Asymmetric truncated Toeplitz operators equal to the zero operator

1. Introduction. Let $H^2$ denote the Hardy space of the unit disk $\mathbb{D} = \{ z : |z| < 1 \}$, that is, the space of functions analytic in $\mathbb{D}$ with square summable Maclaurin coefficients.

Using the boundary values, one can identify $H^2$ with a closed subspace of $L^2(\partial \mathbb{D})$, the subspace of functions whose Fourier coefficients with negative indices vanish. The orthogonal projection $P$ from $L^2(\partial \mathbb{D})$ onto $H^2$, called the Szeg"o projection, is given by

$$Pf(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{f(e^{it})dt}{1-e^{-it}z}, \quad f \in L^2(\partial \mathbb{D}).$$

Note that if $f \in L^1(\partial \mathbb{D})$, then the above integral still defines a function $Pf$ analytic in $\mathbb{D}$.

The classical Toeplitz operator $T\varphi$ with symbol $\varphi \in L^2(\partial \mathbb{D})$ is defined on $H^2$ by

$$T\varphi f = P(\varphi f).$$
It is known that $T_\varphi$ is bounded if and only if $\varphi \in L^\infty(\partial \mathbb{D})$. The operator $S = T_z$ is called the unilateral shift and its adjoint $S^* = T_{\overline{z}}$ is called the backward shift. We have $Sf(z) = zf(z)$ and

$$S^* f(z) = \frac{f(z) - f(0)}{z}.$$

Let $H^\infty$ be the algebra of bounded analytic functions on $\mathbb{D}$ and let $\alpha \in H^\infty$ be an arbitrary inner function, that is, $|\alpha| = 1$ a.e. on $\partial \mathbb{D}$.

By the theorem of A. Beurling (see, for example, [7, Thm. 8.1.1]), every nontrivial, closed $S$-invariant subspace of $H^2$ can be expressed as $\alpha H^2$ for some inner function $\alpha$. Consequently, every nontrivial, closed $S^*$-invariant subspace of $H^2$ is of the form

$$K_\alpha = H^2 \ominus \alpha H^2$$

with $\alpha$ inner. The space $K_\alpha$ is called the model space corresponding to $\alpha$.

The kernel function

$$(1.1) \quad k^\alpha_w(z) = \frac{1 - \alpha(w)\overline{\alpha(z)}}{1 - wz}, \quad w, z \in \mathbb{D},$$

is a reproducing kernel for the model space $K_\alpha$, i.e., for each $f \in K_\alpha$ and $w \in \mathbb{D},$

$$f(w) = \langle f, k^\alpha_w \rangle$$

($\langle \cdot, \cdot \rangle$ being the usual integral inner product). Observe that $k^\alpha_w$ is a bounded function for every $w \in \mathbb{D}$. It follows that the subspace $K^\infty_\alpha = K_\alpha \cap H^\infty$ is dense in $K_\alpha$. If $\alpha(w) = 0$, then $k^\alpha_w = k_w$, where $k_w$ is the Szegö kernel given by $k_w(z) = (1 - wz)^{-1}$.

Let $P_\alpha$ denote the orthogonal projection from $L^2(\partial \mathbb{D})$ onto $K_\alpha$. Then

$$P_\alpha f(z) = \langle f, k^\alpha_z \rangle, \quad f \in L^2(\partial \mathbb{D}), \quad z \in \mathbb{D}.$$  

Just like with the Szegö projection, $P_\alpha f$ is a function analytic in $\mathbb{D}$ for all $f \in L^1(\partial \mathbb{D})$.

A truncated Toeplitz operator with a symbol $\varphi \in L^2(\partial \mathbb{D})$ is the operator $A^\alpha_\varphi$ defined on the model space $K_\alpha$ by

$$A^\alpha_\varphi f = P_\alpha (\varphi f).$$

Densely defined on bounded functions, the operator $A^\alpha_\varphi$ can be seen as a compression to $K_\alpha$ of the classical Toeplitz operator $T_\varphi$ on $H^2$.

The study of truncated Toeplitz operators as a class began in 2007 with D. Sarason’s paper [13]. In spite of similar definitions, there are many differences between truncated Toeplitz operators and the classical ones. One of the first results from [13] states that, unlike in the classical case, a truncated Toeplitz operator is not uniquely determined by its symbol. More precisely, $A^\alpha_\varphi = 0$ if and only if $\varphi \in \overline{\alpha H^2} + \alpha H^2$ ([13, Thm. 3.1]). As a consequence, unbounded symbols can produce bounded truncated Toeplitz operators. Moreover, there exist bounded truncated Toeplitz operators for
which no bounded symbols exist (see [3]). For more interesting results see [6, 9, 10, 11, 12].

