\alpha)} = 6 - 3\log_2(2-\alpha) - \frac{6}{(2-\alpha)} \geq 0$, $\alpha \in (0,1)$.

Indeed, the last inequality has an equivalent form $\log_2(2-\alpha) \leq \frac{2(1-\alpha)}{2-\alpha}$. Let us denote $m(\alpha) := \log_2(2-\alpha) - \frac{2(1-\alpha)}{2-\alpha}$, $\alpha \in (0,1)$. We have $m(0) = m(1) = 0$ and $m'(\alpha) = \frac{2(\ln 2-1)+\alpha}{(2-\alpha)^2\ln 2}$. Hence m is a continuous function decreasing in $(0, 2(1-\ln 2))$ and increasing in $(2(1-\ln 2), 1)$. Therefore $m(\alpha) < 0$, $\alpha \in (0,1)$.

The above considerations imply $s(x) \geq 0$, $x \geq 2$, therefore $t'(x) \geq 0$, $x \geq 2$, hence $\min_{x\geq 2} t(x) = t(2) = \beta^*(\alpha, p)$, $\alpha \in (0,1)$, $p_1(\alpha) \leq p < 1$.

3) Finally let us consider the case when $\alpha = 0$, $p \in (0,1)$. Then $s'(x) = p(p+1)x^{p-2}(x^2+x) > 0$, $x \geq 2$ and $s(2) = 2^p(3p+1) - 1 > 0$, hence $\min_{x\geq 2} t(x) = t(2) = \beta^*(0,p)$, $p \in (0,1)$.

Therefore, we showed that for any point $(\alpha,p) \in A_1$ there is $\min_{x\geq 2} t(x) = t(2)$. From the form of the function t we conclude that $\beta^*(\alpha,p)$, $(\alpha,p) \in A_1$, is the largest constant such that for every fixed $(\alpha,p) \in A_1$ the system of conditions (2.2) holds for any $n = 2, 3, \ldots$.

Thus, for each function of the class $\mathcal{HS}^0(\alpha,p)$, $(\alpha,p) \in A_1$, with any $\beta \in [0, \beta^*(\alpha,p)]$, the coefficient condition (2.1) holds. Hence the proof is completed. □

Proposition 1. *In every class $\mathcal{HS}^0(\alpha,p)$, $(\alpha,p) \in A_1$ there exists a function starlike of the order $\beta^*(\alpha,p)$ in the sense of the restrictive Robertson's definition (*[11]*). The following formula*

$$f_2^*(\alpha,p;z) = z + b_2^*\overline{z^2}, \quad z \in \Delta, \quad (\alpha,p) \in A_1, \quad b_2^* = \frac{1}{U_2(\alpha,p)}. \tag{2.3}$$

gives an example of such function.

Proof. Since

$$|b_1^*| + \sum_{n=2}^{\infty} U_n(\alpha,p)(|a_n^*| + |b_n^*|) = U_2(\alpha,p)|b_2^*| = 1,$$

we have $f_2^* \in \mathcal{HS}^0(\alpha,p)$, $(\alpha,p) \in A_1$. Therefore, according to Theorem 3, the function f_2^* is starlike of the order $\beta^*(\alpha,p) = \frac{1-2b_2^*}{1+b_2^*}$, $\beta^*(\alpha,p) \in [0,1)$. Therefore, for any $r \in (0,1)$, $\theta \in [0,2\pi]$ we have

$$\frac{\partial}{\partial\theta}\Big(\arg f_2^*(re^{i\theta})\Big) = \operatorname{Re}\left\{\frac{1-2b_2^* e^{-3i\theta} r}{1+b_2^* e^{-3i\theta} r}\right\} \geq \beta^*(\alpha,p).$$

Putting $\theta = 0$, we obtain

$$\frac{1-2b_2^* r}{1+b_2^* r} \geq \beta^*(\alpha,p), \quad r \in (0,1).$$

Let us denote $u(r) := \frac{1-2b_2^* r}{1+b_2^* r}$, $r \in [0,1)$. The function u is continuous and decreasing in $(0,1)$, so for any $r \in [0,1)$, $u(r) > \frac{1-2b_2^*}{1+b_2^*}$. Moreover, $\lim_{r\to 1^-} u(r) = \frac{1-2b_2^*}{1+b_2^*}$. Hence the function f_2^* cannot be starlike of the order higher than $\beta^*(\alpha,p)$.