Recently, the authors in [4] and [5] introduced a generalization of truncated Toeplitz operators, the so-called asymmetric truncated Toeplitz operators. Let \( \alpha, \beta \) be two inner functions and let \( \varphi \in L^2(\partial \mathbb{D}) \). An asymmetric truncated Toeplitz operator \( A_{\varphi}^{\alpha,\beta} \) is the operator from \( K_\alpha \) into \( K_\beta \) given by

\[
A_{\varphi}^{\alpha,\beta} f = P_\beta(\varphi f), \quad f \in K_\alpha.
\]

The operator \( A_{\varphi}^{\alpha,\beta} \) is densely defined. Clearly, \( A_{\varphi}^{\alpha,\alpha} = A_{\varphi}^{\alpha} \).

We denote

\[
\mathcal{T}(\alpha,\beta) = \{A_{\varphi}^{\alpha,\beta} : \varphi \in L^2(\partial \mathbb{D}) \text{ and } A_{\varphi}^{\alpha,\beta} \text{ is bounded}\}
\]

and \( \mathcal{T}(\alpha) = \mathcal{T}(\alpha,\alpha) \).

The purpose of this paper is to describe when an operator from \( \mathcal{T}(\alpha,\beta) \) is equal to the zero operator. The description is given in terms of the symbol of the operator. This was done in [4] and [5] for the case when \( \beta \) divides \( \alpha \), that is, when \( \alpha/\beta \) is an inner function. It was proved in [4] and [5] that \( A_{\varphi}^{\alpha,\beta} = 0 \) if and only if \( \varphi \in \overline{\alpha H^2 + \beta H^2} \). Here we show that this is true for all inner functions \( \alpha \) and \( \beta \). We also give some examples of rank-one asymmetric truncated Toeplitz operators.

2. Main result. In this section we prove the following.

**Theorem 2.1.** Let \( \alpha, \beta \) be two nonconstant inner functions and let \( A_{\varphi}^{\alpha,\beta} : K_\alpha \to K_\beta \) be a bounded asymmetric truncated Toeplitz operator with \( \varphi \in L^2(\partial \mathbb{D}) \). Then \( A_{\varphi}^{\alpha,\beta} = 0 \) if and only if \( \varphi \in \overline{\alpha H^2 + \beta H^2} \).

We start with a simple technical lemma.

**Lemma 2.2.** Let \( \alpha, \beta \) be two arbitrary inner functions. If

\[
(2.1) \quad K_\alpha \subset \beta H^2,
\]

then both \( \alpha \) and \( \beta \) have no zeros in \( \mathbb{D} \), or at least one of the functions \( \alpha \) or \( \beta \) is a constant function.

**Proof.** Assume that (2.1) holds. If \( \beta(z_0) = 0 \) for some \( z_0 \in \mathbb{D} \), then \( f(z_0) = 0 \) for every \( f \in K_\alpha \). For \( f = k_{z_0}^\alpha \), we get

\[
k_{z_0}^\alpha(z_0) = ||k_{z_0}^\alpha||^2 = \frac{1 - |\alpha(z_0)|^2}{1 - |z_0|^2} = 0,
\]

which implies that \( |\alpha(z_0)| = 1 \). By the maximum modulus principle, \( \alpha \) is a constant function. Hence, the inclusion (2.1) implies that \( \beta \) has no zeros in \( \mathbb{D} \), or \( \alpha \) is a constant function. But (2.1) is equivalent to

\[
K_\beta \subset \alpha H^2,
\]

and, by the same reasoning, (2.1) also implies that \( \alpha \) has no zeros in \( \mathbb{D} \), or \( \beta \) is a constant function. This completes the proof.
Lemma 2.2 can be rephrased as follows. If $\alpha$, $\beta$ are two nonconstant inner functions and at least one of them has a zero in $D$, then the inclusion $K_\alpha \subset \beta H^2$ does not hold. This allows us to prove the following version of Theorem 2.1.

**Proposition 2.3.** Let $\alpha$, $\beta$ be two nonconstant inner functions such that each of them has a zero in $D$ and let $A^{\alpha,\beta}_\varphi : K_\alpha \to K_\beta$ be a bounded asymmetric truncated Toeplitz operator with $\varphi \in L^2(\partial D)$. Then $A^{\alpha,\beta}_\varphi = 0$ if and only if $\varphi \in \alpha H^2 + \beta H^2$.

**Proof.** The fact that $\varphi \in \alpha H^2 + \beta H^2$ implies $A^{\alpha,\beta}_\varphi = 0$ was proved in [4, Thm. 4.3]. For the convenience of the reader we repeat the reasoning from [4].

Assume that $\varphi = \overline{\alpha h_1 + \beta h_2}$ with $h_1, h_2 \in H^2$. Then, for every $f \in K_\alpha^{\infty}$,

$$A^{\alpha,\beta}_\varphi f = P_\beta(\overline{\alpha h_1 f + \beta h_2 f}) = P_\beta(\overline{\alpha h_1} f).$$

Since $f \perp \alpha H^2$, we see that $\overline{\alpha h_1} f \perp H^2$ and $P_\beta(\overline{\alpha h_1} f) = 0$. The density of $K_\alpha^{\infty}$ implies that $A^{\alpha,\beta}_\varphi = 0$. Note that this part of the proof does not depend on the existence of zeros of $\alpha$ and $\beta$.

Let us now assume that $A^{\alpha,\beta}_\varphi = 0$. By the first part of the proof, we can also assume that $\varphi = \chi + \psi$ for some $\chi \in K_\alpha$, $\psi \in K_\beta$. Let $z_0 \in D$ be a zero of $\alpha$. Then $k_{z_0}^\alpha = k_{z_0}$ and

$$A^{\alpha,\beta}_{\chi + \psi} k_{z_0}^\alpha = P_\beta(\overline{\chi(z_0)} k_{z_0})$$

$$= P_\beta \left( \frac{\overline{\chi(z) - \chi(z_0)}}{z - z_0} + \frac{\overline{\chi(z_0)} k_{z_0}}{\chi(z_0)} \right)$$

$$= \frac{\overline{\chi(z_0)} k_{z_0}}{\chi(z_0)},$$

because the quotient $(\chi(z) - \chi(z_0))/(z - z_0)$ belongs to $K_\alpha$ (see [13, Subsection 2.6]).

Hence,

$$0 = A^{\alpha,\beta}_\varphi k_{z_0}^\alpha = A^{\alpha,\beta}_{\chi + \psi} k_{z_0}^\alpha$$

$$= \frac{\overline{\chi(z_0)} k_{z_0}}{\chi(z_0)} + A^{\alpha,\beta}_\psi k_{z_0}^\alpha = P_\beta \left[ \overline{(\chi(z_0)) + \psi} k_{z_0} \right],$$

which means that

$$\overline{(\chi(z_0)) + \psi} k_{z_0} \in \beta H^2$$

and, consequently,

$$A^{\alpha,\beta}_{\chi + \psi} = \left( A^{\alpha,\beta}_{\chi + \psi} \right)^* = 0,$$

On the other hand ([4, Lem. 3.2]),
and a similar reasoning can be used to show that if \( \beta(w_0) = 0, w_0 \in \mathbb{D} \), then
\[
\chi + \overline{\psi(w_0)} \in \alpha H^2.
\]

By (2.2), (2.3) and the first part of the proof we get
\[
A_{\alpha,\beta}^{\chi + \overline{\psi(w_0)} + \chi(z_0) + \psi} = 0,
\]
and
\[
A_{\alpha,\beta}^{\overline{\psi(w_0)} + \chi(z_0)} = -A_{\alpha,\beta}^{\chi + \psi} = 0.
\]
From this,
\[
P_{\beta} \left( (\psi(w_0) + \chi(z_0)) f \right) = 0
\]
for all \( f \in K_\alpha \).

If \( \psi(w_0) + \chi(z_0) \neq 0 \), then the above equality means that \( P_{\beta}(f) = 0 \) for all \( f \in K_\alpha \), that is, \( K_\alpha \subseteq \beta H^2 \). However, by Lemma 2.2, this cannot be the case here. So
\[
\psi(w_0) + \chi(z_0) = 0
\]
and
\[
\phi = \chi + \psi = \chi + \psi(w_0) + \overline{\chi(z_0)} + \psi \in \alpha H^2 + \beta H^2.
\]

To give a proof of Theorem 2.1 we use the so-called Crofoot transform. For any inner function \( \alpha \) and \( w \in \mathbb{D} \), the Crofoot transform \( J^\alpha_w \) is the multiplication operator defined by
\[
J^\alpha_w f(z) = \frac{\sqrt{1 - |w|^2}}{1 - \overline{w} \alpha(z)} f(z).
\]

The Crofoot transform \( J^\alpha_w \) is a unitary operator from \( K_\alpha \) onto \( K_{\alpha_w} \), where
\[
\alpha_w(z) = \frac{w - \alpha(z)}{1 - \overline{w} \alpha(z)}.
\]
(see, for example, [8, Thm. 10] and [13, pp. 521–523]). Moreover,
\[
(J^\alpha_w)^* f = (J^\alpha_w)^{-1} f = J_w^\alpha f
\]
\[
= \frac{\sqrt{1 - |w|^2}}{1 - \overline{w} \alpha_w} f = \frac{1 - \overline{w} \alpha}{\sqrt{1 - |w|^2}} f.
\]