Let us note that for the function $f_2^*(\alpha,p;\cdot)$ the equality in condition (2.1) holds for $\beta = \beta^*(\alpha,p)$. □

Property 1. *Let $(\alpha,p) \in A_1$, $a \geq 0$. If (α,p) is any point on the curve $p = p^a(\alpha)$, $\alpha \in [0,1]$, where $p^a(\alpha) := p_1(\alpha) + a$, then $\mathcal{HS}^0(\alpha,p)$ is the class of functions starlike of the order $\beta^*(a) := \beta^*(\alpha,p^a(\alpha)) = \frac{2^{1+a}-2}{2^{1+a}+1}$ and in this class there exists a function starlike of the order $\beta^*(a)$, in the sense of the restrictive Robertson's definition (*[11]*).*

If $a = 0$ ($p^0(\alpha) = p_1(\alpha)$), then $\beta^*(0) = \beta^*(\alpha,p_1(\alpha)) = 0$.

Moreover, if $0 \leq a_1 < a_2$, then $0 \leq \beta^*(a_1) < \beta^*(a_2) < 1$.

Indeed, let us note that

$$U_2(\alpha,p^a(\alpha)) = \alpha 2^{1+a-\log_2(2-\alpha)} + (1-\alpha)2^{2+a-\log_2(2-\alpha)} = 2^{1+a},$$

hence $\beta^*(\alpha,p^a(\alpha)) = \frac{2^{1+a}-2}{2^{1+a}+1}$. Moreover, $U_2(\alpha,p^0(\alpha)) = 2$, $\alpha \in [0,1]$, so if (α,p) is a point on the curve $p = p_1(\alpha)$, $\alpha \in [0,1]$, which is the arc of boundary of the set A_1, then functions of the class $\mathcal{HS}^0(\alpha,p)$ are starlike of the order 0, so starlike in Δ.

Moreover, $(\beta^*(a))' = \frac{2^{1+a}3\ln 2}{(2^{1+a}+1)^2} > 0$, $a \geq 0$, therefore, the function β^* is an increasing mapping of variable a. We also have the following result.

Theorem 4. *Let $(\alpha,p) \in A_2$. If $f \in \mathcal{HS}^0(\alpha,p)$, then f is an univalent sense-preserving function convex of the order $\beta^c(\alpha,p) := \frac{U_2(\alpha,p)-4}{U_2(\alpha,p)+2}$ in Δ.*

Proof. Univalence and sense-preservation of the function f in Δ are guaranteed by Theorem 1.

We will use the fact (see [6]) that if f of the form (1.1) ($b_1 = 0$) satisfies the condition

$$\sum_{n=2}^{\infty}\left(\frac{n(n-\beta)}{1-\beta}|a_n| + \frac{n(n+\beta)}{1-\beta}|b_n|\right) \le 1, \quad 0 \le \beta < 1, \tag{2.4}$$

then for any $r \in (0,1)$, f satisfies the condition (1.4), so it is a function convex of the order β in the disc Δ.

Using the fact that for any $\beta \in [0,1)$, $\frac{n-\beta}{1-\beta} \le \frac{n+\beta}{1-\beta}$, $n = 2,3,\dots$, we will show that $\beta^c(\alpha,p)$, $(\alpha,p) \in A_2$ is the largest constant from the range $[0,1)$ such that $\frac{n(n+\beta^c(\alpha,p))}{1-\beta^c(\alpha,p)} \le U_n(\alpha,p)$, $(\alpha,p) \in A_2$ for any $n = 2,3,\dots$.

It is obvious that $\beta^c(\alpha,p) \in [0,1)$ if $(\alpha,p) \in A_2$. Indeed, the inequality $p \ge p_2(\alpha)$, $\alpha \in [0,1]$ is equivalent to $U_2(\alpha,p) \ge 4$.

Let us consider a function of the form $a(x) = a(\alpha,p;x) := \frac{x^p(\alpha+(1-\alpha)x)-x^2}{x^p(\alpha+(1-\alpha)x)+x}$, $x \ge 2$, $(\alpha,p) \in A_2$ and $q := p-1$. Investigation of the behavior of the function $a(\alpha,p;\cdot)$, $(\alpha,p) \in A_2$ in the range $[2,+\infty)$ is equivalent to examination of a function $t(\alpha,q;\cdot)$, $(\alpha,q) \in A_1$ considered in the proof of Theorem 3.

Therefore, we have $\min_{x\ge 2} a(x) = a(2) = \frac{U_2(\alpha,p)-4}{U_2(\alpha,p)+2}$, $(\alpha,p) \in A_2$.