**Lemma 2.4.** Let \( \alpha \) be an inner function and \( w \in \mathbb{D} \). For every \( z \in \mathbb{D} \) we have
\[
k^\alpha_z = \frac{1 - |w|^2}{(1 - w \alpha(z))(1 - \overline{w} \alpha)} k^\alpha_z.
\]
Proof. Fix \( w, z \in \mathbb{D} \). The reproducing kernel \( k_{z}^{\alpha_{w}} \) is given by

\[
 k_{z}^{\alpha_{w}}(\lambda) = \frac{1 - \overline{w(z)} \alpha_{w}(\lambda)}{1 - z\lambda}, \quad \lambda \in \mathbb{D}.
\]

Since

\[
 1 - \overline{w(z)} \alpha_{w}(\lambda) = 1 - \frac{w - \overline{\alpha(z)}}{1 - w\overline{\alpha(z)}} w - \alpha(\lambda)
\]

\[
 = \frac{(1 - |w|^{2})(1 - \overline{\alpha(z)} \alpha(\lambda))}{(1 - w\overline{\alpha(z)})(1 - w\alpha(\lambda))},
\]

we have

\[
 k_{z}^{\alpha_{w}}(\lambda) = \frac{1 - |w|^{2}}{(1 - w\overline{\alpha(z)})(1 - \overline{w\alpha(\lambda)})} \frac{1 - \overline{z}\alpha(\lambda)}{1 - z\lambda} = \frac{(1 - |w|^{2})(1 - \overline{z}\alpha(\lambda))}{(1 - w\overline{\alpha(z)})(1 - w\alpha(\lambda))} k_{z}^{\alpha}(\lambda).
\]

It is known that the map

\[
 A \mapsto J_{w}^{\alpha} A (J_{w}^{\alpha})^{-1}, \quad A \in \mathcal{T}(\alpha),
\]

carries \( \mathcal{T}(\alpha) \) onto \( \mathcal{T}(\alpha_{w}) \) (see [6]). An analogous result can be proved for the asymmetric truncated Toeplitz operators.

**Proposition 2.5.** Let \( \alpha, \beta \) be two inner functions. Let \( a, b \in \mathbb{D} \) and let the functions \( \alpha_{a}, \beta_{b} \) and the operators \( J_{a}^{\alpha} : K_{\alpha} \rightarrow K_{\alpha_{a}}, \ J_{b}^{\beta} : K_{\beta} \rightarrow K_{\beta_{b}} \) be defined as in (2.5) and (2.4), respectively. If \( A \) is a bounded linear operator from \( K_{\alpha} \) into \( K_{\beta} \), then \( A \) belongs to \( \mathcal{T}(\alpha, \beta) \) if and only if \( J_{b}^{\beta} A (J_{a}^{\alpha})^{-1} \) belongs to \( \mathcal{T}(\alpha_{a}, \beta_{b}) \). Moreover, if \( A = A_{\phi}^{\alpha, \beta} \), then \( J_{b}^{\beta} A (J_{a}^{\alpha})^{-1} = A_{\phi}^{\alpha_{a}, \beta_{b}} \) with

\[
 (2.7) \quad \phi = \frac{(1 - |\alpha|)(1 - |\beta|)}{\sqrt{1 - |a|^{2}} \sqrt{1 - |b|^{2}}} \varphi.
\]

**Proof.** Let \( A \) be a bounded linear operator from \( K_{\alpha} \) into \( K_{\beta} \). Assume first that \( A \) belongs to \( \mathcal{T}(\alpha, \beta) \), \( A = A_{\phi}^{\alpha, \beta} \) for \( \varphi \in L^{2}(\partial \mathbb{D}) \). We show that \( J_{b}^{\beta} A (J_{a}^{\alpha})^{-1} = A_{\phi}^{\alpha_{a}, \beta_{b}} \) with \( \phi \) as in (2.7).

For every \( f \in K_{\infty}^{\alpha_{a}} \) and \( z \in \mathbb{D} \) we have

\[
 J_{b}^{\beta} A_{\phi}^{\alpha, \beta} (J_{a}^{\alpha})^{-1} f(z) = \frac{\sqrt{1 - |b|^{2}}}{1 - \overline{b}(z)} P_{\beta} \left( \frac{1 - \overline{\alpha}}{\sqrt{1 - |a|^{2}}} \varphi f(z) \right)
\]

\[
 = \frac{\sqrt{1 - |b|^{2}}}{1 - \overline{b}(z)} \left( \frac{1 - \overline{\alpha}}{\sqrt{1 - |a|^{2}}} \varphi f, k_{z}^{\beta} \right).
\]
By (2.6),
\[
J^\beta_b A^\alpha_\beta (J^\alpha_a)^{-1} f(z) = \frac{\sqrt{1-|b|^2}}{1-b\bar{\beta}} \left\langle \frac{1-\bar{a}\alpha}{\sqrt{1-|a|^2}} \varphi f, \frac{1-b\bar{\beta}}{1-|b|^2} k^\beta_b \right\rangle
\]
\[
= \left\langle \frac{1-b\bar{\beta}}{\sqrt{1-|b|^2}} \frac{1-\bar{a}\alpha}{\sqrt{1-|a|^2}} \varphi f, k^\beta_b \right\rangle
\]
\[
= P^\beta_b \left( \frac{(1-b\bar{\beta})(1-\bar{a}\alpha)}{\sqrt{1-|b|^2} \sqrt{1-|a|^2}} \varphi f \right)(z)
\]
\[
= A^\alpha_\beta \varphi f(z).
\]
Thus \(A \in \mathcal{T}(\alpha, \beta)\) implies that \(J^\beta_b A (J^\alpha_a)^{-1} \in \mathcal{T}(\alpha_a, \beta_b)\).

To prove the other implication assume that \(J^\beta_b A (J^\alpha_a)^{-1} = A^\alpha_\beta \varphi \in \mathcal{T}(\alpha_a, \beta_b)\) for some \(\varphi \in L^2(\partial D)\). Then
\[
A = (J^\beta_b)^{-1} A^\alpha_\beta \varphi \in \mathcal{T}(\alpha_a, \beta_b).
\]

But \((\alpha_a)_a = \alpha\) and \((\beta_b)_b = \beta\), and, by the first part of the proof,
\[
A = J^\beta_b A^\alpha_\beta (J^\alpha_a)^{-1} = A^\alpha_\beta
\]
with
\[
\varphi = \frac{(1-\bar{a}\alpha)(1-b\bar{\beta})}{\sqrt{1-|a|^2} \sqrt{1-|b|^2}} \varphi.
\]
Hence, \(A \in \mathcal{T}(\alpha, \beta)\). An easy computation shows that \(\varphi\) satisfies (2.7). \(\square\)

**Proof of Theorem 2.1.** The fact that \(\varphi \in \overline{\alpha H^2 + \beta H^2}\) implies \(A^\alpha_\beta = 0\) was established in the proof of Proposition 2.3. Assume now that \(\varphi \in L^2(\partial D)\) and \(A^\alpha_\beta = 0\).

If \(\alpha(0) = \beta(0) = 0\), then \(\varphi \in \overline{\alpha H^2 + \beta H^2}\) by Proposition 2.3. If \(\alpha(0) \neq 0\) or \(\beta(0) \neq 0\), put \(a = \alpha(0), b = \beta(0)\). By Proposition 2.5,
\[
0 = J^\beta_b A^\alpha_\beta (J^\alpha_a)^{-1} = A^\alpha_\beta
\]
where
\[
\phi = \frac{(1-\bar{a}\alpha)(1-b\bar{\beta})}{\sqrt{1-|a|^2} \sqrt{1-|b|^2}} \varphi.
\]
Since \(\alpha_a(0) = \beta_b(0) = 0\), by Proposition 2.3,
\[
\phi \in \overline{\alpha_a H^2 + \beta_b H^2}.
\]
Therefore, there exist \(h_1, h_2 \in H^2\) such that
\[
\frac{(1-\bar{a}\alpha)(1-b\bar{\beta})}{\sqrt{1-|a|^2} \sqrt{1-|b|^2}} \varphi = \frac{\bar{a} - \bar{a}}{1-\bar{a}a} h_1 + \frac{b - \beta}{1-b\bar{\beta}} h_2,
\]
and
\[
\varphi = \pi - \overline{\alpha} \frac{1 - |a|^2}{1 - \alpha} \sqrt{1 - |b|^2} \frac{f_1}{1 - \alpha} + b - \beta \frac{1 - |a|^2}{1 - \beta} \sqrt{1 - |b|^2} h_2.
\]
Since \(|\alpha| = 1\) and \(|\beta| = 1\) on the unit circle \(\partial \mathbb{D}\), we have
\[
\frac{\pi - \overline{\alpha}}{1 - \alpha} = -\alpha \quad \text{and} \quad \frac{b - \beta}{1 - \beta} = -\beta.
\]
Consequently,
\[
\varphi = \alpha g_1 + \beta g_2
\]
with
\[
g_1 = -\frac{1 - |a|^2}{1 - \alpha} \sqrt{1 - |b|^2} h_1, \quad g_2 = \frac{1 - |a|^2}{1 - \beta} \sqrt{1 - |b|^2} h_2.
\]
Since \(g_1, g_2 \in H^2\), the proof is complete.

\[\square\]

**Corollary 2.6.** If \(\varphi\) is in \(L^2(\partial \mathbb{D})\), then there is a pair of functions \(\chi \in K_{\alpha^*}, \psi \in K_{\beta}\), such that \(A_{\varphi}^{\alpha, \beta} = A_{\chi + \psi}^{\alpha, \beta}\). If \(\chi, \psi\) is one such pair, then the most general such pair is of the form \(\chi - \overline{\chi}k_0^\beta, \psi + c k_0^\beta\), with \(c\) a scalar.