Hence

$$\begin{aligned}\sum_{n=2}^{\infty}&\left(\frac{n(n-\beta^c(\alpha,p))}{1-\beta^c(\alpha,p)}|a_n| + \frac{n(n+\beta^c(\alpha,p))}{1-\beta^c(\alpha,p)}|b_n|\right)\\ &\le \sum_{n=2}^{\infty}\frac{n(n+\beta^c(\alpha,p))}{1-\beta^c(\alpha,p)}(|a_n|+|b_n|)\\ &\le \sum_{n=2}^{\infty}U_n(\alpha,p)(|a_n|+|b_n|) \le 1, \quad (\alpha,p) \in A_2,\end{aligned}$$

so, if $\beta = \beta^c(\alpha,p)$, then the condition (2.4) holds.

Therefore, for any function of the class $\mathcal{HS}^0(\alpha,p)$, $(\alpha,p) \in A_2$ and for each $\beta \in [0,\beta^c(\alpha,p)]$, the coefficient condition (2.4) holds. It is a sufficient condition for convexity of the order β of a function f in Δ. □

Proposition 2. *In each class $\mathcal{HS}^0(\alpha,p)$, $(\alpha,p) \in A_2$ there exists a function which is convex of the order $\beta^*(\alpha,p)$ in the sense of the restrictive Robertson's definition ([11]). Formula (2.3) gives an example of such function.*

In the proof we use the same method, as in the proof of Proposition 1.

Property 2. *Let $(\alpha,p) \in A_2$, $a \ge 0$. If (α,p) is any point of the curve $p = p^b(\alpha)$, $\alpha \in [0,1]$, where $p^b(\alpha) := p_2(\alpha)+b$, then $\mathcal{HS}^0(\alpha,p)$ is the class of the functions convex of the order $\beta^c(b) := \beta^c(\alpha,p^b(\alpha)) = \frac{2^{2+b}-4}{2^{2+b}+2}$ and in* *this class there exists a function convex of the order $\beta^c(b)$ in the sense of the restrictive Robertson's definition ([11]).*

Moreover, if $0 \le b_1 < b_2$, then $0 \le \beta^c(b_1) < \beta^c(b_2) < 1$.

Justification of validity of this property is analogous to the proof of Property 1.

3. On other properties of functions of the classes $\mathcal{HS}(\alpha, p)$ and $\mathcal{HS}^0(\alpha, p)$. The very well-known Alexander theorem for univalent holomorphic functions shows relationships between starlike and convex functions. P. Duren ([4], p. 108) gave the partial extension of this theorem in the case of complex harmonic functions. We show an analogous extension for functions of the classes investigated in this paper.

Property 3. *Let $\alpha \in [0, 1]$, $p > 1$. If f of the form (1.1) is a function of the class $\mathcal{HS}(\alpha, p-1)$ ($\mathcal{HS}^0(\alpha, p-1)$), then a function K of the form*

$$K(z) = \int_0^1 \frac{f(zt)}{t} dt = \int_0^z \frac{h(u)}{u} du + \overline{\int_0^z \frac{g(u)}{u} du}, \quad z \in \Delta$$

belongs to the class $\mathcal{HS}(\alpha, p)$ ($\mathcal{HS}^0(\alpha, p)$).

Indeed, we have

$$\begin{aligned} K(z) = \int_0^1 \frac{f(zt)}{t} dt &= \int_0^1 \left(\frac{tz + \sum_{n=2}^{\infty} a_n t^n z^n}{t} + \frac{\overline{b_1 z} t + \sum_{n=2}^{\infty} \overline{b_n} t^n \overline{z^n}}{t} \right) dt \\ &= \int_0^1 \left(z + \overline{b_1 z} + \sum_{n=2}^{\infty} a_n z^n t^{n-1} + \sum_{n=2}^{\infty} \overline{b_n z^n} t^{n-1} \right) dt \\ &= z + \sum_{n=2}^{\infty} \frac{a_n z^n}{n} + \overline{b_1 z} + \sum_{n=2}^{\infty} \frac{\overline{b_n z^n}}{n}, \quad z \in \Delta, \end{aligned}$$

therefore, K is of the form (1.1).

Moreover,

$$|b_1| + \sum_{n=2}^{\infty} U_n(\alpha, p) \Big(\frac{|a_n|}{n} + \frac{|b_n|}{n} \Big) = |b_1| + \sum_{n=2}^{\infty} U_n(\alpha, p-1)(|a_n| + |b_n|) \le 1.$$

The next theorems concern the convolutions of complex harmonic functions in Δ.

Definition 7. Let $f_k(z) = h_k(z) + \overline{g_k(z)}$, where $h_k(z) = z + \sum_{n=2}^{\infty} a_n^{(k)} z^n$, $g_k(z) = \sum_{n=1}^{\infty} b_n^{(k)} z^n$ are holomorphic in Δ, $k = 1, 2$.