**Proof.** The proof is analogous to the proofs given in [13] and [4].

The function \(\varphi \in L^2(\partial \mathbb{D})\) can be written as \(\varphi = \varphi_+ + \varphi_-\) with \(\varphi_+, \varphi_- \in H^2\). If \(\chi = P_\alpha(\varphi_-)\) and \(\psi = P_\beta(\varphi_+)\), then \(\varphi - \chi - \psi \in \alpha H^2 + \beta H^2\). By Theorem 2.1, \(A_{\varphi}^{\alpha, \beta} = A_{\chi + \psi}^{\alpha, \beta}\).

Note that for \(f \in K_{\alpha^*}\),
\[
A_{\alpha}^{\alpha, \beta} f = P_\beta \left( f - \overline{\alpha} \beta f \right) = P_\beta f = A_1^{\alpha, \beta} f.
\]
Since \(\overline{\alpha} f \perp H^2\) for \(f \in K_{\alpha^*}\), we get
\[
A_{\alpha}^{\alpha, \beta} f = P_\beta \left( f - \alpha \overline{\alpha} f \right) = P_\beta f = A_1^{\alpha, \beta} f.
\]
Therefore, if \(A_{\varphi}^{\alpha, \beta} = A_{\chi + \psi}^{\alpha, \beta}\) with \(\chi \in K_{\alpha^*}, \psi \in K_{\beta}\) as above and \(\chi_1 = \chi - \overline{\chi} k_0^\beta, \psi_1 = \psi + c k_0^\beta\) for some constant \(c \in \mathbb{C}\), then
\[
A_{\chi + \psi}^{\alpha, \beta} = A_{\chi}^{\alpha, \beta} - c A_{\chi}^{\alpha, \beta} + A_{\psi}^{\alpha, \beta} + c A_{\psi}^{\alpha, \beta} = A_{\varphi}^{\alpha, \beta}.
\]
Moreover, if \(A_{\varphi}^{\alpha, \beta} = A_{\chi + \psi}^{\alpha, \beta} = A_{\chi_1 + \psi_1}^{\alpha, \beta}\) for any other \(\chi_1 \in K_{\alpha^*}, \psi_1 \in K_{\beta}\), then, by Theorem 2.1, there exist \(h_1, h_2 \in H^2\) such that
\[
\chi + \psi - \chi_1 - \psi_1 = \alpha h_1 + \beta h_2.
\]
Hence
\[
\psi - \psi_1 = \beta h_2 + \alpha h_1 + \chi_1 - \chi
\]
and
\[
\psi - \psi_1 = P_\beta(\psi - \psi_1) = P_\beta(\alpha h_1 + \chi_1 - \chi) = c_1 P_\beta 1 = c_1 k_0^\beta
\]
for some constant $c_1$. Similarly,
\[
\chi - \chi_1 = \alpha h_1 + \beta h_2 + \psi_1 - \psi
\]
and
\[
\chi - \chi_1 = P_\alpha(\chi - \chi_1) = P_\alpha(\beta h_2 + \psi_1) = c_2 k^\alpha_0
\]
for some constant $c_2$.

From this,
\[
0 = A^{\alpha,\beta}_\chi - \chi_1 + \psi - \psi_1 = \tau_2 A^{\alpha,\beta}_k + c_1 A^{\alpha,\beta}_k
\]
\[
= (\tau_2 + c_1) A^{\alpha,\beta}_1 = (\tau_2 + c_1) P_{\beta|K_\alpha}.
\]

By Lemma 2.2, $\tau_2 + c_1 = 0$. Putting $c = -c_1 = \tau_2$ we have $\psi_1 = \psi + ck^\beta_0$ and $\chi_1 = \chi - \tau k^\alpha_0$.

3. **Rank-one operators in $\mathcal{S}(\alpha, \beta)$**. Recall that the model space $K_\alpha$ is equipped with a natural conjugation (antilinear, isometric involution) $C_\alpha : K_\alpha \to K_\alpha$, defined in terms of the boundary values by

\[
C_\alpha f(z) = \alpha(z)\overline{f}(z), \quad |z| = 1
\]

(see [13, Subsection 2.3], for more details). A short calculation shows that the conjugate kernel $\tilde{k}^\alpha_w = C_\alpha k^\alpha_w$ is given by

\[
\tilde{k}^\alpha_w(z) = \frac{\alpha(z) - \alpha(w)}{z - w}.
\]

The function $\alpha$ is said to have a nontangential limit at $\eta \in \partial \mathbb{D}$ if there exists $\alpha(z)$ such that $\alpha(z)$ tends to $\alpha(\eta)$ as $z \in \mathbb{D}$ tends to $\eta$ nontangentially (with $|z - \eta| \leq C(1 - |z|)$ for some fixed $C > 0$). We say that $\alpha$ has an angular derivative in the sense of Carathéodory (an ADC) at $\eta \in \partial \mathbb{D}$ if both $\alpha$ and $\alpha'$ have nontangential limits at $\eta$ and $|\alpha(\eta)| = 1$ (for more details see [9, pp. 33–37]). P. R. Ahern and D. N. Clark proved in [1, 2], that $\alpha$ has an ADC at $\eta \in \partial \mathbb{D}$ if and only if every $f \in K_\alpha$ has a nontangential limit $f(\eta)$ at $\eta$. If $\alpha$ has an ADC at $\eta$ and $w$ tends to $\eta$ nontangentially, then the reproducing kernels $k^\alpha_w$ tend in norm to the function $k^\alpha_\eta \in K_\alpha$ given by (1.1) with $\eta$ in place of $w$. Moreover, $f(\eta) = \langle f, k^\alpha_\eta \rangle$ for all $f \in K_\alpha$ and

\[
\tilde{k}^\alpha_\eta(z) = \frac{\alpha(z) - \alpha(\eta)}{z - \eta} = \alpha(\eta)\overline{\eta}k^\alpha_\eta(z).
\]

We can now give some examples of rank-one asymmetric truncated Toeplitz operators (compare with [13, Thm. 5.1]).

**Proposition 3.1.** Let $\alpha$, $\beta$ be two nonconstant inner functions.

(a) For $w \in \mathbb{D}$, the operators $\tilde{k}^\beta_w \otimes k^\alpha_w$ and $k^\beta_w \otimes \tilde{k}^\alpha_w$ belong to $\mathcal{S}(\alpha, \beta)$,

\[
\tilde{k}^\beta_w \otimes k^\alpha_w = A^{\beta,\alpha}_{\tilde{w}} \quad \text{and} \quad k^\beta_w \otimes \tilde{k}^\alpha_w = A^{\beta,\alpha}_{\tilde{w}}.
\]
(b) If both $\alpha$ and $\beta$ have an ADC at the point $\eta$ of $\partial \mathbb{D}$, then the operator $k_\eta^\beta \otimes k_\eta^\alpha$ belongs to $\mathcal{F}(\alpha, \beta)$,

$$k_\eta^\beta \otimes k_\eta^\alpha = A_{k_\eta^\beta + k_\eta^\alpha}^{\alpha, \beta} - 1.$$

**Proof.** (a) Let $w \in \mathbb{D}$ and $f \in K_\alpha$. Since $f (\frac{z-f(w)}{z-w}) \in K_\alpha$ ([(13, Subsection 2.6)]), we have (for $|z| = 1$)

$$A_{\frac{z-f(z)}{z-w}}^{\alpha, \beta} f = P_\beta \left( \frac{\beta(z)f(z)}{z-w} \right)$$

$$= P_\beta \left( \frac{\beta(z)f(z) - f(w)(\beta(z) - \beta(w))}{z-w} \right)$$

$$= f(w)P_\beta \left( \frac{\beta(z) - \beta(w)}{z-w} \right) + f(w)\beta(w)P_\beta \left( \frac{\beta(z)-\beta(w)}{z-w} \right).$$

Similarly,

$$A_{\frac{z-f(z)}{z-w}}^{\alpha, \beta} f = P_\beta \left( \frac{\alpha(z)f(z)}{z-w} \right) = P_\beta \left( \frac{\alpha(z)f(z) - f(w)(\alpha(z) - \alpha(w))}{z-w} \right)$$

$$= f(w)P_\beta \left( \frac{\alpha(z) - \alpha(w)}{z-w} \right) + f(w)\alpha(w)P_\beta \left( \frac{\alpha(z)-\alpha(w)}{z-w} \right).$$

(b) Let $w \in \mathbb{D}$. Then

(3.1) $A_{k_w^\alpha}^{\beta, \alpha} = A_{k_w}^{\alpha, \alpha}$ and $A_{k_w^\beta}^{\alpha, \beta} = A_{k_w}^{\alpha, \beta}$.