Hadamard's convolution of the functions f_1 and f_2 is given by the formula

$$(f_1 * f_2)(z) := z + \sum_{n=2}^{\infty} a_n^{(1)} a_n^{(2)} z^n + \overline{\sum_{n=1}^{\infty} b_n^{(1)} b_n^{(2)} z^n}, \quad z \in \Delta.$$

Definition 8. Let $f_k(z) = h_k(z) + \overline{g_k(z)}$, where $h_k(z) = z + \sum_{n=2}^{\infty} a_n^{(k)} z^n$, $g_k(z) = \sum_{n=1}^{\infty} b_n^{(k)} z^n$ are holomorphic in Δ, $k = 1, 2$.

The integral convolution of the functions f_1 and f_2 is given by the formula

$$(f_1 \diamond f_2)(z) := z + \sum_{n=2}^{\infty} \frac{a_n^{(1)} a_n^{(2)}}{n} z^n + \overline{\sum_{n=1}^{\infty} \frac{b_n^{(1)} b_n^{(2)}}{n} z^n}, \quad z \in \Delta.$$

We have the following result.

Theorem 5. *Let* $\tilde{f}(z) = z + \sum_{n=2}^{\infty} \tilde{a}_n z^n + \overline{\sum_{n=2}^{\infty} \tilde{b}_n z^n}$, $z \in \Delta$ *be a univalent complex harmonic function convex in* Δ.

If $f \in \mathcal{HS}^0(\alpha, p)$, $(\alpha, p) \in A_2$, *then* $f * \tilde{f}$ *is a univalent sense-preserving function starlike in* Δ *and* $f \diamond \tilde{f}$ *is a univalent sense-preserving function convex in* Δ.

If $f \in \mathcal{HS}^0(\alpha, p)$, $(\alpha, p) \in A_3$, *then* $f * \tilde{f}$ *is a univalent sense preserving function convex in* Δ.

Proof. It is obvious that in each considered case $f * \tilde{f}$ and $f \diamond \tilde{f}$ are complex harmonic functions in Δ and have the form required in the class $\mathcal{HS}^0(\alpha, p)$.

Let us assume that $(\alpha, p) \in A_2$. According to the fact that the estimations $|\tilde{a}_n| \le \frac{n+1}{2}$, $|\tilde{b}_n| \le \frac{n-1}{2}$, $n = 2, 3, \ldots$ hold ([3], th. 5.10), we have $\frac{|\tilde{a}_n|}{n} \le 1$, $\frac{|\tilde{b}_n|}{n} \le 1$, $n = 2, 3, \ldots$. Hence

$$\sum_{n=2}^{\infty} n(|a_n \tilde{a}_n| + |b_n \tilde{b}_n|) = \sum_{n=2}^{\infty} n^2 \left(|a_n| \left| \frac{\tilde{a}_n}{n} \right| + |b_n| \left| \frac{\tilde{b}_n}{n} \right| \right)$$
$$\le \sum_{n=2}^{\infty} n^2 (|a_n| + |b_n|) \le \sum_{n=2}^{\infty} U_n(\alpha, p)(|a_n| + |b_n|) \le 1,$$

so $f * \tilde{f} \in \mathcal{HS}^0(1, 1)$, therefore the convolution is a univalent sense-preserving starlike function.

Due to the fact that $(f \diamond \tilde{f})(z) = \int_0^1 \frac{(f * \tilde{f})(zt)}{t} dt$, $z \in \Delta$ and $f * \tilde{f} \in \mathcal{HS}^0(1, 1)$, we obtain from Property 3 that a function $f \diamond \tilde{f}$ belongs to the class $\mathcal{HS}^0(1, 2)$, so it is a univalent and sense-preserving function convex in Δ (see [2]).

Let $(\alpha, p) \in A_3$. Then it can be shown that $U_n(\alpha, p) \ge n^3$, $n = 2, 3, \ldots$. Hence, using an analogous method as above, we can show that for the function $f * \tilde{f}$ the inequalities

$$\sum_{n=2}^{\infty} n^2(|a_n \tilde{a}_n| + |b_n \tilde{b}_n|) \le \sum_{n=2}^{\infty} n^3 (|a_n| + |b_n|) \le \sum_{n=2}^{\infty} U_n(\alpha, p)(|a_n| + |b_n|) \le 1$$

hold. Therefore, for the convolution $f * \tilde{f}$ the condition sufficient for univalence sense-preservation and convexity holds. □

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Received September 21, 2009