Indeed,

$$A_{k_w^\beta}^{\alpha, \beta} f = P_\beta \left( f - \beta(w)k_w f \right) = P_\beta (k_w f) = A_{k_w^\alpha}^{\alpha, \beta} f,$$

for every $f \in K_\alpha$. From this, by Lemma 3.2 in [4],

$$A_{k_w^\beta}^{\alpha, \beta} = \left( A_{k_w^\alpha}^{\beta, \alpha} \right)^* = \left( A_{k_w}^{\alpha, \alpha} \right)^* = A_{k_w}^{\alpha, \beta}.$$

Since for $w \neq 0$ and $|z| = 1$,

$$\frac{\beta(z)}{z-w} = \frac{\beta(z) - \beta(w)}{z-w} + \frac{\beta(w)}{z-w}$$

$$= \overline{k_w^\beta}(z) + \frac{\beta(w)}{w} \frac{wz - 1}{1 - wz} = \overline{k_w^\beta}(z) + \frac{\beta(w)}{w} (K_w(z) - 1),$$

we have, by part (a) and (3.1),

$$\overline{k_w^\beta} \otimes k_w^\alpha = A_{\frac{z-f(z)}{z-w}}^{\alpha, \beta} = A_{k_w^\beta + \frac{\beta(w)}{w} (K_w - 1)}^{\alpha, \beta} = A_{k_w^\beta + \frac{\beta(w)}{w} (1 - k_w^\beta)}^{\alpha, \beta}.$$
Asymmetric truncated Toeplitz operators...

If $\alpha$ and $\beta$ have an ADC at $\eta \in \partial \mathbb{D}$, then $k_{w}^{\alpha}$ and $k_{w}^{\beta}$ converge in norm to $k_{\eta}^{\alpha}$ and $k_{\eta}^{\beta}$, respectively, as $w$ tends to $\eta$ nontangentially. Hence $\tilde{k}_{w}^{\beta} \otimes k_{w}^{\alpha}$ tends to $\tilde{k}_{\eta}^{\beta} \otimes k_{\eta}^{\alpha}$ in the operator norm. On the other hand,

$$
\tilde{k}_{w}^{\beta} + \frac{\beta(w)}{w} \left( k_{w}^{\alpha} - k_{0}^{\beta} \right) \rightarrow \tilde{k}_{\eta}^{\beta} + \frac{\beta(\eta)}{\eta} \left( k_{\eta}^{\alpha} - k_{0}^{\beta} \right)
$$

in $L^{2}(\partial \mathbb{D})$,

which implies that

$$
A_{\tilde{k}_{w}^{\beta} + \frac{\beta(w)}{w} \left( k_{w}^{\alpha} - k_{0}^{\beta} \right)}^{\alpha,\beta} \rightarrow A_{\tilde{k}_{\eta}^{\beta} + \frac{\beta(\eta)}{\eta} \left( k_{\eta}^{\alpha} - k_{0}^{\beta} \right)}^{\alpha,\beta} \text{ in } H^{2},
$$

for every $f \in K_{\alpha}^{\infty}$. Therefore,

$$
\tilde{k}_{\eta}^{\beta} \otimes k_{\eta}^{\alpha} = A_{k_{\eta}^{\alpha} + \frac{\beta(\eta)}{\eta} \left( k_{\eta}^{\alpha} - k_{0}^{\beta} \right)}^{\alpha,\beta}.
$$

Since

$$
\tilde{k}_{\eta}^{\beta}(z) = \frac{\beta(z) - \beta(\eta)}{z - \eta} = \frac{\beta(\eta)}{\eta} k_{\eta}^{\beta}(z),
$$

we get

$$
k_{\eta}^{\beta} \otimes k_{\eta}^{\alpha} = \frac{\eta}{\beta(\eta)} \tilde{k}_{\eta}^{\beta} \otimes k_{\eta}^{\alpha} = \frac{\eta}{\beta(\eta)} A_{\tilde{k}_{\eta}^{\beta} + \frac{\beta(\eta)}{\eta} \left( k_{\eta}^{\alpha} - k_{0}^{\beta} \right)}^{\alpha,\beta} = A_{k_{\eta}^{\alpha} + \frac{\beta(\eta)}{\eta} \left( k_{\eta}^{\alpha} - k_{0}^{\beta} \right)}^{\alpha,\beta}.
$$

□

It was proved in [13, Thm. 5.1] that the only rank-one operators in $\mathcal{J}(\alpha)$ are the nonzero scalar multiples of the operators $\tilde{k}_{w}^{\alpha} \otimes k_{w}^{\alpha}$, $k_{w}^{\alpha} \otimes \tilde{k}_{w}^{\alpha}$ and $k_{\eta}^{\alpha} \otimes k_{\eta}^{\alpha}$. It is still an open question whether the scalar multiples of the operators from Proposition 3.1 are the only rank-one operators in $\mathcal{J}(\alpha, \beta)$ for arbitrary inner functions $\alpha$ and $\beta$.

References


Joanna Jurasik  
Department of Mathematics  
Maria Curie-Skłodowska University  
pl. Marii Curie-Skłodowskiej 1  
20-031 Lublin  
Poland  
e-mail: asia.blicharz@op.pl

Bartosz Łanucha  
Department of Mathematics  
Maria Curie-Skłodowska University  
pl. Marii Curie-Skłodowskiej 1  
20-031 Lublin  
Poland  
e-mail: bartosz.lanucha@poczta.umcs.lublin.pl

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